ISOPARAMETRIC SUBMANIFOLDS AND THEIR COXETER GROUPS

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0. Introduction

In the later 1930's Elie Cartan defined the notion of isoparametric functions on a space form $N$ and began their study [4]–[7]. A smooth function $f: N \to \mathbb{R}$ ($N = \mathbb{R}^{n+1}$, $S^{n+1}$ or $H^{n+1}$), is isoparametric, if $\Delta f$ and $|\nabla f|^2$ are functions of $f$. Among other things Cartan showed that the level hypersurfaces of $f$ are parallel, and each has constant principal curvatures. And conversely, he showed that if $M$ is a hypersurface of $N$ with constant principal curvatures, then there is at least a local isoparametric function having $M$ as a level. Cartan called such a hypersurface isoparametric. In the last ten years, many people carried forward this research [19, 25]. Finally around 1980, Münzner [18] completed the beautiful structure theory of isoparametric hypersurfaces in the spheres, and thereby reduced their classification to a (difficult!) algebraic problem. Many people subsequently made contributions to this classification problem including U. Abresch [1], D. Ferus, H. Karcher, H. F. Münzner [15], et al. While there has been considerable recent progress, it seems much remains to be done. By and large, the theory of isoparametric hypersurfaces has been a special subject by itself; however in recent years there have been applications to the theory of harmonic maps [12], and minimal submanifolds [14, 19, 23]. Recently, Eells [12] gave a definition of isoparametric maps for the purpose of constructing harmonic maps. S. Carter and A. West [3] gave a stronger definition of isoparametric maps from $N^{n+m}$ to $R^m$; their purpose being to generalize Cartan's work to higher codimension. Using their definitions, they were able to show that there is a Coxeter group (i.e., a finite group generated by reflections) associated to each isoparametric map $f: N^{n+2} \to R^2$. However, they did not obtain a similar result for larger $m$. They were also unable to construct a global isoparametric map for a given isoparametric submanifold.

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However, this work of Münzner and Carter-West was instrumental in suggesting to me many of the ideas in this paper, whose main goal is to generalize Cartan's theory of isoparametric functions and hypersurfaces to a general theory of isoparametric maps and submanifolds.

In this paper, we make the following definitions:

**Definition 1.** A complete, connected, smooth \( n \)-dimensional submanifold \( M \) of \( N^{n+m} (N = \mathbb{R}^{n+m}, S^{n+m}, \text{or } H^{n+m}) \) is called isoparametric if the normal bundle is flat and the principal curvatures of \( M \) in the directions of any parallel normal vector field are constant.

**Definition 2.** A smooth map \( f = (f_{n+1}, \cdots, f_{n+m}) : N^{n+m} \to \mathbb{R}^m \) is called isoparametric if

1. \( f \) has a regular point,
2. \( \nabla f_\alpha \cdot \nabla f_\beta \) and \( \Delta f_\alpha \) are functions of \( f \), for all \( \alpha, \beta \),
3. \( [\nabla f_\alpha, \nabla f_\beta] \) is a linear combination of \( \nabla f_{n+1}, \cdots, \nabla f_{n+m} \) with coefficients being functions of \( f \), for all \( \alpha, \beta \).

Our definition of isoparametric map is stronger than Eells' and weaker than Carter-West's. All three definitions agree with Cartan's when \( m = 1 \). We will show that a compact isoparametric submanifold \( M^n \) of \( R^{n+m} \) is isoparametric in \( R^{n+m} \) if and only if \( M^n \) is contained and isoparametric in a standard sphere of \( R^{n+m} \). We also show that a noncompact complete isoparametric submanifold \( M^n \) of \( R^{n+m} \) is always a product \( R^{m_1} \times M^{n-m_1}_1 \), where \( M_1 \) is a compact isoparametric submanifold of \( R^{n+m-m_1} \) for some \( m_1 \). Therefore, in the following discussion, we will assume \( N = R^{n+m} \) and \( M^n \) is a compact isoparametric submanifold of \( R^{n+m} \). Since the set of all isoparametric hypersurfaces of \( S^{n+1} \) is the "same" as the set of all compact isoparametric submanifolds of \( R^{n+2} \) of codimension 2, Münzner's results gave a complete structure theory for them. There are many homogeneous and non-homogeneous codimension 2 isoparametric submanifolds in \( R^{n+2} \) [15, 20, 25]. There are also many examples from transformation group theory for higher codimension [22]. For example, a principal orbit of the adjoint action of a compact Lie group \( G \) on its Lie algebra \( \mathfrak{g} \) is always isoparametric. More generally, if \( G \) acts on \( R^n \) orthogonally with a section [22], i.e. there is a submanifold \( F \) of \( R^n \) which meets every principal orbit orthogonally, then the principal orbit is isoparametric in \( R^n \). In particular, these representations include the isotropy representations of symmetric spaces. Conversely, suppose \( G \) acts on \( R^n \) orthogonally and \( M \) is an orbit of \( G \) such that \( M \) is full and isoparametric in \( R^n \). Then this section admits a section and \( M \) is a principal orbit of \( G \) [22]. Therefore all homogeneous isoparametric submanifolds are obtained from the principal orbits of linear orthogonal actions with sections. A simple example of a linear action with
section is the action of $O(n)$ on the space $S$ of real symmetric trace zero $n \times n$ matrices via conjugation. Then the canonical form in linear algebra for symmetric matrices implies that $F = \{ a \in S | a \text{ is diagonal} \}$ is a section for this action. $F$ is also the normal plane of the principal orbit at $\text{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i$'s are distinct. The $O(n)$-invariant polynomial map $f: S \to R^{n-1}$ defined by $f(a) = (\text{tr} a^2, \ldots, \text{tr} a^n)$ is an isoparametric map. The principal orbit type is $O(n)/H$, where $H = \bigoplus_{i=1}^n Z_2$. The “Weyl group” $N(H)/H = S_n$ is a Coxeter group on $R^n$. One main conclusion of this paper is that these results still hold even if we replace the principal orbit by an arbitrary, not necessarily homogeneous, isoparametric submanifold $M$, and replace the orbit foliation by the family of all parallel submanifolds of $M$. Therefore, this example can be thought as a concrete model for the theory of isoparametric submanifolds.

A submanifold $M^n$ of $R^{n+m}$ is called full if $M$ is not contained in any hyperplane of $R^{n+m}$. The proof of the following theorem is rather straightforward, and it will be given in §2.

**Theorem A.** If $f: N^{n+m} \to R^n$ is isoparametric, then connected components of $f^{-1}(c)$ are isoparametric for regular value $c$, and all regular submanifolds of $f$ are parallel.

Conversely, given an isoparametric submanifold $M^n$ of $R^{n+m}$, we can construct an isoparametric map $f$ with $M$ being a regular level of $f$. And there are many interesting facts that turn up in our proof of the converse.

Let $M^n$ be full and isoparametric in $R^{n+m}$. We show that the holonomy group of $\nu(M)$ is trivial, in particular there is a global orthonormal parallel normal frame field $e_\alpha (n + 1 \leq \alpha \leq n + m)$ on $M$. And there exist $p$ distributions $E_i$ such that $TM = \bigoplus_{i=1}^p E_i$, and the second fundamental form in the direction of $e_\alpha$ restricted to $E_i$ has only one eigenvalue $n_i^\alpha$ of multiplicity $m_i$, and $n_i = (n_i^{n+1}, \ldots, n_i^{n+m})$ are $p$ distinct vectors in $R^m$. In §1, we prove that the group $W$ generated by the $p$ reflections $R_i$ of $R^n$ along $n_1, \ldots, n_p$ is an effective Coxeter group. We call $W$ the Coxeter group associated to $M$. We also show that $E_i$ is integrable, with leaf being an $m_i$-dimensional sphere of radius $1/|n_i|$. And the focal set of $M$ at $q$ is the union of $p$ hyperplanes $l_i(q)$ in the affine normal plane $q + \nu(M)_q$. Each $l_i(q)$ is perpendicular to the leaf of $E_i$ through $q$ at its center. Moreover, we show that the group generated by the $p$ reflections of the affine normal plane at $q$ in $l_i(q)$ is isomorphic to $W$. Therefore, we have

**Theorem B.** Let $M^n$ be a full compact isoparametric submanifold of $R^{n+m}$, $W$ the associated effective Coxeter group on $R^n$. Then

(i) $W$ acts on $M$ via diffeomorphisms; $R_i$ corresponds to a diffeomorphism $\phi_i$,
where $\phi_i(q)$ is the antipodal point of $q$ on the leaf (sphere) of $E_i$ through $q$.

(ii) $W$ acts on the affine normal plane $q + v(M)_q$ via rigid motions, $R_i$ corresponds to the reflection $T_i$ of $q + v(M)_q$ in $l_i(q)$. Moreover, $T_i(q) = \phi_i(q)$.

If $M$ is the principal orbit of the isotropy representation of a symmetric space $G/K$, then the associated Coxeter group in Theorem B for $M$ is the Weyl group (hence is crystallographic) [16], $q + v(M)_q$ is a maximal Lie triple system, $n_i$ is always proportional to some root, and $m_i$ is the number of roots proportional to $n_i$. Geometrically, $m_i$ is the difference of dimensions of the principal orbit and the subprincipal orbits [22]. So the following theorem can be thought as a generalization of Chevalley Restriction Theorem [26] to isoparametric submanifolds.

**Theorem C.** Let $M^n$ be full and isoparametric in $R^{n+m}$, $W$ the associated Coxeter group, $q \in M$ a given point on $M$, and $V$ the affine normal plane $q + v(M)_q$. If $u: V \to R$ is a $W$-invariant homogeneous polynomial of degree $k$, then $u$ can be extended uniquely to a homogeneous degree $k$ polynomial $f$ on $R^{n+m}$, which is constant on $M$.

Now we only need to use another theorem of Chevalley to obtain the converse of Theorem A.

**Theorem (Chevalley [9]).** If $W$ is an effective Coxeter group on $R^n$, then the ring of $W$-invariant polynomials on $R^n$ is a polynomial ring on $m$ generators $u_1, \cdots, u_m$.

Applying Theorem C to these generators $u_1, \cdots, u_m$, we obtain

**Theorem D.** Let $M, W, q,$ and $V$ be as in Theorem C, and let $u_1, \cdots, u_m$ be a set of generators of the $W$-invariant polynomials on $V$. Then $u = (u_1, \cdots, u_m)$ extends uniquely to an isoparametric polynomial map $f: R^{n+m} \to R^m$ having $M$ as a regular level set. Moreover,

1. each regular level set of $f$ is connected,
2. the focal set of $M$ is the set of all critical points of $f$,
3. $V \cap M = Wq$,
4. $f(R^{n+m}) = u(V)$,
5. for $v \in V, f(v)$ is a regular value if and only if $v$ is $W$-regular. We call $f$ the Cartan polynomial map for $M$ corresponding to the $u_i$.

The proof of Theorem C and D are given in §3.

