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# A REMARK ON EXTREMAL KÄHLER METRICS

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## Introduction

In recent years, results from partial differential equations and differential geometry have had striking applications to the study of compact Kähler manifolds. As a notable example, the works of Aubin [1] and Yau [8], [9] on the Calabi conjecture on the existence of Kähler-Einstein metrics imply that an algebraic surface, with ample canonical sheaf and with $c_1^2 = 3c_2$ a so-called $\Phi 2$ surface, is uniformized by the ball in $\mathbf{C}^2$. This union of algebraic and differential geometry is accomplished by the existence of a particularly nice metric, whose differential geometric properties accurately reflect the complex analytic structure of the manifold.

The Kähler-Einstein metrics are solutions of a certain variational problem, introduced by Calabi in [2] and [3]. Specifically, one considers the functional $S$ which assigns to each Kähler metric on a compact complex manifold $M$ the integral over $M$ of the squared scalar curvature. The functional $S$ is restricted to the metrics with a given Kähler class $\omega$ in $H^2(M, \mathbf{R})$, and a critical point for $S_\omega$ is called an extremal Kähler metric. From the Euler equation for $S_\omega$ (see Calabi [4]), one sees that metrics with constant scalar curvature, a fortiori Kähler-Einstein metrics, are extremal. On the other hand, Calabi has exhibited algebraic surfaces which have an extremal metric, but have no metric of constant scalar curvature.

The purpose of this note is to exhibit examples of compact Kähler manifolds which do not admit an extremal Kähler metric. The recent work of Calabi [5] includes a structure theorem for the group of holomorphic automorphisms of a Kähler manifold $M$ which has an extremal metric; in particular, if the dimension of the automorphism group of $M$ is positive, the group must contain a nontrivial compact real Lie subgroup. The examples given here all fail to have such a compact subgroup of their automorphism group. This does not

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give a direct explanation as to why the variational problem of finding an extremal metric does not in general have a solution, and it remains to give general criteria under which the problem can be solved.

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We will retain the notations of [5]. We first extract a useful bit of information from the structure theorem proved there.

**Lemma 1.** *Let $M$ be a compact Kähler manifold with Kähler class $\omega$. Suppose there exists an extremal Kähler metric $(g)$ in the class $\omega$. Suppose further that the connected component of the identity $\mathfrak{H}_0(M)$ of the group of holomorphic automorphisms of $M$ is nontrivial. Then $\mathfrak{H}_0(M)$ contains a compact (real) Lie subgroup of positive dimension.*

*Proof.* Let $R$ be the scalar curvature of the metric $(g)$. If $R$ is constant, then by a theorem of Lichnerowicz and Matsushima [6], [7], $\mathfrak{H}_0(M)$ is reductive, i.e. there is a compact real Lie subgroup $U$ of $\mathfrak{H}_0(M)$ such that $\mathfrak{H}_0(M)$ is the smallest complex Lie subgroup containing $U$. This proves the lemma in this case. If $R$ is not constant, then the vector field $Z_0 = \uparrow \bar{\partial}(\sqrt{-1}\,R)$ is holomorphic and nonzero [4, Theorem 2.1]. It is established in the proof of Theorem 3 of [5] that $Z_0$ is a Killing field on $M$ with respect to $(g)$, hence the (compact) group of holomorphic isometries of $M$ has positive dimension. This completes the proof.

We now proceed with our construction. We note without proof the following simple result.

**Lemma 2.** *Let $M$ be a compact Kähler manifold, $p$ a point of $M$ and $u$: $M_p^* \to M$ the blow-up of $M$ at $p$. Then $u$ induces an isomorphism of $\mathfrak{H}_0(M_p^*)$ with the isotropy subgroup $\mathfrak{H}_0(M)_p$ of $\mathfrak{H}_0(M)$.*

We first construct a series of examples of Kähler manifolds $M_n$ with $\mathfrak{H}_0(M_n)$ the additive group $\mathbf{C}^n$. We restrict ourselves to the case of surfaces; one can easily mimic the construction in higher dimensions.

Fix a positive integer $n$, and let $p$: $E \to \mathbf{CP}^1$ be the rank two holomorphic vector bundle over $\mathbf{P}^1$ whose sheaf of sections is $\mathcal{O}_{\mathbf{P}^1}(n) \oplus \mathcal{O}_{\mathbf{P}^1}$. Let $q$: $S \to \mathbf{P}^1$ be the $\mathbf{P}^1$ bundle $\mathbf{P}(E)$. $S$ is the well-known Hirzebruch surface $\Sigma_n$. Also, $S$ contains a unique section to $q$ with self-intersection $-n$, which we denote by $C_0$.

