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CONFORMAL DEFORMATION OF A RIEMANNIAN METRIC TO CONSTANT SCALAR CURVATURE

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A well-known open question in differential geometry is the question of whether a given compact Riemannian manifold is necessarily conformally equivalent to one of constant scalar curvature. This problem is known as the Yamabe problem because it was formulated by Yamabe [8] in 1960. While Yamabe's paper claimed to solve the problem in the affirmative, it was found by N. Trudinger [6] in 1968 that Yamabe's paper was seriously incorrect. Trudinger was able to correct Yamabe's proof in case the scalar curvature is nonpositive. Progress was made on the case of positive scalar curvature by T. Aubin [1] in 1976. Aubin showed that if dim $M \geqslant 6$ and M is not conformally flat, then M can be conformally changed to constant scalar curvature. Up until this time, Aubin's method has given no information on the Yamabe problem in dimensions 3, 4, and 5. Moreover, his method exploits only the local geometry of M in a small neighborhood of a point, and hence could not be used on a conformally flat manifold where the Yamabe problem is clearly a global problem. Recently, a number of geometers have been interested in the conformally flat manifolds of positive scalar curvature where a solution of Yamabe's problem gives a conformally flat metric of constant scalar curvature, a metric of some geometric interest. Note that the class of conformally flat manifolds of positive scalar curvature is closed under the operation of connected sum, and hence contains connected sums of spherical space forms with copies of $S^1 \times S^{n-1}$.

In this paper we introduce a new global idea into the problem and we solve it in the affirmative in all remaining cases; that is, we assert the existence of a positive solution u on M of the equation

$$\Delta u - \frac{n-2}{4(n-1)} R u + u^{(n+2)/(n-2)} = 0, \tag{0.1}$$

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where $R > 0$ is the scalar curvature of M. We denote the linear part of the operator in (0.1) by L, thus

$$Lu = \Delta u - \frac{n-2}{4(n-1)} Ru.$$

The operator L is a conformally invariant operator in that it changes by a multiplicative factor when the metric of M is multiplied by a positive function. Observe that the question (0.1) is (a normalized version of) the Euler-Lagrange equation for the Sobolev quotient $Q(\varphi)$ for functions φ on M which is given by

$$Q(\varphi) = \frac{\int_M \left(|\nabla\varphi|^2 + (n-2)R\varphi^2/4(n-1)\right) dv}{\left(\int_M |\varphi|^{2n/(n-2)}\, dv\right)^{(n-2)/n}}.$$

The Sobolev quotient $Q(M)$ is then defined by

$$Q(M) = \inf\{Q(\varphi) : \varphi \in C^1(M)\}.$$

The number $Q(M)$ depends only on the conformal class of M. By choosing functions φ which are supported near a point of M, it follows easily that

$$Q(M) \leqslant Q(S^n) \tag{0.2}$$

for any n dimensional manifold M. In this paper we show that equality holds in (0.2) if and only if M is conformally diffeomorphic to S^n with its standard metric. An argument which is by now standard and which originates in [6] shows that if $Q(M) < Q(S^n)$, then there exists a minimum for $Q(\varphi)$ over functions $\varphi \in C^1(M)$. This minimizing function then becomes a positive solution of (0.1) on M.

In order to prove that $Q(M) < Q(S^n)$ for a manifold M conformally different from S^n, we need only exhibit a function φ on M with $Q(\varphi) < Q(S^n)$. Since one can come arbitrarily close to $Q(S^n)$ by a function φ which is supported near a point $o \in M$, it is natural to perturb such a function to make it nonzero but small away from o. Since the nonlinearity of (0.1) involves a higher power of the solution, one expects small solutions to be very close to solutions of $Lu = o$. The only positive solution of $Lu = o$ defined outside a point $o \in M$ is (a multiple of) the Green's function of L. (Note that L is invertible because R is positive.) The question of whether one can satisfy the inequality $Q(\varphi) < Q(S^n)$ reduces to the behavior of the Green's function G near its pole o. If we assume that the metric of M is conformally flat in a neighborhood of o, then G has an expansion in suitable coordinates near o as follows:

$$G(x) = |x|^{2-n} + A + O(|x|).$$

The sign of the constant term A in this expansion is then the crucial ingredient. If A is positive, then one can find a function φ which is a small multiple of G outside a neighborhood of o and which satisfies $Q(\varphi) < Q(S^n)$. On the other hand, it is a theorem of the author and S. T. Yau that $A \geqslant 0$ and $A = 0$ only if M is conformally equivalent to S^n. The case $n = 3$ follows from the positive mass theorem [3] since the metric $\hat{g} = G^{4/(n-2)}g$ is scalar flat and asymptotically Euclidean,

$$\hat{g}_{ij} = \left(1 + A|y|^{2-n}\right)\delta_{ij} + O\left(|y|^{1-n}\right) \quad \text{for } |y| \text{ large},$$

where $y = |x|^{-2}x$. The case $n = 4$ is a consequence of the positive action theorem [4]. The higher dimensional case follows from related techniques and will appear in [5]. In this same paper [5] we have generalized the method of E. Witten [7] (see also [2]) to prove $A > 0$ in case M is a spin manifold. Thus we have $Q(M) < Q(S^n)$ provided M is conformally flat near some point. The same argument works for an arbitrary three dimensional manifold since the Green's function has the above expansion generally in three dimensions. By a delicate perturbation argument (see §2) we are able to handle general compact manifolds of dimensions 4 and 5. Thus, combined with the results of [1], we have an affirmative solution of the Yamabe problem for any compact Riemannian manifold of dimension greater than two.

We recently learned that Rui-Tao Dong showed that solutions blow up near a single point at most.