We also obtain a structure theory for isoparametric submanifolds. An isoparametric submanifold $M^n$ of $R^{n+m}$ is called irreducible if $M$ cannot be written as a product of two lower dimensional isoparametric submanifolds. We show that every isoparametric submanifold can be written uniquely up to permutation as a product of irreducible ones. Moreover, $M$ is irreducible if and only if the associated Coxeter group is irreducible. Therefore it suffices to
classify all irreducible ones. It is obvious that if \( f : \mathbb{R}^{n+m} \to \mathbb{R}^m \) is isoparametric and \( \phi : \mathbb{R}^m \to \mathbb{R}^m \) is a diffeomorphism, then \( \phi \circ f \) is also isoparametric. However, if we fix a set of generators for each Coxeter group, then the polynomial isoparametric map obtained in Theorem D is unique. The proof of Theorem D also provides an algebraic system of equations (completely determined by the Coxeter group \( W \) and the multiplicities \( m_i \)), of which \( f \) is a solution. Moreover, given any irreducible Coxeter group \( W \) on \( \mathbb{R}^m \) with the positive root system \( \Delta^+ = \{ r_1, \ldots, r_p \} \) and positive integers \( m_i \) associated to each positive root \( r_i \) (such that \( m_i = m_j \) if \( r_i \) and \( r_j \) are in the same orbit of \( W \)) the solutions of the algebraic system associated to \( W \) and \( m_i \) are isoparametric maps from \( \mathbb{R}^{n+m} \) to \( \mathbb{R}^n \), where \( n = \sum_{i=1}^{p} m_i \).

Let \( q_0 \) be a fixed point in an isoparametric submanifold \( M^n \) of \( \mathbb{R}^{n+m} \), and \( q = q_0 + v \in q_0 + \nu(M)_{q_0} \). Then there is a unique parallel normal field \( V \) on \( M \) such that \( V(q_0) = v \), and the parallel set through \( q \), \( M_q = \{ x + V(x) | x \in M \} \), is always a smooth submanifold. Moreover, \( \{ M_q \} \) defines a singular foliation of \( \mathbb{R}^{n+m} \), which has \( M \) as one regular leaf, and all other regular leaves are also isoparametric. If \( M \) is a principal orbit of a linear action \( \rho \) with a section on \( \mathbb{R}^{n+m} \), then this singular foliation is the orbit foliation of \( \rho \). To describe the general case in more detail, let \( U \) be the fundamental region of the Coxeter group \( W \) action on \( q_0 + \nu(M)_{q_0} \). Then \( U \) is a simplicial cone, and \( U \) is the leaf space. If \( q \in \text{Int}(\sigma) \), where \( \sigma \) is a \( k \)-dimensional simplex of \( U \) and \( \sigma \) is open in \( \bigcap_{i=1}^k l_i \), then the leaf \( M_q \) through \( q \) has dimension \( n - \sum_{j=1}^{k} m_{ij} \). Since we construct this singular foliation of \( \mathbb{R}^{n+m} \) from \( M \) by a purely geometric construction, the geometry of this foliation is part of the geometry of \( M \). There are also applications to the theory of harmonic maps and minimal submanifolds just as in the homogeneous case [14, 17, 21], even though in general there is no Lie group action around. We show in §4 that the Gauss map for each leaf is harmonic. We also prove that for each \( k \)-dimensional simplex \( \sigma \) as above, there is a point \( q_k \in \sigma \cap S^{m-1} \) such that the leaf through \( q_k \) is a minimal submanifold of \( S^{n+m-1} \) of dimension \( n - \sum_{j=1}^{k} m_{ij} \).

So it is rather clear that isoparametric submanifolds and their singular foliations can be viewed as geometric generalizations of the principal orbits and the orbit foliations of linear actions with section. By studying the geometry of isoparametric submanifolds, we obtain a more clear geometric picture of these homogeneous spaces and their actions. It seems that Riemannian geometry also gives a different method to study invariant theory. For example, as a consequence of Theorem C, we see that if \( G \) acts on \( \mathbb{R}^n \) orthogonally with a section then the ring of \( G \)-invariant polynomials on \( \mathbb{R}^n \) is a polynomial ring.

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Added in revision. Professor G. Schwartz showed us a preprint of a paper by J. Dadok titled "polar coordinates induced by actions of compact Lie groups" [11], and pointed out to us that Dadok's definition of polar representation is equivalent to our definition of a linear action with a section. Moreover, Dadok classified polar representations in this paper and using his classification he showed that given any polar representation $\rho: G \to O(n)$ there is a symmetric space $G/K$ and an isometry $A: R^n \to T(G/K)_{eK}$ such that the orbits of $\rho$ are mapped under $A$ to the orbits of $\text{Ad}(K)$. We have also since noted a paper by L. Conlon [10], which has many of Dadok's results. Therefore, all the homogeneous isoparametric submanifolds are obtained from the principal orbits of the isotropy representations of symmetric spaces.

Some further progress has been made in understanding the nature of isoparametric submanifolds. For example, we have shown that the associated Coxeter group $W$ for an isoparametric submanifold must be crystallographic, i.e. $W$ is a Weyl group, and that in each irreducible example at most two distinct integers occur as multiplicities. Moreover, the homology and cohomology of isoparametric submanifolds can be computed from their Dynkin diagrams and multiplicities. These results will be reported in a separate paper by W. Y. Hsiang, R. S. Palais, and the author. It is also shown by the author in another paper that the convexity theorem of Kostant can be generalized to isoparametric submanifold $M^n$ in $R^{n+m}$, namely the image by orthogonal projection of $M$ to a fixed normal plane $v(M)_{x_0}$ is the convex hull of the orbit of $x_0$ under the Weyl group of $M$.

1. Isoparametric submanifolds

In this section, we will study the geometry of isoparametric submanifolds and associate to such manifolds Coxeter groups.

To set notations, we will review briefly the local geometry of submanifolds of Euclidean space. Let $M$ be an $n$-dimensional submanifold of $R^{n+m}$. We choose a local orthonormal frame field $e_1, \ldots, e_{n+m}$ on $R^{n+m}$ such that $e_1, \ldots, e_n$ are tangent to $M$. Let $X$ be the position vector of $M$, i.e., $X$ is the inclusion map from $M$ to $R^{n+m}$. Let $w_1, \ldots, w_{n+m}$ be the dual coframe. We can write

$$dX = \sum_i w_i e_i, \quad de_A = \sum_B w_A e_B.$$

Henceforth, we shall agree on the index ranges

$$1 \leq i, j, k \leq n, \quad n + 1 \leq \alpha, \beta, \gamma \leq n + m, \quad 1 \leq A, B, C \leq n + m.$$
The structure equations of $\mathbb{R}^{n+m}$ are
\begin{equation}
\begin{aligned}
    dw_A &= \sum_B w_{AB} \wedge w_B, \\
    w_{AB} + w_{BA} &= 0, \\
    dw_{AB} &= \sum_C w_{AC} \wedge w_{CB}.
\end{aligned}
\end{equation}

Restricting these forms to $M$, we have
\begin{equation}
\begin{aligned}
    dw_i &= \sum_j w_{ij} \wedge w_j, \\
    w_{ij} + w_{ji} &= 0, \\
    dw_{ij} - \sum_k w_{ik} \wedge w_{kj} &= -\Omega_{ij} = -\sum_\alpha w_{i\alpha} \wedge w_{j\alpha},
\end{aligned}
\end{equation}

where $(w_{ij})$ is the Levi-Civita connection and $\Omega$ is the Riemann tensor of $M$.

There is also an induced connection on the normal bundle $\nu(M)$, namely
\[ De_a = \sum\beta w_{a\beta} e_\beta. \]
Then
\[ dw_{i\beta} + \sum_j w_{aj\gamma} \wedge w_{i\beta} = -\Omega_{i\beta}, \]
where $\Omega_{i\beta} = \sum_\alpha w_{i\alpha} \wedge w_{i\beta}$ is the normal curvature of $M$. We say that $\nu(M)$ is flat if the normal curvature is zero. A normal vector field $\nu$ is parallel if $D\nu = 0$.

Suppose $\nu(M)$ is flat, then locally we can choose a normal orthonormal frame field $e_a$ on $M$ such that $w_{a\beta} = 0$. The two fundamental forms are
\[ I = \sum_\alpha w_{i\alpha}^2, \]
\[ II = \sum_{i, j, \alpha} h_{i\alpha}^a w_j e_a = \sum_\alpha II_a e_a. \]

The eigenvalues of $II_a$ with respect to $I$ are called the principal curvatures of $M$ in the directions of $e_a$. And $H = \sum_\alpha H_a e_a$ is the mean curvature vector, where $H_a = \sum_i h_{i\alpha}^a$ is the mean curvature in the direction $e_a$.

1.1. Proposition. Suppose $M^n \subset \mathbb{R}^{n+m}$ has flat normal bundle, then locally there exists an orthonormal frame field $e_A$ such that $e_A$ is parallel and the $II_a$ are diagonalized simultaneously, i.e., $w_{a\beta} = 0$ and $w_{i\alpha} = \lambda_{i\alpha} w_j$.

Proof. We have noted that there is a local orthonormal normal frame $e_a$ such that $w_{a\beta} = 0$. Then
\[ 0 = dw_{a\beta} = \sum_k w_{ak} \wedge w_{k\beta} = -\sum_{k, i < j} \left( h_{ik}^a h_{ij}^\beta - h_{ij}^\alpha h_{ki}^\beta \right) w_j \wedge w_j. \]

So $A_a A_\beta = A_\beta A_a$, where $A_a = (h_{ij}^\alpha)$. Hence there is an orthonormal local tangent frame field $e_1, \cdots, e_n$ such that $A_a$ is diagonal for all $a$, i.e. $h_{ij}^\alpha = \lambda_{ij}^\alpha$, or equivalently $w_{i\alpha} = \lambda_{i\alpha} w_j$. q.e.d.

It follows from Definition 1 that if $M^n$ is isoparametric in $S^{n+m}$, then $M^n$ is also isoparametric in $\mathbb{R}^{n+m+1}$. Later in this section we will show that a compact isoparametric submanifold of $\mathbb{R}^{n+m}$ is contained and isoparametric in a suitable standard sphere. So we will concentrate on isoparametric submanifolds of Euclidean space.
We will derive in the following some geometric properties of an isoparametric submanifolds $M$ in $\mathbb{R}^{n+m}$. Since $\nu(M)$ is flat, Proposition 1 gives a frame field $e_\alpha$ such that $w_\alpha\beta = 0$ and $w_\alpha = \lambda^\alpha_i w_i$, where $\lambda^\alpha_i$ are constant on $M$. So there exist uniquely eigen-distributions $E_1, \cdots, E_p$ for $II$ with dimensions $m_1, \cdots, m_p$ respectively, such that $II_{\alpha\beta} E_i$ has only one eigenvalue $n^\alpha_i$ with multiplicity $m_i$, and $n_i = (n_{i+1}^\alpha, \cdots, n_{i+m}^\alpha)$ are $p$ distinct vectors in $\mathbb{R}^m$. We arrange our indices so that $\{e_k \mid \Sigma_{i=1}^{m_i} n^{\alpha}_i < k \leq \Sigma_{i=1}^{m_{i-1}} m_i \}$ is a base for $E_i$. Next, we consider the following normal vectors $\nu_i = \Sigma_{\beta} n^\alpha_i e_\alpha$. The $\nu_i$'s are independent of the choice of the parallel frame $e_\alpha$. To see this, let $e_p^* = \Sigma_\beta s_{\alpha\beta} e_\beta$ be another local parallel normal frame. Then $(s_{\alpha\beta})$ is a constant $m \times m$ orthogonal matrix, and

$$\lambda^\alpha_i = \Sigma_\beta s_{\alpha\beta} \lambda^\beta_i.$$

So it follows that

$$\Sigma_\alpha \lambda^\alpha_i e_\alpha^* = \Sigma_\alpha \lambda^\alpha_i e_\alpha.$$

We call these $\nu_i$'s the curvature normal vectors of $M$. If $M^n$ is isoparametric in $\mathbb{R}^{n+m}$, then $M$ is obviously also isoparametric in $\mathbb{R}^{n+m+1}$. To avoid this redundancy, we make the following definition:

1.2. Definition. A submanifold $M^n$ of $\mathbb{R}^{n+m}$ is full if $M$ is not contained in any hyperplane of $\mathbb{R}^{n+m}$.