$\mathfrak{H}_0(S)$ fits into an exact sequence

$$1 \to \mathbf{P}(\mathrm{Aut}(E)) \to \mathfrak{H}_0(S) \xrightarrow{i} \mathrm{Aut}(\mathbf{P}^1) \to 1,$$

where the subgroup $\mathbf{P}(\mathrm{Aut}(E))$ of $\mathfrak{H}_0(S)$ is the subgroup which acts trivially on the base $\mathbf{P}^1$. Each element of $\mathfrak{H}_0(S)$ fixes $C_0$, and we can view the map $i$ as the restriction of $\mathfrak{H}_0(S)$ to $C_0$.

The group $\mathrm{Aut}(E)$ is the group of invertible matrices of the form $\left(\begin{smallmatrix}\alpha & \beta\\ 0 & \gamma\end{smallmatrix}\right)$ with $\alpha \in \mathrm{Aut}(\mathcal{O}(n)) = \mathbf{C}^*$, $\gamma \in \mathrm{Aut}(\mathcal{O}) = \mathbf{C}^*$ and $\beta \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}(n))$. This last group is the same as $H^0(\mathbf{P}^1, \mathcal{O}(n))$, which we regard as the group of homogeneous forms of degree $n$ on the base $\mathbf{P}^1$.

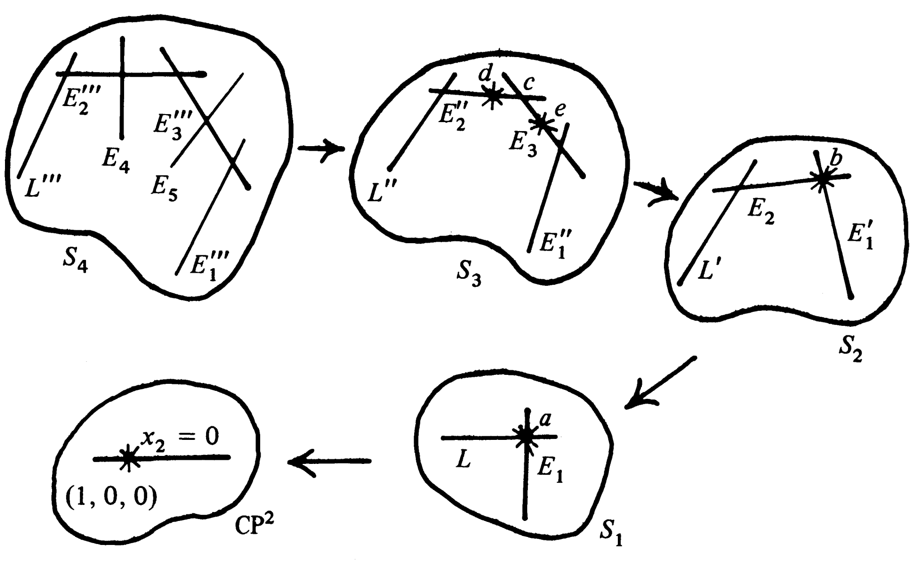
Let $a$, $b$ and $c$ be distinct points on $C_0$, and let $d$ and $e$ be distinct points on the fiber $q^{-1}(q(a))$, different from $a$. Let $M_n$ be the blow-up of $S$ at $b$, $c$, $d$ and $e$. We note that any element of $\mathfrak{H}_0(S)$ which fixes $b$ must fix the fiber $q^{-1}(q(b)) = q^{-1}(q(a))$, and as the point $a$ is the intersection of this fiber with $C_0$, $a$ must be fixed as well. Thus $\mathfrak{H}_0(M_n)$ is the subgroup of $\mathfrak{H}_0(S)$ fixing $a$, $b$, $c$, $d$ and $e$. In particular, every element of $\mathfrak{H}_0(M_n)$ fixes three points on $C_0$ and on $q^{-1}(q(a))$, so each element must act as the identity of these two curves. $\mathfrak{H}_0(M_n)$ is therefore the subgroup of $\mathbf{P}(\mathrm{Aut}(E))$ acting as the identity on $q^{-1}(q(a))$. Thus $\mathfrak{H}_0(M_n)$ is the group of matrices

$$\left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \middle| \beta \in H^0(\mathbf{P}^1, \mathcal{O}(n)), \beta(q(a)) = 0 \right\} \cong \mathbf{C}^n.$$

Our next example is a surface with automorphism group the Heisenberg group

$$\left\{ \begin{pmatrix} 1 & \alpha & \beta \\ & 1 & \gamma \\ 0 & & 1 \end{pmatrix} \right\} \subseteq M_3(\mathbf{C}).$$

This surface is obtained by blowing up $\mathbf{CP}^2$ according to the following diagram (the points to be blown up are starred).