1. The Sobolev quotient of a conformally flat manifold

In this section we assume M is a compact Riemannian manifold with metric g, o is a point of M, and g is conformally flat in a neighborhood of o. Let $Q(\varphi)$ denote the Sobolev quotient of a function φ on M, and let $E(\varphi)$ denote the energy associated with L, that is

$$E(\varphi) = \int_M \left(|\nabla\varphi|^2 + \frac{n-2}{4(n-1)}R\varphi^2\right) dv,$$

$$Q(\varphi) = \frac{E(\varphi)}{\left(\int_M |\varphi|^{2n/(n-2)}\, dv\right)^{(n-2)/n}}.$$

By changing the metric g conformally we may assume that g is flat near o. Let x be Euclidean coordinates centered at o so that $g_{ij} = \delta_{ij}$ in the x coordinates. Observe that the functions

$$u_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{(n-2)/2}$$

for $\varepsilon > 0$, are solutions on $\mathbf{R}^n$ of the equation

$$\Delta u_\varepsilon + n(n-2)u_\varepsilon^{(n+2)/(n-2)} = 0.$$

Multiplying this equation by u_ε and integrating by parts gives

$$\int_{\mathbf{R}^n} |\nabla u_\varepsilon|^2\, dx = n(n-2)\int_{\mathbf{R}^n} u_\varepsilon^{2n/(n-2)}\, dx.$$

From here we can express $Q(S^n)$ in terms of u_ε:

$$Q(S^n) = \frac{\int_{\mathbf{R}^n}|\nabla u_\varepsilon|^2\, dx}{\left(\int_{\mathbf{R}^n} u_\varepsilon^{2n/(n-2)}\, dx\right)^{(n-2)/n}} = n(n-2)\left(\int_{\mathbf{R}^n} u_\varepsilon^{2n/(n-2)}\, dx\right)^{2/n}. \tag{1.1}$$

Let G be the positive solution (a multiple of the Green's function) of $LG = 0$ on $M - \{o\}$ which behaves like $|x|^{2-n}$ near 0. Since G is a harmonic function near o, it has an expansion for $|x|$ small

$$G(x) = |x|^{2-n} + A + \alpha(x), \tag{1.2}$$

where $\alpha(x)$ is a smooth harmonic function (near 0) with $\alpha(0) = 0$. Let ρ_0 be a small radius, and $\varepsilon_0 > 0$ a number to be chosen small relative to ρ_0. Let $\psi(x)$ be a piecewise smooth decreasing function of $|x|$ which satisfies $\psi(x) = 1$ for $|x| \leqslant \rho_0$, $\psi(x) = 0$ for $|x| \geqslant 2\rho_0$, and $|\nabla\psi| \leqslant \rho_0^{-1}$ for $\rho_0 \leqslant |x| \leqslant 2\rho_0$. We now construct a piecewise smooth test function φ on M as follows:

$$\varphi(x) = \begin{cases} u_\varepsilon(x) & \text{for } |x| \leqslant \rho_0, \\ \varepsilon_0(G(x) - \psi(x)\alpha(x)) & \text{for } \rho_0 \leqslant |x| \leqslant 2\rho_0, \\ \varepsilon_0 G(x) & \text{for } x \in M - B_{2\rho_0}(0). \end{cases}$$

In order for the function φ to be continuous across $\partial B_{\rho_0}(0)$ we must require ε to satisfy

$$\varepsilon_0\left(\rho_0^{2-n} + A\right) = \left(\frac{\varepsilon}{\varepsilon^2 + \rho_0^2}\right)^{(n-2)/2}. \tag{1.3}$$

We compute $E(\varphi)$ as a sum of the energy in $B_{\rho_0}(0)$ and the energy in $M - B_{\rho_0}(0)$. Using the equation for u_ε we have, after an integration by parts,

$$\int_{B_{\rho_0}(0)} |\nabla u_\varepsilon|^2\, dx = n(n-2)\int_{B_{\rho_0}(0)} u_\varepsilon^{2n/(n-2)}\, dx + \int_{\partial B_{\rho_0}(0)} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r}.$$

Using (1.1) and the definition of φ we have

$$\int_{B_{\rho_0}(0)} |\nabla\varphi|^2\, dx \leqslant Q(S^n)\left(\int_{B_{\rho_0}} \varphi^{2n/(n-2)}\, dx\right)^{(n-2)/n} + \int_{\partial B_{\rho_0}(0)} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r}. \tag{1.4}$$

Evaluating the energy of φ on $M - B_{\rho_0}$ we have

$$\int_{M-B_{\rho_0}} \left(|\nabla\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2 \right) dv = \varepsilon_0^2 \int_{M-B_{\rho_0}} \left(|\nabla G|^2 + \frac{n-2}{4(n-1)} RG^2 \right) dv$$
$$+ \varepsilon_0^2 \int_{B_{2\rho_0} - B_{\rho_0}} \left(|\nabla \psi\alpha|^2 - 2\nabla G \cdot \nabla(\psi\alpha) \right) dx.$$

Since $|\alpha(x)| \leqslant c|x|$, we see that $|\nabla(\psi\alpha)| \leqslant c$ for $\rho_0 \leqslant |x| \leqslant 2\rho_0$. Therefore we have for a constant c

$$\int_{M-B_{\rho_0}} \left(|\nabla\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2 \right) dv$$
$$\leqslant \varepsilon_0^2 \int_{M-B_{\rho_0}} \left(|\nabla G|^2 + \frac{n-2}{4(n-1)} RG^2 \right) dv + c\rho_0\varepsilon_0^2.$$

Since G satisfies $LG = 0$, the first term on the right becomes a boundary integral

$$\int_{M-B_{\rho_0}} \left(|\nabla\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2 \right) dv \leqslant -\varepsilon_0^2 \int_{\partial B_{\rho_0}} G \frac{\partial G}{\partial r} + c\rho_0\varepsilon_0^2.$$

Combining this with (1.4) we have

$$E(\varphi) \leqslant Q(S^n) \left(\int_{B_{\rho_0}} \varphi^{2n/(n-2)}\, dv \right)^{(n-2)/2}$$
$$(1.5) \qquad + \int_{\partial B_{\rho_0}} \left(u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0^2 G \frac{\partial G}{\partial r} \right) + c\rho_0\varepsilon_0^2.$$