1.3. Proposition. An isoparametric submanifold $M^n$ in $\mathbb{R}^{n+m}$ is full if and only if the curvature normals $\nu_1, \cdots, \nu_p$ span $\nu(M)$. In particular, if $M^n$ is full and isoparametric in $\mathbb{R}^{n+m}$, then $m \leq n$.

Proof. It is obvious that $\nu_1, \cdots, \nu_p$ span $\nu(M)$ if and only if the rank of the $m \times p$ matrix $N = (n_i, \cdots, n_p)$ is $m$. Suppose $M$ is contained in a hyperplane normal to $(1,0, \cdots, 0)$ in $\mathbb{R}^{n+m}$. Then we can choose $e_{n+1} = (1,0, \cdots, 0) \in \mathbb{R}^{n+m}$, so $\lambda_i^{n+1} = 0$ for all $i$, and rank$(n_1, \cdots, n_p) \leq m - 1$. Conversely, if rank$(n_1, \cdots, n_p) < m$, then there is a nonzero vector $c \in \mathbb{R}^m$ such that $c \cdot n_i = 0$ for all $1 \leq i \leq p$. We claim $\Sigma_\alpha c^\alpha e_\alpha$ is a constant vector $b$ in $\mathbb{R}^{n+m}$. To see this, we calculate the differential of the map $\Sigma_\alpha c^\alpha e_\alpha$ on $M$:

$$d \left( \Sigma_\alpha c^\alpha e_\alpha \right) = \Sigma_\alpha c^\alpha de_\alpha = - \Sigma_\alpha c^\alpha \lambda^\alpha_i w_i e_i = - \Sigma c \cdot n_i \text{id}_{E_i} = 0.$$

Then it follows that

$$d( X \cdot b) = dX \cdot b = \Sigma_i w_i e_i \cdot b = 0.$$

Hence $X \cdot b = a$ constant $c_0$, i.e. $M$ is contained in a hyperplane. q.e.d.

Next, we study the holonomy of $\nu(M)$. 
1.4. Proposition. Let $M^n$ be full and isoparametric in $R^{n+m}$, and $v_1, \cdots, v_p$ its curvature normal vectors. Then the holonomy group $G$ of $\nu(M)$ is a subgroup of the permutation group $S_p$. Moreover, if $v_1, \cdots, v_p$ have distinct length, then $G = \{\text{id}\}$.

Proof. Let $r: [0,1] \to M$ be a closed curve in $M$ with $r(0) = r(1) = q$. Suppose parallel translation of $e_\alpha(q)$ along $r$ is given by $\Sigma_{\beta} a_{\alpha\beta} e_\beta(q)$, i.e. $A = (a_{\alpha\beta})$ is in $G$. From the definition of isoparametric, there exists a permutation $\sigma$ of $1,2,\cdots,p$ such that

$$\lambda_{\sigma(i)}^\alpha = \sum_{\beta} a_{\alpha\beta} \lambda_i^\beta.$$ 

Let $n_1, \cdots, n_p$ be as before, then we have $An_j = n_{\sigma(j)}$, i.e. $G \subset S_p$. If $n_1, \cdots, n_p$ have distinct length, then since $A$ is orthogonal we conclude that $A = \text{id}$. q.e.d.

Remark. In the end of §3, we will show that the holonomy of $\nu(M)$ is always trivial.

For the remainder of this paper until 3.5, we will make the following standing assumptions:

(1) $M^n$ is always assumed to be full and isoparametric in $R^{n+m}$ with the inclusion map $X: M \to R^{n+m}$, and the holonomy of $\nu(M)$ is trivial.

(2) $e_i$ is a local orthonormal tangent frame field and $e_\alpha$ is a global orthonormal parallel normal frame field on $M$. Such that

$$w_{\alpha\beta} = 0, \quad w_{i\alpha} = \lambda_i^\alpha w_i,$$

where $\lambda_i^\alpha$ are constant.

(3) Let $E_1, \cdots, E_p$ be the unique eigendistributions on $M$ for II. We let $m_j$ be the rank of $E_j$, and arrange our indices so that $E_j$ is spanned by $\{e_k|\Sigma_{j=1}^{j-1} m_j < k \leq \Sigma_{j=1}^j m_j\}$. Obviously $\Sigma_{j=1}^p m_j = n$.

(4) Let

$$n_1^\alpha = \lambda_1^\alpha = \cdots = \lambda_{m_1}^\alpha,$$

$$n_{p}^\alpha = \lambda_{\mu}^\alpha = \cdots = \lambda_\alpha^n, \quad \mu_p = 1 + \sum_{j=1}^{p-1} m_j.$$ 

Then $n_1, \cdots, n_p$ are $p$ distinct vectors in $R^m$. The vectors $v = \Sigma_{\alpha} n_\alpha^a e_\alpha$ are the curvature normal vectors of $M$.

For a given global orthonormal parallel normal frame $e_\alpha$, we define a normal bundle map

$$Y: M^n \times R^m \to R^{n+m}, \quad Y(q, z) = q + \sum_{\alpha} z_\alpha e_\alpha(q).$$
Geometrically, \( Y(q + \mathbb{R}^m) = q + \nu(M)_q \) is the affine normal plane at \( q \), we denote it by \( N_q \) and \( Y(M \times \{z\}) \) is an \( n \)-dimensional submanifold parallel to \( M \) in \( \mathbb{R}^{n+m} \) for almost all \( z \). The differential of \( Y \) is

\[
dY = d\left(X + \sum \alpha z_a e_a\right) = dX + \sum \alpha z_a de_a + \sum dz_a e_a
\]

(1.5)

\[
= \sum_{i=1}^p (1 - z \cdot n_i)\text{id}_{E_i} + \sum_{a} dz_a e_a.
\]

So we have

1.5. Proposition. The critical point set of \( Y \) is \( M \times \Lambda \), where \( \Lambda \) is the union of \( p \) hyperplanes \( l_i \) in \( \mathbb{R}^m \) defined by the linear equations \( 1 = n_i \cdot z \) for \( i = 1, \ldots, p \).

1.6. Definition. The focal set \( \Sigma \) of \( M \) in \( \mathbb{R}^{n+m} \) is the set of all critical values of the normal bundle map \( Y \).

Focal set is well defined and independent of the choice of \( e_a \). To see this, we let \( \pi: \nu(M) \to M \) be the normal bundle, and \( \tilde{Y}: \nu(M) \to \mathbb{R}^{n+m} \) the map defined by \( \tilde{Y}(v) = \pi(v) + v \). Then we have a canonical isomorphism \( \phi: M \times \mathbb{R}^m \to \nu(M) \) with \( \phi(x, z) = \Sigma_a z_a e_a(x) \), and \( Y = \tilde{Y} \circ \phi \). Then the focal set of \( M \) is the set of all critical values of \( Y \), i.e. the focal variety of \( M \) in the classical sense.

It is obvious that we have

1.7. Proposition. \( \Sigma = \bigcup_{q \in M} \Sigma_q \), when \( \Sigma_q \) is the union of the \( p \) hyperplanes \( l_i(q) = Y(q \times l_i) \) in \( N_q = q + \nu(M) \), where \( l_i \) is the hyperplane of \( \mathbb{R}^m \) defined by \( z \cdot n_i = 1 \).

In order to understand the eigendistributions, we need some formulas for the Riemannian connection of \( M \) in terms of \( E_t \). Using (1.4) and the structure equations, we have

\[
dw_{ia} = d(\lambda^i_a w_j) = \lambda^i_a dw_i = \lambda^i_a \sum_{j} w_{ij} \wedge w_j
\]

\[
= \sum_{j} w_{ij} \wedge w_{ja} = \sum_{j} \lambda^a_j w_{ij} \wedge w_j,
\]

so

\[
\sum_j \left( \lambda^i_a - \lambda^a_i \right) w_{ij} \wedge w_j = 0.
\]

Suppose

\[
w_{ij} = \sum_k r_{ijk} w_k,
\]

then we have

\[
\sum_{j,k} \left( \lambda^a_i - \lambda^a_j \right) r_{ijk} w_k \wedge w_j = 0,
\]
Therefore we have

1.8. Proposition. Let \( w_{ij} = \sum_k r_{ijk} w_k \). If \( e_{i_1}, e_{i_2} \in E_{i_1}, \ e_j \in E_{i_2}, \text{and} \ i_1 \neq i_2, \) then \( r_{ijk} = 0. \)

Using this proposition, we can obtain the following results for the eigendistributions:

1.9. Theorem. Suppose \( M^n \) is full and isoparametric in \( R^{n+m} \), then for all \( 1 \leq i \leq p, \) we have

(i) \( E_i \) is integrable.

(ii) If \( n_i \neq 0, \) then the leaf \( L_i \) of \( E_i \) through \( q \in M \) is an \( m_i \)-dimensional standard sphere of radius \( 1/|n_i| \) with center \( c_i \). Moreover \( \bar{q} + v_i(\bar{q})/|n_i|^2 = c_i \), for all \( \bar{q} \in L_i \), and \( l_i(q) \) intersects the \( (m_i + 1) \)-dimensional plane spanned by \( L_i \) orthogonally at \( c_i \).

(iii) If \( n_i = 0, \) then \( L_i \) is an \( m_i \)-dimensional plane of \( R^{n+m} \).

(iv) If \( M \) is compact, then \( n_i \neq 0 \) for all \( 1 \leq i \leq p. \)

(v) Let \( T_i \) denote the reflection of \( N_q = q + v(M)q \) in the hyperplane \( l_i(q) \) if \( n_i \neq 0, \) and \( T_i = \text{id} \) if \( n_i = 0. \) Then \( T_i(q) \in M. \)

Proof. For simplicity, we assume \( i = 1. \) \( E_1 \) is defined by the following 1-form equations on \( M: \)

\[
 w_k = 0, \quad m_1 < k \leq n. 
\]

Using the structure equations and Proposition 1.8, we have

\[
 dw_k = \sum_{i=1}^{m_1} w_{ki} \wedge w_i + \sum_{i,j=1}^{m_1} r_{kij} w_j \wedge w_i = 0. 
\]

By Frobenius theorem, \( E_1 \) is integrable.

Let \( L_1 \) be the leaf of \( E_1 \) through \( q. \) Then the above calculation also shows that \( w_{ik} = 0 \) for \( i \leq m_1, \ k > m_1. \) To compute the curvature of \( L_1, \) we use (1.3)

\[
 \Omega_{ij} = \sum_{k>m_1} w_{ik} \wedge w_{kj} + \sum_{\alpha} w_{i\alpha} \wedge w_{j\alpha} = |n_1|^2 w_i \wedge w_j. 
\]

So \( L_1 \) has constant sectional curvature \( |n_1|^2. \) If \( n_1 = 0, \) then we have \( w_{ik} = 0, \ w_{i\alpha} = 0 \) for \( k > m_1, \ n < \alpha \leq n + m; \) i.e. \( L_1 \) is totally geodesic in \( R^{n+m}. \) So if \( n_1 = 0, \) then \( L_1 \) is an \( m_1 \)-dimensional plane of \( R^{n+m}, \) which proves (iii) and (iv). To obtain (ii), we divide the proof into four steps.