In words, we first blow up $\mathbf{P}^2$ at $(1, 0, 0)$ to obtain the surface $S_1$. Let $E_1$ be the exceptional curve, and let $L$ be the proper transform of the line $x_2 = 0$ (we use homogeneous coordinates $(x_0 : x_1 : x_2)$ on $\mathbf{P}^2$). $E_1$ and $L$ intersect at a single point $a$; let $S_2$ be the blow-up of $S_1$ at $a$. We let $E_2$ be the new exceptional curve, and denote the proper transforms of $E_1$ and $L$ by $E_1'$ and $L'$. One easily sees that an element of $\mathfrak{H}_0(S_1)$ fixes $a$ if and only if it fixes $L$, whence $\mathfrak{H}_0(S_2)$ is the subgroup of $\mathrm{PGL}_3(\mathbf{C})$ of upper triangular matrices. The curves $E_1'$ and $E_2$ are also fixed by $\mathfrak{H}_0(S_2)$, hence the point of intersection, $b$, is also fixed. Let $S_3$ be the blow-up of $S_2$ at $b$, let $E_3$ be the exceptional curve over $b$, and let $E_1''$, $E_2''$ and $L''$ be the proper transforms of $E_1'$, $E_2$ and $L'$. Let $c$ be the point of intersection of $E_2''$ and $E_3$, and let $d$ and $e$ be points on $E_3 - (E_2'' \cup E_1'')$ and $E_2'' - (E_3 \cup E_1'')$, respectively. Finally, we let $S_4$ be the blow-up of $S_3$ at $d$ and $e$.

Since $b$ is fixed by $\mathfrak{H}_0(S_2)$, $\mathfrak{H}_0(S_3)$ equals $\mathfrak{H}_0(S_2)$. The curves $E_2''$ and $E_3$ are stable under $\mathfrak{H}_0(S_3)$; we now compute the action on these curves.

In a neighborhood of the point $c$, $S_3$ has local coordinates $u = x_2^2 x_0 / x_1^3$ and $v = x_1^2 / x_2 x_0$. $E_2''$ is defined near $c$ by $u = 0$, $E_3$ is defined near $c$ by $v = 0$, $v$ restricts to a parameter on $E_2''$, and $u$ restricts to a parameter on $E_3$. We see that a matrix

$$T = \begin{pmatrix} 1 & \alpha & \beta \\ & \gamma & \delta \\ 0 & & \varepsilon \end{pmatrix}$$

sends $v$ to

$$T(v) = (\gamma x_1 + \delta x_2)^2 / \varepsilon x_2 (x_0 + \alpha x_1 + \beta x_2),$$

and sends $u$ to

$$T(u) = \varepsilon^2 x_2^2 (x_0 + \alpha x_1 + \beta x_2) / (\gamma x_1 + \delta x_2)^3.$$

When restricted to $E_2''$, $E_3$ respectively, this gives

$$T(v) = (\gamma^2/\varepsilon) v; \qquad T(u) = (\varepsilon^2/\gamma^3) u.$$

Since $\mathfrak{H}_0(S_3)$ already fixed two points on $E_2''$ and $E_3$, $\mathfrak{H}_0(S_4)$ consists of those $T$ as above which act as the identity on $E_2''$ and $E_3$. For this to occur, we must have $\gamma^2 = \varepsilon$ and $\gamma^3 = \varepsilon^2$, i.e. $\gamma = \varepsilon = 1$, and $\mathfrak{H}_0(S_4)$ is the Heisenberg group, as desired.

There are many simple examples of surfaces for which the existence of an extremal Kähler metric is not known. It was pointed out to me by G. Schumacher that all the examples described here have nonpositive first Chern class; among surfaces with positive first Chern class (Del Pezzo surfaces) only

$\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$, and the blow-up of $\mathbf{P}^2$ at one point are known to support an extremal metric. In the first two cases, the metrics are Einstein; in the third, the metric has nonconstant scalar curvature (see Calabi [4, §3]). For the blow-up of $\mathbf{P}^2$ at two points, nothing is known except that an extremal metric would have nonconstant scalar curvature.

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