If M is not conformally equivalent to S^n, it follows from [3], [4] and [5] that $A > 0$. We use this to show that the last two terms in (1.5) are negative if ρ_0, ε_0 are chosen small. For $|x| = \rho_0$ we have from (1.2) and (1.3)

$$\frac{\partial u_\varepsilon}{\partial r} - \varepsilon \frac{\partial G}{\partial r} \leqslant -(n-2) \left[\left(\frac{\varepsilon}{\varepsilon^2 + \rho_0^2} \right)^{(n-2)/2} \frac{\rho_0}{\varepsilon^2 + \rho_0^2} - \varepsilon_0 \rho_0^{1-n} \right] + c\varepsilon_0$$
$$= -(n-2)\varepsilon_0\rho_0^{-1} \left[\left(\rho_0^{2-n} + A \right) \left(\left(\frac{\varepsilon}{\rho_0} \right)^2 + 1 \right)^{-1} - \rho_0^{2-n} \right] + c\varepsilon_0.$$

Using the inequality $(t^2 + 1)^{-1} \geqslant 1 - t^2$, we get

$$\frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0 \frac{\partial G}{\partial r} \leqslant -(n-2)\varepsilon_0 A\rho_0^{-1} + c\rho_0^{-1-n}\varepsilon^2\varepsilon_0 + c\varepsilon_0$$
$$\leqslant -(n-2)\varepsilon_0 A\rho_0^{-1} + c\rho_0^{-1-n}\varepsilon_0^{n/(n-2)} + c\varepsilon_0,$$

where the second inequality follows from (1.3). Using this in (1.5) we have

$$(1.6)\qquad \begin{aligned} E(\varphi) \leqslant Q(S^n)\left(\int_{B_{\rho_0}} \varphi^{2n/(n-2)}\,dv\right)^{(n-2)/n} \\ -(n-2)\sigma_{n-1}A\varepsilon_0^2 + c\rho_0^{-n}\varepsilon_0^{1+n/(n-2)} + c\rho_0\varepsilon_0^2, \end{aligned}$$

where σ_{n-1} denotes the volume of S^{n-1}. Since $A > 0$, by choosing ρ_0 small and ε_0 much smaller than ρ_0, we have $Q(\varphi) < Q(S^n)$. Thus we have shown the following result.

Theorem 1. *If M^n is a compact Riemannian manifold which is conformally flat in an open set and is not conformally diffeomorphic to S^n, then the Sobolev quotient of M is strictly less than that of S^n.*

The proof given above works for $n = 3$ without the conformally flat assumption because the function $G(x)$ has the expansion (1.2) generally in this case.

Proposition 1. *If M is a compact three-dimensional manifold which is conformally different from S^3, then the Sobolev quotient of M is strictly less than that of S^3.*

Proof. Let x be normal coordinates centered at 0, and note that $G(x)$ satisfies

$$G(x) = |x|^{-1} + A + O(|x|)$$

for x small. Let u_ε, ψ, φ be as above, and apply the same argument. Correction terms must be introduced in B_{ρ_0} to account for the difference between g and the Euclidean metric. We see easily that

$$\int_{B_{\rho_0}} \left(|\nabla u_\varepsilon|^2 + 8^{-1}Ru_\varepsilon^2\right) dv \leqslant \int_{B_{\rho_0}} |\nabla u_\varepsilon|^2\,dx + c\rho_0\varepsilon_0^2$$

$$\int_{B_{\rho_0}} u_\varepsilon^6\,dx \leqslant \int_{B_{\rho_0}} u_\varepsilon^6\,dv + c\varepsilon_0^4.$$

Since the error terms are allowable, we see that the above proof succeeds for $n = 3$.

2. The Sobolev quotient of a general four and five dimensional manifold

For $n = 4, 5$ we will remove the restriction that M be conformally flat in an open set. First observe that there is no loss of generality in assuming that $R \equiv 0$ in a neighborhood of a point $o \in M$ since this can be accomplished by multiplication of the metric by a function. Let x denote a normal rectangular coordinate system centered at o, and $r = |x|$, $\xi = x/|x|$ a corresponding polar coordinate system. The metric g of M can then be written

$$g = dr^2 + r^2 h_r,$$

where h_r is a metric on S^{n-1} with h_0 being the standard metric. Given $\rho > 0$ with ρ small, let $\zeta(r)$ be a smooth nonincreasing function satisfying $\zeta(r) = 1$ for $r \leqslant \rho$, $\zeta(r) = 0$ for $r \geqslant 2\rho$, $|\zeta'(r)| \leqslant c\rho^{-1}$, and $|\zeta''(r)| \leqslant c\rho^{-2}$ for all $r > 0$. We now define a modified metric ${}^{\rho}g$ on M by setting ${}^{\rho}g = g$ on $M - B_{2\rho}$, and

$${}^{\rho}g = dr^2 + r^2\big(\zeta(r)h_0 + (1 - \zeta(r))h_r\big) \quad \text{for } r \leqslant 2\rho.$$

Thus ${}^{\rho}g$ is Euclidean in B_ρ and agrees with g outside $B_{2\rho}$. It is easy to check that the curvature tensor of ${}^{\rho}g$ is bounded independent of ρ. Let L_ρ denote the linear operator taken in terms of ${}^{\rho}g$,

$$L_\rho = \Delta_\rho - \frac{n-2}{4(n-1)} R_\rho.$$

Let λ_ρ denote the lowest eigenvalue of L_ρ, and λ the lowest eigenvalue of L. Since (M, g) is conformal to a metric of positive scalar curvature, we have $\lambda > 0$. Let G denote the multiple of the Green's function of L with pole at o normalized so that $\lim_{|x| \to 0} |x|^{n-2} G(x) = 1$. We need the following lemma.