Step 1. The map \( X + (1/|n_1|^2)\sum_\alpha n_1^\alpha e_\alpha \) maps \( L_1 \) to a constant vector \( c_1 \) in \( R^{n+m}. \) In particular, \( c_1 = \bar{q} + v_1(\bar{q})/|n_1|^2 \) for all \( \bar{q} \in L_1. \)
This can be seen directly from the differential of this map. We have
\[ d \left( X + \sum_{a} \frac{n_{1}^{a} e_{a}}{|n_{1}|^{2}} \right) = dX + \sum_{a} \frac{n_{1}^{a}}{|n_{1}|^{2}} de_{a} \]
\[ = \sum_{i} w_{i} e_{i} - \sum_{i, a} \frac{n_{1}^{a}}{|n_{1}|^{2}} \lambda_{i}^{a} w_{i} e_{i} \]
\[ = \sum_{i=1}^{p} \left( 1 - \frac{n_{1} \cdot n_{i}}{|n_{i}|^{2}} \right) id_{E_{i}}, \]
which vanishes on \( E_{1} \).

**Step 2.** \( \xi_{1} = E_{1} + \nu(M) | L_{1} \) is a fixed linear \((m_{1} + m)\)-subspace of \( R^{n+m} \).

We define a map \( g : L_{1} \to \text{Gr}(m_{1} + m, n + m) \) by \( g(x) = (m_{1} + m) - \text{plane} \ E_{1}(M)_{x} \oplus \nu(M)_{x} \). Then locally, we can write
\[ g = e_{1} \wedge \cdots \wedge e_{m_{1}} \wedge e_{n+1} \cdots \wedge e_{n+m}. \]

Using (1.1), we have
\[ dg = \sum_{i \leq m_{1}} \sum_{k > m_{1}} w_{ik} e_{1} \wedge \cdots \wedge e_{i} \wedge \cdots \wedge e_{m_{1}} \wedge e_{n+1} \cdots \wedge e_{n+m} + \sum_{k > m_{1}} w_{ak} e_{1} \wedge \cdots \wedge e_{m_{1}} \wedge e_{n+1} \cdots \wedge e_{k} \cdots \wedge e_{n+m}. \]

But \( w_{ik} = 0, w_{ak} = 0 \) on \( L_{1} \) for \( i \leq m_{1}, k > m_{1} \). So \( dg = 0 \).

**Step 3.** \( L_{1} \) is contained in the \((m_{1} + m)\)-plane \( V_{1} = c_{1} + \xi_{1} \). This follows from Steps 1 and 2.

**Step 4.** The second fundamental form \( II \) of \( L_{1} \) in \( V_{1} \) is \( (\sum_{1}^{m_{1}} w_{i}^{2}) v_{i} \).

We have \( w_{ik} = 0 \) on \( L_{1} \) for \( k > m_{1} \), so \( II \cdot e_{k} = 0 \). Suppose \( \Sigma_{a} z_{a} e_{a} \) is perpendicular to \( v_{1} = \Sigma_{a} n_{1}^{a} e_{a} \), i.e., \( z \cdot n_{1} = 0 \), then \( II \cdot \Sigma_{a} z_{a} e_{a} = +\Sigma_{i=1}^{m_{1}} z_{a} w_{ia} w_{i} = \Sigma_{i=1}^{m_{1}} (n_{1} \cdot z) w_{i}^{2} = 0 \). Similarly
\[ II \cdot \frac{v_{1}}{|n_{1}|} = \sum_{i=1}^{m_{1}} \left( n_{1} \cdot \frac{n_{1}}{|n_{1}|} \right) w_{i}^{2} = |n_{1}| \sum_{1}^{m_{1}} w_{i}^{2}. \]

Finally to prove (v), we note that \( T_{i}(q) \in L_{1} \subset M \). q.e.d.

Because \( T_{i}(q) \) is the antipode of \( q \) in the sphere \( L_{j} \), \( q \) goes to \( T_{i}(q) \) gives a well-defined involution on \( M \), i.e. we have

1.10. **Corollary.** Suppose \( n_{j} \neq 0 \), and \( \phi_{i} : M \to M \) maps \( q \) to the antipodal point of \( q \) in \( L_{j} \), where \( L_{i} \) is the leaf of \( E_{j} \) through \( q \). Then \( \phi_{i}^{2} = id \). In particular, we have
\[ \phi_{i} = X + \frac{2}{|n_{j}|} \sum_{a} n_{a} e_{a} \]
is a diffeomorphism.
If \( n_i = 0 \), we let \( \phi_i = \text{id} \).

Next we study the general parallel submanifold of \( M \) given by the normal bundle map \( Y \) of \( M \).

1.11. **Proposition.** Let \( X^* \) be the map from \( M \) to \( R^{n+m} \) given by \( X^* = Y|M \times \{z\} \), i.e. \( X^* = X + \sum_a z_a e_a \). Then \( M^* = X^*(M) \) is an \( n \)-dimensional immersed isoparametric submanifold of \( R^{n+m} \) if and only if \( 1 - z \cdot n_i \neq 0 \) for all \( 1 \leq i \leq p \). Moreover, if \( q^* = X^*(q) \), then \( M^* \) and \( M \) have the same normal plane and focal sets at \( q \) and \( q^* \) respectively.

**Proof.** Using (1.5), we have

\[
dX^* = \sum_{i=1}^p (1 - z \cdot n_i) \text{id}_{E_i}.
\]

So \( dX^* \) has rank \( n \) if and only if \( 1 - z \cdot n_i \neq 0 \) for all \( 1 \leq i \leq p \).

We may choose the following local frame on \( M^* \):

\[
e_a^* = e_a, \quad e_i^* = e_i,
\]

\[
w_i^* = \left( 1 - \sum_a z_a \lambda_i^a \right) w_i.
\]

Then

\[
w_{a\beta} = d\epsilon_{a}^* \cdot e_{\beta}^* = w_{a\beta} = 0,
\]

\[
w_{i\alpha} = d\epsilon_i^* \cdot e_{\alpha}^* = w_{i\alpha} = \frac{\lambda_i^\alpha}{1 - \sum_{\beta} z_{\beta} \lambda_i^\beta} w_i^*.
\]

which implies that \( M^* \) is isoparametric, \( e_{a}^* \) is a global parallel normal frame on \( M^* \), and

\[
n_i^* = \frac{n_i}{1 - z \cdot n_i}.
\]

Since we have

\[
X^* + \sum_a y_a e_a^* = \left( X + \sum_a z_a e_a \right) + \sum_a y_a e_a
\]

\[
= X + \sum_a (y + z)_a e_a,
\]

the focal sets at \( q \) and \( q^* \) are the same. q.e.d.

We call an \( M^* \) as in Proposition 1.11 a **parallel submanifold** of \( M \). Using Corollary 1.10 and Proposition 1.11 we have

1.12. **Proposition.** If \( n_i \neq 0 \), then \( 1 - 2(n_i \cdot n_j)/|n_i|^2 \) never vanishes for any \( 1 \leq j \leq p \).
By examining more carefully the involutive diffeomorphism \( \phi \), we obtain

**1.13. Theorem.** Suppose \( n_i \neq 0 \). Let \( \phi_i(q) = q + 2v_i(q)/|n_i|^2 \) be the diffeomorphism associated to the distribution \( E_i \). Then the following hold:

(i) There exists a permutation \( \sigma_i \) of \( 1, 2, \cdots, p \), such that \( \sigma_i(i) = i \), and \( E_j(\phi_i(q)) = E_{\sigma_i(j)}(q) \) for all \( q \in M \). In particular, we have \( m_j = m_{\sigma_i(j)} \).

(ii) Let \( \tilde{e}_\alpha \) denote the normal frame field defined by \( \tilde{e}_\alpha(\phi_i(q)) = e_\alpha(q) \). Then \( \tilde{e}_\alpha \) is again a global parallel normal frame on \( M \), and

\[
\tilde{e}_\alpha = \sum_\beta \left( \delta_{\alpha\beta} - 2 \frac{n^\alpha_i n^\beta_i}{|n_i|^2} \right) e_\beta,
\]

i.e., \( \tilde{e}_\alpha = S_\alpha e_\alpha \), where \( S_\alpha \) is the reflection of \( v(M)_q \) along \( v_i(q) \).

(iii)

\[
v_{\sigma_i(j)}(q) = \left( 1 - 2 \frac{n_i \cdot n_{\sigma_i(j)}}{|n_i|^2} \right) v_j(\phi_i(q)).
\]

(iv)

\[
S_i(v_j(q)) = v_j(\phi_i(q)) \quad \text{for all } 1 \leq j \leq p.
\]

(v) Let \( R_i \) be the reflection of \( R^m \) along \( n_i \). Then

\[
R_i(n_j) = \left( 1 - 2 \frac{n_i \cdot n_{\sigma_i(j)}}{|n_i|^2} \right) ^{-1} n_{\sigma_i(j)}.
\]

**Proof.** We will assume \( i = 1 \), and \( q' = \phi_1(q) \). It follows from Proposition 1.11 that \( TM_q = TM_{q'} \), and \( E_1(q), \cdots, E_p(q) \) are eigenspaces of the second fundamental form \( II \) of \( M \) at \( q' \). Hence there is a permutation \( \sigma \) (which may depend on \( q \)) of \( 1, 2, \cdots, p \) such that \( E_{\sigma(j)}(q) = E_j(q') \). It follows from Proposition 1.11 that \( \tilde{e}_\alpha = e_\alpha \circ \phi_1 \) is a global parallel normal frame on \( M \), so \( \tilde{e}_\alpha, e_\alpha \), differ by a constant \( O(m) \) matrix. To determine this matrix, we compare \( \tilde{e}_\alpha(q') \) and \( e_\alpha(q') \). Note that \( e_\alpha(q') \) is the parallel translation of \( e_\alpha(q) \) on \( M \) along a path \( \gamma \) joining \( q \) and \( q' \) with respect to the normal connection of \( \nu(M) \). But parallel translation in \( \nu(M) \) is independent of the path, so we can choose \( \gamma \) to lie in the leaf \( L_1 \) of \( E_1 \) through \( q \). From Theorem 1.9, there is an \( (m_1 + m) \)-dimensional plane \( V_1 \) such that \( L_1 \subseteq V_1 \), and the normal bundle of \( L_1 \) in \( V_1 \) is equal to \( \nu(M)_q |L_1 \). Therefore, parallel translation of \( e_\alpha(q) \) to \( q' \) along \( \gamma \) with respect to the normal connection of \( \nu(M) \) is the same as with respect to the normal connection of \( L_1 \) in \( V_1 \). Then it is easily seen that \( e_\alpha(q') \) is the reflection of \( e_\alpha(q) \) in \( \nu(M)_q |V_1(q) \), i.e.,

\[
\tilde{e}_\alpha = S_1 e_\alpha = e_\alpha - 2 \frac{v_1 \cdot e_\alpha}{|n_1|^2} v_1 = \sum_\beta \left( \delta_{\alpha\beta} - 2 \frac{n^\alpha_1 n^\beta_1}{|n_1|^2} \right) e_\beta.
\]
So we have
\[ \bar{n}_i^a = \sum_{\beta} \left( \delta_{\alpha\beta} - 2 \frac{n^a_{\alpha} n^\beta_{1}}{|n_1|^2} \right) n_i^\beta = n_i^a - 2 \frac{n_1 \cdot n_i^a}{|n_1|^2 n_i^a}, \]
\[ \bar{n}_i = R_i(n_i). \]
For the corresponding curvature normal vectors we have
\[ v_i(q') = \sum a e_a(q') = \sum a e_a(q) \]
\[ = v_i(q) - 2 \frac{v_1 \cdot v_i}{|n_1|^2} v_i(q). \]
On the other hand, \( E_i(q') = E_{\sigma(i)}(q) \) and (1.8) imply that
\[ \bar{n}_i = \frac{1}{1 - z \cdot n_{\sigma(i)}} n_{\sigma(i)} \]
where \( z = \frac{2n_1}{|n_1|^2} \).
Therefore we obtain (v). This also proves that \( \sigma \) is independent of \( q \). The rest of the theorem follows. q.e.d.

According to (v) of Theorem 1.3 the \( p \) reflections \( R_1, \ldots, R_p \) permute the corresponding \( p \) reflection hyperplanes, so the root system of the group \( W \) generated by \( R_1, \ldots, R_p \) is finite. Then it follows from the basic theory of Coxeter groups [2, Proposition 4.1.3, p. 37] that \( W \) is a Coxeter group.

1.14. Theorem. Let \( M^n \) be full and isoparametric in \( R^{n+m} \), \( e_a \) a global parallel normal frame, and \( v_i = \sum a e_a \) the curvature normal vectors. Then the group \( W \) generated by the \( p \) reflections \( R_1, \ldots, R_p \) along \( n_i \) in \( R^m \) is an effective Coxeter group. We call \( W \) the Coxeter group associated to \( M \).

1.15. Corollary. Let \( M^n \) be full and isoparametric in \( R^{n+m} \), \( M* \) parallel to \( M \). Then the Coxeter groups of \( M \) and \( M* \) are the same.

Proof. Using (1.8), \( W \) and \( W* \) have the same root system. q.e.d.

Now we are ready to prove Theorem B.

1.16. Proof of Theorem B. The group generated by \( S_1, \ldots, S_p \) is obviously isomorphic to \( W \). So to prove (ii), it suffices to prove that \( S_i \cdots S_i = \text{id} \) is equivalent to \( T_i \cdots T_i = \text{id} \). We proceed as follows: Let \( q' = T_i \cdots T_i(q) \).
Applying Theorem 1.13(iii) and (iv) repeatedly, we obtain
\[ v_j(q') = S_i \cdots S_i(v_j(q)), \quad v_j(q') \text{ is proportional to } v_{\sigma(j)}(q), \]
where \( \sigma = \sigma_i \cdots \sigma_i \). Let \( z \in R^m \) such that \( q' - q = \sum a z_a e_a \). Define \( X* = X + \sum a z_a e_a \). Then \( X* \) is a diffeomorphism, because \( X* \) is the composition of diffeomorphisms \( \phi_i \phi_i \cdots \phi_i \). The calculation in Proposition 1.11 shows that
\[ v_j(q') = \frac{v_{\sigma(j)}(q)}{1 - n_{\sigma(j)} \cdot z}. \]
Suppose $T_i \cdots T_i = \text{id}$, then $q' = q$. $v_j(q') = v_j(q) = S_i \cdots S_i(v_j(q))$. Hence $S_i \cdots S_i = \text{id}$. Conversely, if $S_i \cdots S_i = \text{id}$, then

$$v_j(q') = v_j(q) = \frac{v_j(q)}{1 - n_j \cdot z},$$

which implies that $n_j \cdot z = 0$ for all $1 \leq j \leq p$. Because $M$ is full, Proposition 1.3 implies that $z = 0$, so $T_i \cdots T_i(q) = q$. From Proposition 1.11 we have an open ball $B$ centered at $q$ in the affine normal plane $q + \nu(M)$ so that for any $\bar{q} = q + \sum_\alpha z_\alpha e_\alpha(q) \in B$, $\bar{X} = X + \sum_\alpha z_\alpha e_\alpha$ defines a full isoparametric submanifold $\bar{M} = \bar{X}(M)$ of $R^{n+m}$. Moreover, $\bar{q} + \nu(M) = q + \nu(M)_\bar{q}$ and $l_j(\bar{q}) = l_j(q)$, and $n_j$ is proportional to $n_j$. So $q$ in the above argument can be replaced by $\bar{q} \in B$. Therefore we have proved that the affine transformation $T_i \cdots T_i$ is the identity on $B$ and hence everywhere. Since $\phi_i(q) = T_i(q)$, (i) is a consequence of (ii).

1.17. Corollary. There exists a vector $a \in R^m$ such that $a \cdot n_i = 1$ for all $n_i \neq 0$.

Proof. The group $G$ generated by $T_i, \cdots, T_p$ on the affine normal plane $N_q = q + \nu(M)_q$ is isomorphic to $W$, so in particular the order $|G|$ of $G$ is finite. For a finite subgroup $G$ of the affine group $N_q$, $c = 1/|G|\sum_{g \in G} g(q)$ is a fixed point of $G$. Suppose $c = q + \sum_\alpha a_\alpha e_\alpha(q)$. Then $T_i(c) = c$, i.e. $c \in l_i(q)$, which implies that $a \cdot n_i = 1$ for all $n_i \neq 0$. q.e.d.

Now suppose $n_i \neq 0$ for all $1 \leq i \leq p$, and $a \cdot n_i = 1$ for all $i$. We claim that the map $X + \sum_\alpha a_\alpha e_\alpha$ is a constant vector $c \in R^{n+m}$ on $M$, because

$$d\left(X + \sum_\alpha a_\alpha e_\alpha\right) = \sum_{i=1}^p (1 - a \cdot n_i) \text{id}_{E_i} = 0.$$

In particular, we have $|X - c|^2 = |a|^2$, i.e., $M$ is contained in the sphere of radius $|a|$ and center $c$ in $R^{n+m}$. Therefore we have proved

1.18. Theorem. Suppose $M^n$ is full isoparametric in $R^{n+m}$, and $n_i \neq 0$ for all $1 \leq i \leq p$. Then there exist vectors $a \in R^m$, $c \in R^{n+m}$, such that $M$ is contained in the sphere of radius $|a|$ and center at $c$. In particular $M$ is compact and $\cap_{i=1}^p l_i(q) = \{c\}$. 

1.19. Corollary. Suppose $M^n$ is contained in a sphere of $R^{n+m}$ centered at the origin, and $M$ is full and isoparametric. Then there exists a vector $a \in R^m$ such that

(i) $X = -\sum_\alpha a_\alpha e_\alpha$.

(ii) All the affine normal planes $q + \nu(M)_q$ pass through the origin of $R^{n+m}$.

Theorems 1.9 and 1.18 imply that a full isoparametric submanifold is compact if and only if $n_i \neq 0$ for all $1 \leq i \leq p$. Next, we will discuss the case when some $n_i$ is zero.
If $M^n$ is full and isoparametric in $\mathbb{R}^{n+m}$, then $M^n \times R^l$ is also full and isoparametric in $R^{n+m+l}$ with one of the curvature normal vectors being zero. The following theorem states the converse is also true.

1.20. Theorem. If $M^n$ is full and isoparametric in $\mathbb{R}^{n+m}$, and $n_1 = 0$, then there is a full isoparametric submanifold $M_1$ of $\mathbb{R}^{n+m-m_1}$ such that $M = M_1 \times R^{m_1}$.

Proof. By Corollary 1.17, $a \cdot n_i = 1$ for all $2 \leq i \leq p$. Consider

$$X^* = X + \sum_{\alpha} a_\alpha e_\alpha.$$ 

Then Proposition 1.11 implies that

$$dX^* = \sum_i (1 - n_i \cdot a) \text{id}_{E_i} = \text{id}_{E_i},$$

and $M^* = X^*(M)$ is a flat totally geodesic submanifold of $\mathbb{R}^{n+m}$. So $M^*$ is an $m$-dimensional plane and $X^*: M \to M^*$ is a submersion. We claim that $\bigoplus_{i=2}^p E_i$ is integrable. For if $X^*(q) = q^*$ then $X^*(L_i(q)) = q^*$ for all $i \geq 2$, i.e., $L_i(q) \subset (X^*)^{-1}(q^*)$. Hence $(X^*)^{-1}(q^*)$ is the integral submanifold of $\bigoplus_{i=2}^p E_i$. Note that $\bigoplus_{i=2}^p E_i$ is also defined by

$$w_i = 0, \quad i \leq m_1,$$

so

$$0 = dw_i = \sum_{k > m_1} w_{ik} \wedge w_k = \sum_{k, l > m_1} r_{ikl} w_i \wedge w_k.$$

Hence

$$r_{ikl} = r_{ilk} \quad \text{for } i \leq m_1, k, l > m_1.$$ 

From (1.6), we have

$$\lambda^\alpha_l r_{ikl} = \lambda^\alpha_i r_{ilk}.$$ 

If $e_k, e_l \in E_j$ for some $j \geq 2$, then Proposition 1.8 implies that $r_{ikl} = 0$. If $e_k$ and $e_l$ belong to two different eigenspaces, then (1.9) and (1.10) imply that $r_{ikl} = 0$. Therefore we have that

$$w_{ik} = 0, \quad w_{ia} = 0, \quad i \leq m_1, k > m_1$$

on $M$. Then the fundamental theorem for submanifolds of Euclidean space implies that $M = R^{m_1} \times \tilde{M}$, where $\tilde{M} = X^*^{-1}(q^*)$ is a compact isoparametric submanifold of $\mathbb{R}^{n+m-m_1}$. q.e.d.

Next we discuss the irreducibility of the associated Coxeter group of an isoparametric submanifold, which leads to a decomposition theorem for isoparametric submanifolds.

If $M^n_1$ is isoparametric in $\mathbb{R}^{n_i+l_i}$ with Coxeter group $W_i$ on $\mathbb{R}^{l_i}$, for $i = 1, 2$, then $M_1 \times M_2$ is isoparametric in $\mathbb{R}^{n_1+n_2+l_1+l_2}$ with Coxeter group $W_1 \times W_2$ on $\mathbb{R}^{l_1+l_2}$. The converse is also true.
1.21. Theorem. Let $M^n$ be a compact full isoparametric submanifold of $\mathbb{R}^{n+m}$, $W$ the associated Coxeter group. Suppose $\mathbb{R}^m = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, and $W = W_1 \times W_2$, where $W_i$ is a Coxeter group on $\mathbb{R}^{k_i}$ for $i = 1, 2$. Then there exist two lower-dimensional isoparametric submanifolds $M_1, M_2$, such that $M = M_1 \times M_2$.

Proof. We may assume that $n_i \in \mathbb{R}^{k_i} \times 0$, $R_i \in W_1$ for $i \leq p_1$, and $n_j \in 0 \times \mathbb{R}^{k_2}$, $R_j \in W_2$ for $j > p_1$. Let $a \in \mathbb{R}^{k_1} \times 0$ such that $a \cdot n_i = 1$ for all $i \leq p_1$. Consider $X^* = X + \sum a_i e_i$, then $dX^* = \sum_{i > p_1} id E_i$. So a similar argument as in Theorem 1.20 will show that $\bigoplus_{i \leq p_1} E_i$ is integrable, and $M$ is the product of the leaf of $\bigoplus_{i \leq p_1} E_i$ and $X^*(M)$. q.e.d.