Lemma 1. *The eigenvalues λ_ρ converge to λ as ρ tends to 0, and hence $\lambda_\rho > 0$ for ρ sufficiently small. Thus L_ρ has a positive solution G_ρ with pole at o normalized so that $\lim_{x \to 0} |x|^{n-2} G_\rho(x) = 1$. The functions G_ρ converge (as $\rho \downarrow 0$) uniformly to G in C^2 norm on compact subsets of $M - \{o\}$. For $n \geqslant 4$ the following estimates hold for $x \in B_{2\rho}$:*

$$\left|G_\rho(x) - |x|^{2-n}\right| \leqslant c|x|^{\alpha - n},$$

$$\left|\nabla\left(G_\rho(x) - |x|^{2-n}\right)\right| \leqslant c|x|^{\alpha - n - 1}$$

for any $\alpha \in (3, 4)$, where c depends on α but not ρ. (Actually one can take $\alpha = 4$ provided $n > 4$.) In the second inequality above, ∇ denotes the gradient with respect to g.

Proof. The first statement follows from the fact that ${}^{\rho}g$ converges in C^1 norm to g as $\rho \downarrow 0$ and R_ρ is uniformly bounded. We omit the details. We next prove the inequalities by observing that if $\rho_1 > 0$ is fixed and sufficiently small, we have $\Delta_\rho |x|^{\alpha - n} \leqslant -\varepsilon(\alpha)|x|^{2-n}$ and $|\Delta_\rho |x|^{2-n}| \leqslant c_1 |x|^{2-n}$ for $x \in B_{\rho_1}(0)$. Therefore we can choose c sufficiently large so that for $x \in B_{\rho_1}(0)$,

$$\Delta_\rho\left(|x|^{2-n} + c|x|^{\alpha - n}\right) \leqslant 0,$$

$$\Delta_\rho\left(|x|^{2-n} - c|x|^{\alpha - n}\right) \geqslant 0,$$

$$\rho_1^{2-n} - c\rho_1^{\alpha - n} \leqslant 0.$$

It can then be seen from the maximum principle that

$$|x|^{2-n} - c|x|^{\alpha-n} \leqslant G_\rho(x) \leqslant |x|^{2-n} + c|x|^{\alpha-n}.$$

The second inequality then follows from the gradient estimate for elliptic equations noting that the metrics ${}^\rho g$ are all uniformly equivalent to g. The convergence of G_ρ to G now follows because we have a uniform upper bound on G_ρ and hence on its derivatives on compact subsets of $M - \{o\}$. Any limit of G_ρ is forced to have the correct singularity from the above inequality and hence G_ρ converges to G. This completes a sketch of the proof of Lemma 1.

Since the metrics ${}^\rho g$ are Euclidean in $B_\rho(0)$, the function G_ρ is harmonic in B_ρ and hence has an expansion for $|x|$ small,

$$G_\rho(x) = |x|^{2-n} + A_\rho + O(|x|), \qquad A_\rho \geqslant 0.$$

Therefore we have inequality (1.6) for $(M, {}^\rho g)$ for $\rho_0 < \rho$. To relate this information back to (M, g) we let φ be as in §1 (constructed from $(M, {}^\rho g)$) and observe

$$\int_{B_{\rho_0}} |\nabla\varphi|^2\, dv = \int_0^{\rho_0} \left|\frac{d\varphi}{dr}\right|^2 \operatorname{Vol}_g(\partial B_r)\, dr \tag{2.1}$$

since $\varphi = u_\varepsilon$ in B_{ρ_0} is a function of r. The following observation about $\operatorname{Vol}_g(\partial B_r)$ is necessary.

Lemma 2. *For r small, the asymptotic formula*

$$r^{1-n}\operatorname{Vol}_g(\partial B_r) = \sigma_{n-1} - \frac{\sigma_{n-1}r^2}{6n}R(0) + O(r^4)$$

holds, where $\sigma_{n-1} = \operatorname{Vol}(S^{n-1})$.

Proof. Let g_{ij} be the expression of g in terms of rectangular normal coordinates x, and let $g = \det(g_{ij})$. By direct calculation

$$\frac{\partial\sqrt{g}}{\partial x^i} = \frac{1}{2}\sqrt{g}\sum_{k,l} g^{kl}g_{kl,i},$$

$$\frac{\partial^2\sqrt{g}}{\partial x^i\partial x^j}(0) = \frac{1}{2}\sum_k g_{kk,ij}.$$

Therefore, by Taylor's theorem,

$$\sqrt{g}(x) = 1 + \frac{1}{4}\sum_{i,j,k} g_{kk,ij}x^ix^j + \sum_{i,j,k} a_{ijk}x^ix^jx^k + O(|x|^4),$$

where a_{ijk} depends on the third derivatives of $\sqrt{g}$ at $x = 0$. Setting $r = |x|$, $x = r\xi$, $\xi \in S^{n-1}$ we have

$$\operatorname{Vol}_g(\partial B_r) = r^{n-1}\int_{S^{n-1}} \sqrt{g}(r\xi)\, d\xi,$$

where $d\xi$ is the volume element for the unit S^{n-1}. Then we have from above

$$r^{1-n}\operatorname{Vol}_g(\partial B_r) = \sigma_{n-1} + \frac{\sigma_{n-1}r^2}{4n}\sum_{i,k} g_{kk,ii} + O(r^4),$$

where we have used the obvious facts

$$\int_{S^{n-1}} \xi_i\xi_j\,d\xi = \frac{\sigma_{n-1}}{n}\delta_{ij}, \qquad \int_{S^{n-1}} \xi_i\xi_j\xi_k\,d\xi = 0.$$

Direct calculation in normal coordinates shows

$$R(0) = -\frac{3}{2}\sum_{i,j} g_{ii,jj}(0),$$

thus completing the proof of Lemma 2.