Therefore we make the following:

1.22. Definition. An isoparametric submanifold $M^n$ of $\mathbb{R}^{n+m}$ is called irreducible if $M$ is not the product of two lower-dimensional isoparametric submanifolds.

As a consequence of Theorem 1.20 and 1.21, we have

1.23. Proposition. An isoparametric submanifold of Euclidean space is irreducible if and only if the associated Coxeter group is irreducible.

1.24. Theorem. Every isoparametric submanifold of Euclidean space can be written as the product of irreducible ones, and such decomposition is unique up to permutation.

2. Isoparametric map

In this section, we will prove Theorem A.

If $m = 1$, Definition 2 reduces to that of an isoparametric function given by E. Cartan [4]. For $m \leq n$, S. Carter and A. West [3] gave another definition of isoparametric map as follows: $f: N^{n+m} \to \mathbb{R}^m$ is isoparametric if $A \cup (\ast A)$ is closed under exterior differentiation and wedge product, where $\ast$ is the $\ast$-operator for the Riemannian metric on $N^{n+m}$ and $A = f^*(\wedge^* \mathbb{R}^m)$. They show that when $m = 2$, their definition is equivalent to ours. However, for general $m$, their definitions seem to require stronger conditions than necessary, because of the following standard equalities:

$$\ast d\ast (df_i) = \pm \Delta f_i, \quad \ast (df_i \wedge \ast df_j) = \langle \nabla f_i, \nabla f_j \rangle,$$

$$\ast d\ast (df_i \wedge df_j) = (\Delta f_i) df_j - (\Delta f_j) df_i + \text{dual of } [\nabla f_i, \nabla f_j].$$

Suppose $f: N^{n+m} \to \mathbb{R}^m$ is isoparametric, then we may assume that at any regular point of $f$, there is a local orthonormal frame field $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+m}$ with dual coframe $w_1, \cdots, w_{n+m}$, such that

$$df_\alpha = \sum_{\beta} c_{\alpha \beta} w_\beta$$

(2.1)
with \( \text{rank}(c_{a\beta}) = m \), and \( c_{a\beta} \) are functions of \( f \), so
\[
dc_{a\beta} \equiv 0 \pmod{w_{n+1}, \ldots, w_{n+m}}.
\]

Hereafter we agree on the same indices notion as in §1. It is obvious that \( w_\alpha = 0 \) defines the level submanifolds of \( f \). Condition (2) implies that the normal distribution defined by \( w_i = 0 \) is completely integrable.

2.1. Proposition. Let \( f: N^{n+m} \to R^m \) be isoparametric, \( c = f(q) \) a regular value, \( M = f^{-1}(c) \), and \( F \) the leaf of the normal distribution through \( q \). Then

(i) \( F \) is totally geodesic.

(ii) \( \nu(M) \) is flat and has trivial holonomy group.

Proof. Take the exterior differential of (2.1), and using the structure equations, we obtain
\[
(2.2) \quad \sum_\beta dc_{a\beta} \wedge w_\beta + \sum_\beta c_{a\beta}w_{\beta i} \wedge w_i + \sum_\gamma c_{a\beta}w_{\beta \gamma} \wedge w_\gamma = 0.
\]

Because \( dc_{a\beta} \equiv 0 \pmod{w_{n+1}, \ldots, w_{n+m}} \) and the coefficient of \( w_i \wedge w_\gamma \) of (2.2) is zero, we obtain
\[
\sum_\beta c_{a\beta}(-w_{\beta i}(e_\gamma) + w_{\beta \gamma}(e_i)) = 0.
\]

But \( \text{rank}(c_{a\beta}) = m \), hence
\[
(2.3) \quad w_{\beta \gamma}(e_i) = w_{\beta i}(e_\gamma).
\]

From condition (2) of Definition 2, we have
\[
[e_\alpha, e_\beta] = \sum_\gamma \gamma_{a\beta \gamma} e_\gamma = \nabla_{e_\alpha} e_\beta - \nabla_{e_\beta} e_\alpha
\]
\[
= \sum_i (w_{\beta i}(e_\alpha) - w_{ai}(e_\beta)) e_i + \sum_\gamma (w_{\beta \gamma}(e_\alpha) - w_{ar}(e_\beta)) e_\gamma.
\]

Hence \( w_{\beta i}(e_\alpha) = w_{ai}(e_\beta) \). Using (2.3) and (2.4), we have
\[
w_{\beta i}(e_\alpha) = w_{\beta a}(e_i) = w_{\alpha i}(e_\beta) = w_{a\beta}(e_i) = -w_{\beta a}(e_i).
\]

So
\[
(2.4) \quad w_{a\beta}(e_i) = 0 \quad \text{and} \quad w_{ai}(e_\beta) = 0,
\]

i.e.,
\[
w_{a\beta} = 0 \quad \text{on} \ M, \quad w_{ai} = 0 \quad \text{on} \ F.
\]

We note that \( e_\alpha \) on \( M \) can be obtained by applying the Gram-Schmidt process to \( \nabla f_{n+1}, \ldots, \nabla f_{n+m} \), so \( e_\alpha \) is a global parallel normal frame on \( M \), hence the holonomy of \( \nu(M) \) is trivial. q.e.d.
If \( v(M) \) is flat, then Proposition 1.1 states that there is a local orthonormal tangent frame such that \( \Pi_\alpha \) are diagonalized simultaneously for all \( \alpha \).

2.2. Proposition. The same notation as in Proposition 2.1. Then we have

(i) The mean curvature vector \( H = \sum_\alpha H_\alpha e_\alpha \) on \( M \) is parallel.
(ii) The principal curvatures of \( M \) in the direction of \( e_\alpha \) are constant for all \( \alpha \).

Proof. We choose a local frame field \( e_A \) as in Proposition 1.1 such that \( w_i = \lambda_i^a w_i \), and \( w_{\alpha\beta} \equiv 0 \) (mod \( w_{n+1}, \cdots, w_{n+m} \)).

For a smooth function \( \phi \) on \( N \), the gradient, the Hessian, and the Laplacian of \( \phi \) are given by

\[
\begin{align*}
d\phi &= \sum_A \phi_A w_A, \\
\sum_B \phi_{AB} w_B &= d\phi_A + \sum_B \phi_B w_{BA}, \\
\Delta \phi &= \sum_A \phi_{AA}.
\end{align*}
\]

Then a direct computation using the given frame gives us

\[
df_\alpha = \sum_\beta c_{\alpha\beta} w_\beta, \quad \Delta f_\alpha = \sum_\beta d c_{\alpha\beta}(e_\beta) - \sum_\beta c_{\alpha\beta} H_\beta,
\]

where \( H_\beta \) is the mean curvature of level submanifolds in the direction of \( e_\beta \). Since \( \Delta f_\alpha, c_{\alpha\beta} \) are functions of \( f \), \( \sum_\beta c_{\alpha\beta} H_\beta \) is a function of \( f \). However \( \text{rank}(c_{\alpha\beta}) = m \), so \( H_\alpha \)'s are functions of \( f \), i.e. \( H_\alpha \)'s are constant on \( M \). \( e_A \) is chosen so that \( w_{\alpha\beta} = 0 \), so (i) is proved.

To prove (ii), we use the same method as used by Nomizu [13] for codimension one. We assume \( N = R^{n+m} \), the other two cases are similar. Let \( X \) be the position vector of \( M \) in \( R^{n+m} \), then \( dX = \sum_i w_i e_i \). For given \( e_\alpha \), and constant \( t \in R \), we define a map on an open neighborhood \( U \) of \( M \):

\[
X^* = X + te_\alpha,
\]

then

\[
dX^* = \sum_i (1 - t\lambda_i^\alpha) w_i e_i.
\]

Therefore \( X^*(U) = U^* \) is an open neighborhood of a level submanifold of \( f \) for small \( t \). Moreover, using the same calculation as in Proposition 1.11, we have

\[
(2.5) \quad \lambda_i^\alpha = \frac{\lambda_i^\alpha}{1 - t\lambda_i^\alpha}, \quad H_\alpha^* = \sum_i \frac{\lambda_i^\alpha}{1 - t\lambda_i^\alpha} = \sum_{k=0}^\infty \left( \sum_i (\lambda_i^\alpha) k+1 \right) t^k.
\]
The left-hand side of (2.5) depends on $t$ alone because of (i). Therefore $\sum_i (\lambda_i')^{k+1}$ is a function of $t$ for all $k$, i.e. $\lambda_i'$'s are constant on $M$. q.e.d.

As a consequence of Propositions 2.1 and 2.2 we obtain Theorem A.

3. Construction of polynomial isoparametric maps

In this section, given an isoparametric submanifold $M^n$ of $\mathbb{R}^{n+m}$, we will construct a polynomial isoparametric map on $\mathbb{R}^{n+m}$ which has $M$ as a level submanifold. This construction is a generalization of the Chevalley Restriction Theorem [26]. We also obtain a structure theory for all isoparametric submanifolds.

Because of the decomposition Theorem 1.24, we may assume $M^n$ is a compact, full, and irreducible isoparametric submanifold of $\mathbb{R}^{n+m}$, and $W$ is the associated irreducible Coxeter group on $\mathbb{R}^m$. We will use the same notations as in §1.

Let $Y: M^n \times \mathbb{R}^m \to \mathbb{R}^{n+m}$ be the normal bundle map associated to the global parallel normal frame $e_\alpha$ as in §1. Then there is a small ball $B$ centered at the origin of $\mathbb{R}^m$ such that $Y|\mathbb{R}^m \times B$ is a local coordinate system for $\mathbb{R}^{n+m}$. In particular, $z \cdot n_i < 1$ for all $z \in B$, $1 \leq i \leq p$. We denote $Y(M \times B)$ by $\mathcal{O}$. In fact, $\mathcal{O}$ is a tubular neighborhood of $M$ in $\mathbb{R}^{n+m}$. From Theorem 1.18 and Corollary 1.19, we may assume that $M$ is contained in a sphere centered at the origin in $\mathbb{R}^{n+m}$, and that there is a vector $a \in \mathbb{R}^m$ such that $X = -\sum_\alpha a_\alpha e_\alpha$.

Then

$$Y = X + \sum_\alpha z_\alpha e_\alpha = \sum_\alpha (z_\alpha - a_\alpha) e_\alpha.$$}

We let $y = z - a$. Then $y_\alpha$ is a smooth function defined on the tubular neighborhood $\mathcal{O}$ of $\mathbb{R}^{n+m}$.

3.1. Proposition. Let $u: \mathbb{R}^m \to \mathbb{R}$ be a $W$-invariant smooth function, where $W$ is the Coxeter group associated to a full isoparametric compact submanifold $M$ of $\mathbb{R}^{n+m}$. Then the map $f: \mathcal{O} \to \mathbb{R}$ defined by $f(Y(q, z)) = u(z - a)$ is a smooth function, and $f|M$ is constant. We call this $f$ the extension of the $W$-invariant function $u$.

In order to construct a global isoparametric map for $M$, we need the following lemmas.

3.2. Lemma. Suppose $u: \mathbb{R}^m \to \mathbb{R}$ is a $W$-invariant homogeneous polynomial of degree $k$, then the function

$$\phi(y) = \sum_{i=1}^p m_i \frac{\nabla u(y) \cdot n_i}{y \cdot n_i}$$

is a $W$-invariant homogeneous polynomial of degree $k - 2$. 


Proof. Since \( u(R_i y) = u(y) \), \( \nabla u(R_i y) = R_i(\nabla u(y)) \). We claim that \( \nabla u(y) \cdot n_i = 0 \) if \( y \cdot n_i = 0 \). For if \( y \cdot n_i = 0 \), then \( R_i(y) = y \), so \( \nabla u(y) = R_i(\nabla u(y)) \), i.e. \( \nabla u(y) \cdot n_i = 0 \). Therefore \( \phi(y) \) is a homogeneous polynomial of degree \( k - 2 \). To check that \( \phi \) is \( W \)-invariant, we calculate

\[
\phi(R_i y) = \sum_j m_j \frac{\nabla u(R_i y) \cdot n_j}{R_i y \cdot n_j}
\]

\[
= \sum_j m_j \frac{R_i(\nabla u(y)) \cdot n_j}{R_i y \cdot n_j}
\]

\[
= \sum_j m_j \frac{\nabla u(y) \cdot R_i n_j}{y \cdot R_i n_j}. 
\]

Using Theorem 1.13(i) and (v), we are done.

3.3. Lemma. Let \( u: R^m \to R \) be a \( W \)-invariant homogeneous polynomial of degree \( k \), \( f: \Theta \to R \) its extension. Then

(i) \( \Delta f \) is the extension of a \( W \)-invariant homogeneous polynomial of degree \( k - 2 \) on \( R^m \).

(ii) \( |\nabla f|^2 \) is the extension of a \( W \)-invariant homogeneous polynomial of degree \( 2(k - 1) \) on \( R^m \).

Proof. Using (1.5), we may choose a local frame field \( e^*_A = e_A \) on \( \Theta \subset R^{n+m} \), and the dual coframe is

\[
w_j^* = (1 - z \cdot n_i)w_j, \quad \text{for } \sum_{r=1}^{i-1} m_r + 1 \leq j \leq \sum_{r=1}^{i} m_r, \quad w_\alpha^* = dz_\alpha.
\]

The Levi-Civita connection 1-form on \( \Theta \) is \( w^{*}_{AB} = w_{AB} \). Then we have

\[
dy_\alpha = w_\alpha^*, \quad \Delta y_\alpha = -\sum_{i=1}^{p} m_i n_\alpha^* \frac{m_i n_\alpha}{1 - z \cdot n_i} = \sum_{i=1}^{p} m_i n_\alpha^* \frac{m_i}{y \cdot n_i}.
\]

Since \( f = u(y_{a+1}, \ldots, y_{a+k}) \), we have

\[
df = \sum_\alpha u_\alpha w_\alpha^*, \quad |\nabla f|^2 = |\overline{\nabla} u|^2,
\]

\[
\Delta f = \overline{\Delta} u + \sum_i m_i \frac{\overline{\nabla} u \cdot n_i}{y \cdot n_i},
\]

where \( \overline{\Delta}, \overline{\nabla} \), are the standard Laplacian and gradient on \( R^m \). (i) follows from Lemma 3.2. To prove (ii), we note that \( \overline{\nabla} u(R_i y) = R_i(\overline{\nabla} u(y)) \), so \( |\overline{\nabla} u|^2 \) is a \( W \)-invariant polynomial of degree \( 2(k - 1) \) on \( R^m \).
3.4. Proof of Theorem C. We prove this theorem on $\mathcal{O}$ by using induction on the degree $k$ of $u$. The theorem is obvious for $k = 0$. Suppose it is true for all $l < k$. Given a degree $k$ $W$-invariant homogeneous polynomial $u$ on $R^m$, using Lemma 3.3(ii), $|df|^2$ is again the extension of a $W$-invariant homogeneous polynomial of degree $2k - 2$ on $R^m$. Applying Lemma 3.3(i) repeatedly, we have $\Delta^{k-1}(|df|^2)$ is the extension of a degree zero $W$-invariant polynomial on $R^m$, hence it is a constant. Therefore

$$0 = \Delta^k(|df|^2)$$

$$= \sum_{l=0}^k \sum_{p+p' = k-l} c_{l,p}(\Delta^p f)_{i_1,...,i_l}(\Delta^{p'} f)_{i_1,...,i_l},$$

where $c_{l,p}$ are constants depending on $l$ and $p$. We claim that

$$(\Delta^p f)_{i_1,...,i_l}(\Delta^{k-l-p} f)_{i_1,...,i_l}, \quad p' = k - l - p,$$

is zero if $l < k$. For we may assume that $p \geq k - l - p$, so $p \geq 1$. By Lemmas 3.3(i), $\Delta^p f$ is the extension of a degree $k - 2p$ $W$-invariant polynomial on $R^m$. By the induction hypothesis, $\Delta^p f$ is a homogeneous polynomial on $\mathcal{O} \subset R^{n+m}$ of degree $k - 2p$, hence all the partial derivatives of order bigger than $k - 2p$ will be zero. We have $l + 1 > k - 2p$ by assumption, so we obtain

$$0 = \sum_{i_1,...,i_l} f_{i_1, i_2,...,i_l}^2,$$

i.e., $D^a f = 0$ in $O$ for $|a| = k + 1$. This proves that $f$ is a homogeneous polynomial of degree $k$ in $O$. There is a unique polynomial extension to $R^{n+m}$, which we still denote by $f$.

3.5. Proof of Theorem D. By a theorem of Chevalley [9] the ring of $W$-invariant polynomials on $R^m$ is a polynomial ring with $m$ generators $u_1, \cdots, u_m$. Let $f_1, \cdots, f_m$ be their extended polynomials on $R^{n+m}$. Then because $u_1, \cdots, u_m$ are generators, $f = (f_1, \cdots, f_m)$ will automatically satisfy conditions (0) and (1) of Definition 2. Since $[\nabla y_a, \nabla y_b] = 0$ and $f$ is a function of $y$, condition (2) of Definition 2 is also satisfied. The rest of the theorem follows from the fact that $u_1, \cdots, u_m$ separate orbits of $W$ and regular points of the map $u = (u_1, \cdots, u_m)$ are the $W$-regular points. q.e.d.

The above proof also gives us a constructive method for finding all compact irreducible isoparametric submanifolds of Euclidean space. To be specific, given an irreducible Coxeter group $W$ on $R^m$, we denote the root system by $\Delta$, i.e.,

$$\Delta = \{ a | a \text{ is a unit vector in } R^m \text{ such that the reflection in } R^m \text{ along } a \text{ is in } W \}.$$
Choose \( t \in R^m \) with \( t \cdot a \neq 0 \) for all \( a \in \Delta \), and let \( \Delta^+ \) denote the set of positive roots relative to \( t \), i.e. \( \Delta^+ = \{ a \in \Delta | (a, t) \geq 0 \} \). Suppose \( \Delta^+ = \{ a_1, \ldots, a_p \} \). Next associate a positive integer \( m_i \) to each \( a_i \) so that \( m_i = m_j \) if \( a_i \) and \( a_j \) are in the same \( W \) orbit. Set \( n = \sum_m^p m_i \). Let \( u_1, \ldots, u_m \) be a fixed set of generators for the ring of \( W \)-invariant polynomials on \( R^m \), which can be chosen to be homogeneous of degree \( k_i \). Then there are \( W \)-invariant polynomials \( V_i, \Phi_i, U_{ij}, \) and \( \psi_{ijk} \) such that

\[
\Delta u_i = V_i(u), \quad \Delta u_i \cdot \Delta u_j = U_{ij}(u),
\]

\[
\sum_j m_j \frac{\nabla u_i \cdot a_j}{y \cdot a_j} = \Phi_i(u), \quad [\nabla u_i, \nabla u_j] = \sum_k \psi_{ijk}(u) \nabla u_k.
\]

Then any polynomial solution \( f = (f_1, \ldots, f_m) : R^{n+m} \rightarrow R^m \), with \( f_i \) being homogeneous of degree \( k_i \), of the following system is an isoparametric map:

\[
\Delta f_i = V_i(f) + \Phi_i(f),
\]

\[
\nabla f_i \cdot \nabla f_j = U_{ij}(f),
\]

\[
[\nabla f_i, \nabla f_j] = \sum_k \psi_{ijk}(f) \nabla f_k.
\]

Moreover, if \( M \) is any regular level submanifold of such an \( f \), then the associated Coxeter group of \( M \) is \( W \) and the rank of the eigendistributions \( E_1, \ldots, E_p \) of \( M \) are \( m_1, \ldots, m_p \), respectively.

Since \( u_1 \) can be chosen to be \( \sum_j x_1^j \), the extension \( f_1 \) is \( \sum_x x_2^j \). So (3.1) is a system of equations for \( (m - 1) \) functions. Because both the coefficients and the admissible solutions for the system (3.1) are homogeneous polynomials, the problem of classifying isoparametric submanifolds becomes a purely algebraic one.

However, it is still not known for which irreducible Coxeter group \( W \) and multiplicities \( m_i \), (3.1) has polynomial solutions. By a remarkable result of Münzner [18] (that \( p = 1, 2, 3, 4, 6 \) \( W \) must be crystallographic if \( m = 2 \), and it is natural to conjecture that the two remaining noncrystallographic irreducible Coxeter groups do not arise from nonhomogeneous isoparametric submanifolds.

Similar results can be proved for the hyperbolic space \( H^n \subset R^{n,1} \). Moreover because of the algebraic nature of this problem, the author believes that the study of isoparametric submanifolds in the flat pseudo-Riemannian space \( R^{k,l} \) will yield to techniques similar to those we have used for the Euclidean case.

So far, the results in §§1 and 3 are proved under the assumption that the holonomy of \( \nu(M) \) is trivial. Now we will show that this assumption is automatically satisfied.
3.6. Proposition. If $M^n$ is full and isoparametric in $R^{n+m}$, then the holonomy of $v(M)$ is trivial.

Proof. Given a nonzero vector $v_0 \in v(M)_{q_0}$, let $M^*$ denote the set of all points in $R^{n+m}$ of the form $q + v$, where $v$ is the parallel translation of $v_0$ along some path in $M$, joining $q_0$ to $q$. If $v_0$ is small enough, then $M^*$ is an immersed submanifold. By Proposition 1.4, $M^*$ is a finite cover of $M$. Note that Proposition 1.11 is a local result, hence $M^*$ is isoparametric with $n^*_i = n_i/(1 - z \cdot n_i)$, and $1 - z \cdot n_i \neq 0$ for all $i$. Moreover,

$$S = \{ z \in R^m | |1 - z \cdot n_i|/|n_i| = |1 - z \cdot n_j|/|n_j| \\
\text{for some } i \neq j \text{ and } n_i \neq 0, n_j \neq 0 \}$$

is the union of finitely many hyperplanes of $R^m$. So there is a small $z \not\in S$. Let $v_0 = \Sigma_a z_a e_a(q_0)$, then $n^*_1, \cdots, n^*_p$ are distinct. Proposition 1.4 implies that $v(M^*)$ has trivial holonomy. By Theorem D, there is a Cartan polynomial map $f$ having $M^*$ as a regular level. However, from the construction of $f$, we see that $M$ is an open subset of regular level of $f$. But $M$ is also compact, hence $M$ is a regular level of $f$. This proves that the holonomy of $v(M)$ is trivial, and $M^*$ is diffeomorphic to $M$. q.e.d.