Since we have chosen $R(0) = 0$, we now have from (2.1), letting ∇^ρ denote the gradient with respect to ${}^\rho g$,

$$\int_{B_{\rho_0}} |\nabla\varphi|^2\,dv \leqslant \int_{B_{\rho_0}} |\nabla^\rho\varphi|^2\,dx + c\int_{B_{\rho_0}} |\nabla\varphi|^2(x)|x|^4\,dx. \tag{2.2}$$

From the definition of φ we have

$$\int_{B_{\rho_0}} |\nabla\varphi|^2(x)|x|^4\,dx = (n-2)^2\varepsilon^{n-2}\int_{B_{\rho_0}} \frac{|x|^6}{\left(\varepsilon^2+|x|^2\right)^n}\,dx.$$

The change of coordinates $y = \varepsilon^{-1}x$ and simple computation gives

$$\int_{B_{\rho_0}} |\nabla\varphi|^2(x)|x|^4\,dx \leqslant c\varepsilon^{n-2}\rho_0^{6-n}$$

for $n = 4, 5$. Using this in (2.2) then gives (for $n = 4, 5$)

$$\int_{B_{\rho_0}} |\nabla\varphi|^2\,dv \leqslant \int_{B_{\rho_0}} |\nabla^\rho\varphi|^2\,dx + c\rho_0^{6-n}\varepsilon_0^2, \tag{2.3}$$

where we have used (1.3) which says $\varepsilon_0 \approx \varepsilon^{(n-2)/2}$.

Observe generally that if $f(r)$ is any nonnegative radial function on $B_{2\rho}$, it follows from Lemma 2 that

$$\int_{B_\rho} f\,dv \leqslant \int_{B_\rho} f\,dv_\rho + c_1\int_{B_\rho} f(|x|)|x|^4\,dx, \tag{2.4}$$

where dv_ρ denotes the volume element of ${}^\rho g$. We use this to estimate the energy of φ on $B_{2\rho} - B_{\rho_0}$. First observe that for $\rho_0 \leqslant |x| \leqslant 2\rho_0$ we have

$$\frac{\partial G_\rho}{\partial r} = (2-n)r^{1-n} + O(1), \qquad |\hat{\nabla}G_\rho| \leqslant c,$$

where c depends on ρ and $\hat{\nabla}$ denotes the spherical gradient, that is, the gradient with respect to $r^2 h_r$. From (2.4) we then have

$$\int_{B_{2\rho_0}-B_{\rho_0}} |\nabla\varphi|^2\,dv \leqslant \int_{B_{2\rho_0}-B_{\rho_0}} |\nabla^\rho\varphi|^2\,dv_\rho + c\varepsilon_0^2 \int_{B_{2\rho_0}-B_{\rho_0}} |x|^{6-2n}\,dx + c\rho_0\varepsilon_0^2.$$

For $n = 4, 5$ this gives

$$\int_{B_{2\rho_0}-B_{\rho_0}} |\nabla\varphi|^2\,dv \leqslant \int_{B_{2\rho_0}-B_{\rho_0}} |\nabla^\rho\varphi|^2\,dv_\rho + c\rho_0\varepsilon_0^2, \tag{2.5}$$

where c depends on ρ but not on ρ_0.

We now estimate the integral over $B_{2\rho} - B_{2\rho_0}$. This is more delicate because we need constants independent of ρ for which we employ Lemma 1. Throughout the following argument we use c_1 to denote a constant independent of both ρ and ρ_0. For $2\rho_0 \leqslant |x| \leqslant 2\rho$ we have $\varphi = \varepsilon_0 G_\rho$, so we estimate the square gradient of G^ρ. We first do the radial derivative

$$\begin{aligned}\left(\frac{\partial G_\rho}{\partial r}\right)^2 &= \left(\frac{\partial}{\partial r}\left(G_\rho - r^{2-n}\right) + \frac{\partial}{\partial r}\left(r^{2-n}\right)\right)^2 \\ &= \left(\frac{\partial}{\partial r}\left(G_\rho - r^{2-n}\right)\right)^2 + 2(2-n)r^{1-n}\frac{\partial}{\partial r}\left(G_\rho - r^{2-n}\right) + (2-n)^2 r^{2-2n}.\end{aligned}$$

Using (2.4) on the third function on the right we find, by Lemma 1 and the fact that we are using normal coordinates,

$$\begin{aligned}\int_{B_{2\rho}-B_{2\rho_0}} \left(\frac{\partial G_\rho}{\partial r}\right)^2 dv \leqslant{}& \int_{B_{2\rho}-B_{2\rho_0}} \left(\frac{\partial G_\rho}{\partial r}\right)^2 dv_\rho \\ &+ c_1 \int_{B_{2\rho}-B_{2\rho_0}} \left(r^{2\alpha-2n} + r^{\alpha+2-2n} + r^{6-2n}\right) dx\end{aligned}$$

for any $\alpha \in (3,4)$. In particular, for $n = 4, 5$ we get (choosing $\alpha = 3.5$)

$$\int_{B_{2\rho}-B_{2\rho_0}} \left(\frac{\partial G_\rho}{\partial r}\right)^2 dv \leqslant \int_{B_{2\rho}-B_{2\rho_0}} \left(\frac{\partial G_\rho}{\partial r}\right)^2 dv_\rho + c_1\rho^{1/2}.$$

From Lemma 1 we have $|\hat{\nabla} G_\rho| \leqslant c_1 r^{\alpha-n-1}$, from which we easily see

$$\int_{B_{2\rho}-B_{2\rho_0}} |\hat{\nabla} G_\rho|^2\,dv \leqslant \int_{B_{2\rho}-B_{2\rho_0}} |\hat{\nabla}^\rho G_\rho|^2\,dv_\rho + c_1\rho.$$

Combining these with (2.3) and (2.5) we have

$$\int_{B_{2\rho}} |\nabla\varphi|^2\,dv \leqslant \int_{B_{2\rho}} |\nabla^\rho\varphi|^2\,dv_\rho + c\rho_0\varepsilon_0^2 + c_1\rho^{1/2}\varepsilon_0^2. \tag{2.6}$$

Note that R_ρ does not vanish in $B_{2\rho} - B_\rho$, so for $\sigma \in [\rho, 2\rho]$ we estimate its integral on B_σ. First observe that in rectangular normal coordinates ${}^\rho g_{ij} = \zeta\delta_{ij} + (1 - \zeta)g_{ij}$. We see, by direct calculation,

$$R_\rho = \sum_{i,j} \left({}^\rho g_{ij,ij} - {}^\rho g_{ii,jj}\right) + O\left(|x|^2\right)$$

and hence, by Stokes theorem,

$$\int_{B_\sigma} R_\rho \, dv_\rho = \int_{\partial B_\sigma} \sum_{i,j} \left({}^\rho g_{ij,i} r_j - {}^\rho g_{ii,j} r_j\right) + O(\sigma^{n+2}).$$

Since $R \equiv 0$ in B_σ, we have

$$\int_{\partial B_\sigma} \sum_{i,j} \left(g_{ij,i} r_i - g_{ii,j} r_j\right) = O(\sigma^{n+2}).$$

Therefore

$$\int_{B_\sigma} R_\rho dv_\rho = \zeta'(\sigma) \int_{\partial B_\sigma} \left(\sum_{i,j} (\delta_{ij} - g_{ij}) r_i r_j - \sum_i (\delta_{ii} - g_{ii})\right) + O(\sigma^{n+2}).$$

Since x^i are normal coordinates, the first term in the integral vanishes, and, as in Lemma 2, Taylor's theorem gives

$$\sum_i (g_{ii} - \delta_{ii}) = \frac{1}{2} \sum_{i,j,k} g_{ii,jk}(0) x^j x^k + O(r^4).$$

After integration the quadratic term vanishes and since $\zeta'(\sigma) = O(\sigma^{-1})$ we finally have

$$\int_{B_\sigma} R_\rho dv_\rho = O(\sigma^{n+2}). \tag{2.7}$$

We now compute by Lemma 1 for $\alpha \in (3, 4)$

$$\left| \int_{B_{2\rho} - B_\rho} R_\rho G_\rho^2 \, dv_\rho \right| \leqslant \left| \int_{B_{2\rho} - B_\rho} R_\rho |x|^{4-2n} \, dv_\rho \right| + c_1 \int_{B_{2\rho} - B_\rho} |x|^{2+\alpha-2n} \, dx.$$

On the other hand we have

$$\int_{B_{2\rho} - B_\rho} R_\rho |x|^{4-2n} \, dx = \int_\rho^{2\rho} \sigma^{4-2n} \frac{d}{d\sigma} \left(\int_{B_\sigma} R_\rho \, dv_\rho \right) d\sigma$$
$$= (2\rho)^{4-2n} \int_{B_{2\rho}} R_\rho \, dv_\rho + (2n - 4) \int_\rho^{2\rho} \sigma^{3-2n} \left(\int_{B_\sigma} R_\rho \, dv_\rho \right) d\sigma.$$

Applying (2.7) we therefore have

$$\left|\int_{B_{2\rho}-B_\rho} R_\rho |x|^{4-2n}\, dv_\rho\right| \leq c_1\rho^{6-n}.$$

Thus for $n = 4, 5$ we have shown (again $\alpha = 3.5$)

$$\left|\int_{B_{2\rho}-B_\rho} R_\rho \varphi^2\, dv_\rho\right| \leq c_1\rho^{1/2}\varepsilon_0^2,$$

which combined with (2.6) shows

$$E(\varphi) \leq E_\rho(\varphi) + c\rho_0\varepsilon_0^2 + c_1\rho^{1/2}\varepsilon_0^2.$$

Combined with (1.6) we then have

$$E(\varphi) \leq Q(S^n)\left(\int_{B_{\rho_0}} \varphi^{2n/(n-2)}\, dv_\rho\right)^{(n-2)/n} - (n-2)\sigma_{n-1}A_\rho\varepsilon_0^2$$
$$+ c\rho_0^{-n}\varepsilon_0^{1+n/(n-2)} + c\rho_0\varepsilon_0^2 + c_1\rho^{1/2}\varepsilon_0^2.$$

Arguing as above, we can replace dv_ρ by dv in the integral to obtain

$$(2.8)\qquad E(\varphi) \leq Q(S^n)\left(\int_M \varphi^{2n/(n-2)}\, dv\right)^{(n-2)/n} - (n-2)\sigma_{n-1}A_\rho\varepsilon_0^2$$
$$+ c\rho_0^{-n}\varepsilon_0^{1+n/(n-2)} + c\rho_0\varepsilon_0^2 + c_1\rho^{1/2}\varepsilon_0^2,$$

where c depends on ρ but c_1 does not. If we can establish the inequality

$$(2.9)\qquad \underline{\lim}_{\rho\to 0} A_\rho > 0,$$

then we can finish the proof by *fixing* ρ small, then fixing ρ_0, and finally choosing ε_0 sufficiently small. The following lemma gives a condition under which (2.9) holds.

Lemma 3. *If the metric $G^{4/(n-2)}g$ is not Ricci flat on $M - \{o\}$, then* (2.9) *holds.*

Proof. For notational simplicity we let $\bar{g}$ denote ${}^\rho g$ for any ρ, and $\bar{G}$ the corresponding Green's function. Suppose $G^{4/(n-2)}g$ is not Ricci flat on $M - \{o\}$, and let K be a compact subset of $M - \{o\}$ on which $G^{4/(n-2)}g$ is not Ricci flat. From Lemma 1 we see that for ρ small the metric $\bar{G}^{4/(n-2)}\bar{g}$ is not Ricci flat on K. Let χ be a smooth nonnegative function with compact support in $M - \{o\}$ with $\chi \equiv 1$ on K. For a tensor $S = S_{ij}$ with compact support in $M - \{o\}$ we introduce the notation

$$g^t = \bar{G}^{4/(n-2)}\bar{g} + tS, \qquad R^t_{ij} = \mathrm{Ric}(g^t)$$

and let R^t denote the scalar curvature of g^t. Let u_t denote the solution of

$$\Delta_t u_t - \frac{n-2}{4(n-1)} R^t u_t = 0 \quad \text{on } M - \{0\}, \qquad u_t(0) = 1. \tag{2.10}$$

Such a solution exists for $|t| < \delta_0$, with δ_0 depending only on g and S. In fact, we can write $u_t = H_t \bar{G}^{-1}$, where H_t is the normalized Green's function for the metric $\bar{g} + t\bar{G}^{-4/(n-2)}S$ which exists for $|t| < \delta$ by Lemma 1.