3.7. Corollary. Let $M^n$ be full and isoparametric in $R^{n+m}$, $e_a$ a global parallel normal frame, and $v = q_0 + \Sigma_a z_a e_a(q_0)$ a $W$-regular point. Then $X = X + \Sigma_a z_a e_a$ maps $M$ diffeomorphically onto the parallel submanifold $M^*$ of $M$ through $v$.

Remark. The author would like to thank S. Carter and A. West for pointing out a gap in the original proof of Proposition 3.6.

Next, we give an example to demonstrate that the theory of isoparametric submanifolds in an arbitrary flat manifold can be rather different. Let $N$ be the flat 2-dimensional Mobius strip:

$$N = \{ (x, y)/x \in [0, 1], y \in R \}/(0, y) \sim (1, -y).$$

Then $f(x, y) = y^2$ is isoparametric on $N$ (i.e., $\Delta f$ and $|\nabla f|^2$ are functions of $f$), and 0 is the only singular value of $f$. Let $M_t = f^{-1}(t)$. Then the holonomy of $v(M_t)$ is $\{id\}$ for $t > 0$, and $Z_2$ for $t = 0$; $M_t$ is isoparametric; $M_t$, $t > 0$, is a double cover of $M_0$; and the normal bundle map has no focal points.

4. Applications to variational problems

Isoparametric submanifolds provide many solutions to natural variational problems in Riemannian geometry. In particular, we are interested in the following well-known functionals:

1. The energy functional $E$ for maps $f: M \to N$ [10], i.e.

$$E(f) = \int_M |df|^2 d\text{vol}(M).$$
Critical points of $E$ are by definition harmonic maps.

(2) The area functional $A_k$ for immersions $f: M^k \to N$, i.e. $A_k(f) = k$-dimensional volume of $M$ with respect to the metric on $M$ induced by $f$. A critical point of $A_k$ is called a minimal immersion. The gradient of $A_k$ at $f$ is the mean curvature vector of $f(M)$ in $N$ [8].

In this section, we will make the following assumptions:

Let $X: M^n \to R^{n+m}$ be a full and isoparametric submanifold, $W$ the Coxeter group associated to $M$ with multiplicities $m_i$. Suppose $M$ is contained in the unit sphere of $R^{n+m}$, then by Corollary 1.19 there is a unit vector $a \in R^m$ such that $X = -\sum a_\alpha e_\alpha$. Moreover, the affine normal plane $N_q = q + v(M)_q$ is a linear $m$-plane of $R^{n+m}$ and $W$ acts on $N_q$ orthogonally. Let $U$ be the fundamental region of $W$ on $N_q$ containing $q$. Then $U$ is a simplicial cone with $m$-faces [2, Chapter 4]. Using notation as in §1, we have

$$U = \left\{ q + \sum z_\alpha e_\alpha(q) | z \cdot n_i < 1 \text{ for all } 1 \leq i \leq p \right\},$$

where $y = z - a$ is a linear coordinate system on $N_q$, and $y \cdot n_i = z \cdot n_i - 1$.

4.1. Theorem. Suppose $z \cdot n_i = 1$ for $i = i_1, \ldots, i_r$ and $z \cdot n_i < 1$ otherwise. Let $X' = X + \sum a_\alpha e_\alpha, M' = X'(M)$. Then the following hold:

(0) $M'$ is a submanifold of $R^{n+m}$ with dimension $n - \sum_{r=1}^{s} m_i$.

(i) If $z = n_i/|n_i|^2$, then $M$ is a $S^m$-sphere bundle over $M'$.

(ii) Let $q' = X'(q)$. Then $v(M')_q = v(M)_q \oplus_{r=1}^{s} E_i(q)$, and the mean curvature vector of $M'$ at $q'$ is

$$H'(q) = \sum_{\alpha} H'_\alpha e_\alpha(q),$$

where

$$H'_\alpha = \sum_{j \neq i_1, \ldots, i_r} \frac{m_j n_j}{1 - z \cdot n_j}.$$

(iii) $H'(q') \cdot v_i(q) = 0$ for all $1 \leq r \leq s$. In particular, we have the identities

$$\sum_{j \neq i_1, \ldots, i_r} \frac{m_j n_j \cdot v_i(q)}{1 - z \cdot n_j} = 0 \quad \text{for all } 1 \leq r \leq s.$$

Proof. $dX' = \sum_{i=1}^{s}(1 - z \cdot n_i)\text{id}_{E_i}$ implies that $dX'$ has constant rank $n - \sum_{r=1}^{s} m_i$, and $X'(L_i(q)) = X(q)$ for all $i = i_1, \ldots, i_r$, which prove (0), (i) and the first part of (ii). It follows from the calculation in Proposition 1.11 that

$$H'(q) \cdot e_\alpha(q) = H'_\alpha = \sum_{j \neq i_1, \ldots, i_r} \frac{m_j n_j}{1 - z \cdot n_j},$$

$$H'(q') \cdot e_i(q) = \sum_{j \neq i_1, \ldots, i_r} \frac{m_j \gamma_{i j j}}{1 - z \cdot n_j} \quad \text{for } e_i(q) \in \bigoplus_{r=1}^{s} E_i(q),$$

where $w_{j} = \sum_k \gamma_{i j k} w_k$. 

By Proposition 1.8, we have $\gamma_{i;j} = 0$, which proves the second part of (ii). It remains only to show that $H'(q') \cdot v_i(q) = 0$. It suffices to show that $H'(q') \cdot v$ is a constant for all unit vector $v \in E_i(q) \oplus Rv_i(q)$. To prove this, we note that from Theorem 1.9 $v_i/|n_i|$ defines a diffeomorphism from the leaf $L_i$ of $E_i$ to the unit sphere of $E_i(q) \oplus Rv_i(q)$, and the principal curvatures of $M'$ in these directions can be calculated as follows:

$$\left( \frac{dv_i}{|n_i|}, \frac{v_i}{|n_i|} \right) = \left( \frac{dv_k}{|n_i|}, \frac{v_k}{|n_i|} \right) = \frac{n_i \cdot n_k}{|n_i|} w_k,$$

$$H'(q') \cdot \frac{v_i(q)}{|n_i|} = \frac{1}{|n_i|} \sum_{k \neq i, \ldots, i} m_k n_i \cdot n_k \frac{1}{1 - z \cdot n_k}.$$ q.e.d.

Applying the above theorem (ii) to the harmonic maps, we obtain

**4.2. Theorem.** Let $f: \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a full and isoparametric map, $c \in \mathbb{R}^k$, then

(i) $f^{-1}(c)$ is always a submanifold of $\mathbb{R}^{n+m}$.

(ii) The Gauss map of $M = f^{-1}(c)$ is a harmonic map from $M$ to the appropriate Grassmann manifold.

**Proof.** A theorem of Ruh and Vilms [24] states that if the mean curvature vector of a submanifold $M^k$ of $\mathbb{R}^n$ is parallel, then the Gauss map $g: M \to \text{Gr}(k, n)$ is harmonic. So to prove our theorem, it suffices to prove that the mean curvature vector of $M'$ in Theorem 4.1 is parallel with respect to the normal connection of $v(M')$. To see this, we note that $H' = \sum H' \alpha$, where $H'_\alpha$ is constant, and $w_\alpha = 0$ on $M'$ for $e_j \in \Theta_{r=1} E_i$, so $H'$ is parallel. q.e.d.

Using Theorem 4.1(iii), we also find many minimal submanifolds from isoparametric maps.

**4.3. Theorem.** Let $X: M^n \to \mathbb{R}^{n+m}$ be full and isoparametric, $W$ the Coxeter group of $M$ on $\mathbb{R}^m$, and $U = \{ y \in \mathbb{R}^m \mid y \cdot n_i < 0 \}$ the fundamental region of $W$ on $\mathbb{R}^m$. Suppose $M$ is contained in the unit sphere of $\mathbb{R}^{n+m}$, i.e., there is a unit vector $a$ such that $X = -\sum a_n e_a$. Then the following hold:

(i) $\Gamma = \overline{U} \cap S^{m-1}$ is a polyhedron of $m$ faces in $S^{m-1}$, and each face is totally geodesic. In fact, $S^{m-1}$ is invariant under $W$, and $\Gamma$ is the closure of the fundamental region of the induced $W$-action on $S^{m-1}$.

(ii) If $\sigma_i$ is the interior of an $i$-dimensional simplex of $\Gamma$, then there exist $n_{i_1}, \ldots, n_{i_\ell}$ such that $y \in \sigma_i$ if and only if $y \cdot n_{i_r} = 0$ for $r = 1, \ldots, s$ and $y \cdot n_i < 0$ otherwise. In particular, $M' = Y(M \times \{ z \})$ are diffeomorphic for all $z \in a + \sigma_i$ with $\dim M' = v_i$, where $v_i = n_i - \sum_{r=1}^{s} m_i$. 

(iii) Let $P_i: \overline{\sigma_i} \to \mathbb{R}$ be defined by $P_i(y) = A_y(M_{a+y}^i)$. If $y_i \in \sigma_i$ is a critical point of $P_i$, then $M_{y_i+a}$ is minimal in $S^{n+m-1}$.
(iv) $P_i$ assumes a maximum in $\sigma_i$. So there exists a $\nu_i$-dimensional minimal submanifold of $S^{n+m-1}$. In particular, if $\sigma_0 = \{y_0\}$ is a vertex of $\Gamma$, then $M'_{y_0 + a}$ is minimal.

**Proof.** (i) and (ii) are obvious. Suppose $y_i \in \sigma_i$ is a critical point of $P_i$. Let $y(t)$ be a curve in $\sigma_i$ through $y_i$ at $t = 0$, and $(dy/dt)(0) = b$. Then

$$0 = \left. \frac{dP_i}{dt} (y(t)) \right|_{t=0} = H' \cdot \left( \sum_{\alpha} b_{\alpha} e_{\alpha} \right),$$

where $H'$ denotes the mean curvature vector of $M'_{y_i + a}$ in $R^{n+m}$. (4.1) is true for all $b \in R^m$ such that $b \cdot y_i = 0$ and $b \cdot n_i = 0$ for $1 \leq r \leq s$. By Theorem 4.1, we also have $H' \cdot v_i = 0$ for $1 \leq r \leq s$, and $H' = \sum_{\alpha} H'_\alpha e_{\alpha}$. Hence $H'$ is proportional to

$$\sum_{\alpha} (y_i)_{\alpha} e_{\alpha} = \sum_{\alpha} (-a + y_i + a)_{\alpha} e_{\alpha} = q + \sum_{\alpha} (y_i + a)_{\alpha} e_{\alpha}(q),$$

which is the position vector of $M'_{y_i + a}$. So $M'_{y_i + a}$ is minimal in $S^{n+m-1}$, which proves (iii). Now $P_i$ is continuous in $\sigma_i$, positive on $\sigma_i$ and zero on the boundary of $\sigma_i$, hence $P_i$ assumes a maximum in the interior. q.e.d.

Note that if $M^n \subset R^{n+m}$ is a principal orbit of a $G$-action on $R^{n+m}$ with a section, then Theorems 4.2 and 4.3 are just applications of the symmetric criticality principal [21]. In particular, Theorem 4.3(ii) was proved by W. Y. Hsiang [17] for the homogeneous case. For general isoparametric submanifold $M$, even though there does not exist a group of isometries acting on $M$ transitively, we still obtain the same results.

**References**


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