Since this metric is Euclidean near 0 we have for $|x|$ small

$$H_t(x) = |x|^{2-n} + \bar{A}_t + O(|x|),$$

$$\bar{G}(x) = |x|^{2-n} + \bar{A} + O(|x|),$$

from which it follows that for $|x|$ small

$$u_t(x) = 1 + (\bar{A}_t - \bar{A})|x|^{n-2} + O(|x|^{n-1}).$$

Integrating (2.10) with respect to dv^t, the volume element of g^t, and using Stoke's theorem we find

$$\bar{A} - \bar{A}_t = (4(n-1)\sigma_{n-1})^{-1} \int_{M-\{o\}} R^t u_t \, dv^t.$$

Differentiating the integral on the right and evaluating at $t = 0$ we find

$$\frac{d}{dt} \int_{M-\{o\}} R^t u_t \, dv^t \bigg|_{t=0} = -\int_{M-\{o\}} \langle \operatorname{Ric}(g^0), S \rangle \, dv^0,$$

where we have used $R \equiv 0$, $u_0 \equiv 1$. Taking $S = -\chi \operatorname{Ric}(g^0)$ we make the term on the right positive; in fact, since we assume $G^{4/(n-2)}g$ is not Ricci flat on K, and $G_\rho = \bar{G}$ is close to G in C^2 norm (Lemma 1) on the support of χ, we find

$$\frac{d}{dt} \int_{M-\{o\}} R^t u_t \, dv^t \bigg|_{t=0} \geqslant \delta_1 > 0,$$

where δ_1 is independent of ρ. Also by Lemma 1 the metrics g^t vary smoothly in t up to any order on the support of χ uniformly in ρ. Thus there exists t_0 small so that $\bar{A} - \bar{A}_{t_0} \geqslant \delta_2$ with $\delta_2 > 0$ independent of ρ. Thus we have for ρ small

$$\bar{A} = A_\rho = \bar{A}_{t_0} + (\bar{A} - \bar{A}_{t_0}) \geqslant \delta_2.$$

This establishes (2.9), and proves Lemma 3.

Finally we must analyze the case when $G^{4/(n-2)}g$ is Ricci flat. We will show that this can only hold if M is conformally equivalent to S^n. First observe that $(M - \{o\}, G^{4/(n-2)}g)$ is asymptotically Euclidean in the sense that the metric $\hat{g} = G^{4/(n-2)}g$ satisfies for $|y|$ large, $y \in \mathbf{R}^n$,

$$\hat{g}_{ij}(y) = \delta_{ij} + O(|y|^{-2}), \tag{2.11}$$

where $y = |x|^{-2}x$, x near 0 in normal coordinates. The following result is of independent interest.

Proposition 2. *Let $(N, \hat{g})$ be a Riemannian manifold which is asymptotically Euclidean in the sense of* (2.11). *If $\hat{g}$ is Ricci flat, then $(N, \hat{g})$ is isometric to $\mathbf{R}^n$ with its Euclidean metric.*

Proof. The conclusion of Proposition 2 is not difficult if $\hat{g}$ is sufficiently near the Euclidean metric at infinity. We use the Ricci flat assumption to improve the decay of $\hat{g}$. A standard argument asserts the existence of harmonic coordinates $v^1, \cdots, v^n$ defined near infinity such that $v^i = y^i + O(|y|^{-1})$ so that in terms of v coordinates the metric satisfies (2.11). We simply rename v^i to be y^i, so that without loss of generality we can take y^i to be harmonic coordinates. The Ricci flat condition then implies for all i, j

$$|\Delta \hat{g}_{ij}| \leqslant c|\nabla \hat{g}|^2.$$

This shows that $\Delta \hat{g}_{ij} = O(|y|^{-6})$ and hence elliptic theory gives an improvement on the decay of $\hat{g}$; in fact, one derives in a standard way

$$\hat{g}_{ij}(y) = \delta_{ij} + A_{ij}|y|^{2-n} + O\left(|y|^{1-n}\right),$$

where (A_{ij}) is a constant $n \times n$ matrix. By rotating the y coordinates we may assume (A_{ij}) is diagonal. The condition that y^j be harmonic implies $A_{jj} = \frac{1}{2}\Sigma_i A_{ii}$. Since this holds for each j and $n > 2$, we conclude that $A_{ij} = 0$ and hence we have

$$\hat{g}_{ij}(y) = \delta_{ij} + O\left(|y|^{1-n}\right).$$

We may assume that each y^j is a global harmonic function on N, possibly linearly dependent away from infinity. The Bochner formula for y^i then shows $\Delta|\nabla y^i|^2 = 2|\nabla\nabla y^i|^2$ since $\hat{g}$ is Ricci flat. We clearly have

$$|\nabla y^i|^2 = 1 + O\left(|y|^{1-n}\right), \qquad \nabla|\nabla y^i|^2 = O\left(|y|^{-n}\right).$$

Integrating the Bochner formula over a large ball gives

$$2\int_{B_R} |\nabla\nabla y^i|^2\, dv = \int_{\partial B_R} \frac{\partial}{\partial \vec{n}}|\nabla y^i|^2 = O(R^{-1}),$$

where $\vec{n}$ is the outer normal to ∂B_R. Letting R go to infinity then shows $\nabla\nabla y^i \equiv 0$ and hence ∇y^i is a parallel vector field for each i. It follows immediately that $(y^1, \cdots, y^n)$: $N \to \mathbf{R}^n$ defines an isometry. This proves Proposition 2.

Combining Lemma 3 and Proposition 2 with our previous work, and taking into account the results of Aubin [1] we get our main theorem on the Sobolev quotient.

Theorem 2. *Let M^n be a compact Riemannian manifold with $n \geqslant 3$. If M is not conformally diffeomorphic to S^n, then the Sobolev quotient $Q(M)$ is strictly less than $Q(S^n)$.*

3. Conformal deformation to constant scalar curvature

We now prove our main theorem concerning conformal deformation. The argument used here is that of Trudinger [6]. Essentially the same argument was used later by Aubin [1]. We repeat the argument for the sake of completeness since it is very simple.

Theorem 3. *Any compact Riemannian manifold (M^n, g) has a conformally related metric $u^{4/(n-2)}g$, $u > 0$, of constant scalar curvature.*

Proof. Let $\alpha \in [1, (n+2)/(n-2)]$, $\alpha_0 = (n+2)/(n-2)$, and consider the ratio

$$Q_\alpha(\varphi) = \frac{E(\varphi)}{\left(\int_M |\varphi|^{\alpha+1}\right)^{2/\alpha+1}}, \qquad \varphi \in H_1(M),$$

where $H_1(M)$ is the Sobolev space of function with L^2 first derivatives. It is elementary from the Sobolev embedding theorem, and was proved by Yamabe [8] that there exists, for any $\alpha \in (1, \alpha_0)$, smooth functions $u_\alpha > 0$, $\int_M u_\alpha^{\alpha+1}\,dv = 1$, satisfying

$$Q_\alpha(u_\alpha) = \min\{Q_\alpha(\varphi) \colon \varphi \in H_1(M)\}.$$

We denote this value by $Q_\alpha(M)$ so that $Q_{\alpha_0}(M) = Q(M)$. Moreover, u_α satisfies the Euler-Lagrange equation

$$\Delta u_\alpha - \frac{n-2}{4(n-1)} R u_\alpha + Q_\alpha(M) u_\alpha^\alpha = 0. \tag{3.1}$$

One attempts to take the limit as $\alpha \uparrow \alpha_0$. Since we have a uniform bound on the H_1 norm of u_α, by weak compactness we can find a weakly convergent sequence $\{u_{\alpha_i}\}$. The weak form of (3.1) is

$$\int_M \left(\nabla\eta \cdot \nabla u_\alpha + \frac{n-2}{4(n-1)} R u_\alpha \eta - Q_\alpha(M) u_\alpha^\alpha \eta \right) dv = 0$$

for any $\eta \in C^\infty(M)$. Since H_1 is compactly contained in L^p for any $p < 2n/(n-2)$, it follows easily that the weak H_1 limit u of the sequence u_{α_i} satisfies the limiting equation. (Note that one sees immediately that $\lim_{\alpha \to \alpha_0} Q_\alpha(M) = Q(M)$.) A regularity result of Trudinger [6] then implies that u is smooth. One need only show that u is nonzero, and this is where Theorem 2 enters. Given $P \in M$ and $\rho > 0$ and small, let η be a smooth function on M which is equal to one in $B_\rho(P)$ and zero outside $B_{2\rho}(P)$. Multiply (3.1) by $\eta^2 u_\alpha$ and integrate by parts to get

$$\int_M \eta^2 |\nabla u_\alpha|^2 \, dv \leqslant -2\int_M \eta u_\alpha \nabla\eta \cdot \nabla u_\alpha + c\int_M \eta^2 u_\alpha^2 + Q_\alpha(M)\int_M \eta^2 u_\alpha^{\alpha+1}.$$

This easily implies for any $\varepsilon > 0$

$$(1-\varepsilon)\int_M |\nabla \eta u_\alpha|^2 \, dv \leqslant c(\varepsilon)\rho^{-2}\int_M u_\alpha^2 + Q_\alpha(M)\int_M \eta^2 u_\alpha^{\alpha+1},$$

where $c(\varepsilon)$ depends on ε and M. The Sobolev inequality in $B_{2\rho}$ holds with the Euclidean Sobolev constant $Q(S^n)$ plus an error term which is of order ρ^2 because the metric is Euclidean up to second order. Therefore we have

$$\begin{aligned}(3.2)\qquad &(1-\varepsilon)\left(Q(S^n) - c\rho^2\right)\left(\int_M (\eta u_\alpha)^{2n/(n-2)} \, dv\right)^{(n-2)/n} \\ &\qquad \leqslant c(\varepsilon)\rho^{-2}\int_M u_\alpha^2 + Q_\alpha(M)\int_M \eta^2 u_\alpha^{\alpha+1}.\end{aligned}$$

Now observe that $\eta^2 u_\alpha^{\alpha+1} = (\eta u_\alpha)^2 u_\alpha^{\alpha-1}$ and hence

$$\begin{aligned}\int_M \eta^2 u_\alpha^{\alpha+1} &\leqslant \left(\int_M (\eta u_\alpha)^{2n/(n-2)}\right)^{(n-2)/n} \left(\int_M u_\alpha^{(\alpha-1)n/2}\right)^{2/n} \\ &\leqslant \left(\int_M (\eta u_\alpha)^{2n/(n-2)}\right)^{(n-2)/n},\end{aligned}$$

where we have used Hölder's inequality twice, have normalized g so that $\mathrm{Vol}_g(M) = 1$, $\int_M u_\alpha^{\alpha+1} = 1$, and the fact that $(\alpha - 1)n/2 \leqslant \alpha + 1$. Since our theorem is trivial if M is conformally diffeomorphic to S^n, we assume that is not the case, and hence by Theorem 2 we have $Q(M) < Q(S^n)$. In particular, for α near α_0 we have $Q_\alpha(M) < Q(S^n)$. Now we fix ε, ρ small enough to absorb the last term on the right of (3.2) to the left to get

$$\left(\int_M (\eta u_\alpha)^{2n/(n-2)} \, dv\right)^{(n-2)/n} \leqslant c\int_M u_\alpha^2 \, dv.$$

Since η is one on $B_\rho(P)$, we can take a finite covering of M by balls of radius ρ and sum these inequalities to obtain

$$\left(\int_M u_\alpha^{2n/(n-2)} \, dv\right)^{(n-2)/n} \leqslant c\int_M u_\alpha^2 \, dv.$$

Since $\alpha + 1 \leqslant 2n/(n-2)$, this implies

$$1 \leqslant c\int_M u_\alpha^2 \, dv.$$

This gives a uniform lower bound on the L^2 norm of u_α. Since H_1 is compactly contained in L^2, the same lower bound holds on u, and hence u is nonzero. This completes the proof of Theorem 3.

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