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KOSZUL COHOMOLOGY AND THE GEOMETRY OF PROJECTIVE VARIETIES. II

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0. Introduction

This paper continues the search begun in [2] for some new techniques to use in computing Koszul cohomology. The same notation will be used as in that paper.

One extremely natural question is to determine the Koszul cohomology groups $\mathscr{K}_{p,q}(\mathbf{P}^r, H^k, H^d, W)$, where $H \to \mathbf{P}^r$ is the hyperplane bundle, $d \geqslant 1$ and $W \subseteq H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^n}(d))$ is a base-point free linear system. The simplest case of this is to ask when the multiplication map

$$W \otimes H^0(\mathbf{P}^r, \mathcal{O}(k)) \to H^0(\mathbf{P}^r, \mathcal{O}(k+d)) \tag{0.1}$$

must be surjective. Indeed, the surjectivity of (0.1) comes up in a conjecture of Carlson, Green, Griffiths & Harris [1]. Let

$$S_k \subset \mathbf{P}_{\binom{d+3}{3}-1}$$

be the variety of smooth surfaces in $\mathbf{P}^3$ of degree d which contain a curve C of degree k which is not a complete intersection. Is

$$\operatorname{codim} S_k \geqslant d - 3? \tag{0.2}$$

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This conjecture will be answered in the affirmative in §4 as a consequence of an analysis of the Koszul cohomology groups of projective space.

We will establish the following vanishing theorem: For $W \subseteq H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(d))$ a base-point free linear subspace, $d \geqslant 1$,

$$\mathcal{K}_{p,q}(\mathbf{P}^r, H^k, H^d, W) = 0 \tag{0.3}$$

if

$$\begin{aligned} k + (q-1)d &\geqslant p + \operatorname{codim}(W, H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(d))), \\ d + 1 &\geqslant \operatorname{codim}(W, H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(d))). \end{aligned} \tag{0.4}$$

The other main result we prove is as follows: Let X be a smooth complete algebraic variety, $E \to X$ an analytic vector bundle and $p_0 \geqslant 0$ an integer. Then there exists an ample line bundle $L_0 \to X$ so that

$$\mathcal{K}_{p,q}(X, E, L) = 0 \quad \text{for } p \leqslant p_0, q \geqslant 2, \tag{0.5}$$

for any analytic line bundle $L \to X$ such that $L \otimes L_0^{-1}$ is ample. Two special cases of this result were needed in [3], and it gives a partial answer to Problem 5.13 of [2].

1. Algebraic preliminaries

(a) **Truncation.** Consider

$$\begin{cases} V & \text{a vector space,} \\ S(V) & \text{the symmetric algebra of } V, \\ B = \bigoplus_{q \in \mathbf{Z}} B_q & \text{a graded } S(V)\text{-algebra.} \end{cases} \tag{1.a.1}$$

We define the *kth truncation of* B, denoted $T_k(B)$, by

$$T_k(B) = \bigoplus_{q \in \mathbf{Z}} T_k(B)_q, \tag{1.a.2}$$

where

$$T_k(B)_q = \begin{cases} 0, & q < k, \\ B_q, & q \geqslant k. \end{cases} \tag{1.a.3}$$

We note that directly from the definition of Koszul cohomology,

$$\begin{cases} \mathcal{K}_{p,q}(T_k(B), V) \simeq \mathcal{K}_{p,q}(B, V), & q \geqslant k+1, \\ \mathcal{K}_{p,q}(T_k(B), V) = 0, & q < k. \end{cases} \tag{1.a.4}$$

From the exact sequence of $S(V)$-modules

$$0 \to T_{k+1}(B) \to T_k(B) \to B_k \to 0,$$

we conclude from the long exact sequence for Koszul cohomology that

$$\mathcal{K}_{p,q}(T_k(B), V) \simeq \ker\left(\Lambda^p V \otimes B_k \to \Lambda^{p-1} \otimes B_{k+1}\right). \tag{1.a.5}$$

This is particularly interesting in the case of the *truncated symmetric algebra*

$$T_k(S(V)) = \bigoplus_{q \geqslant k} S^q V.$$

We conclude that, for $k > 0$,

$$\begin{aligned} &\mathcal{K}_{p,q}(T_k(S(V)), V) \\ &\quad \simeq \begin{cases} \ker(\Lambda^p \otimes S^k V \to \Lambda^{p-1} V \otimes S^{k+1} V), & q = k, \\ 0, & q \neq k. \end{cases} \end{aligned} \tag{1.a.6}$$

Using the *Littlewood-Richardson rule* from representation theory, we conclude that

$$\ker(\Lambda^p V \otimes S^k V \to \Lambda^{p-1} V \otimes S^{k+1} V) \simeq V^{(k,1^p)} \tag{1.a.7}$$

as GL(V)-modules, where in general $V^{(\lambda_1^{k_1}, \ldots, \lambda_m^{k_m})}$ denotes the representation of GL(V) whose Young diagram has k_1 rows of λ_1 elements each, k_2 rows of λ_2 elements each, etc. (see [4]). For our purposes, we may regard (1.a.7) as a definition, noting that

$$V^{(k,1^p)} = 0 \quad \text{if } p \geqslant \dim V. \tag{1.a.8}$$

We now have, for $k > 0$,

$$\mathcal{K}_{p,q}(T_k(S(V)), V) \simeq \begin{cases} V^{(k,1^p)}, & q = k, \\ 0, & q \neq k, \end{cases} \tag{1.a.9}$$

and thus $T_k(S(V))$ has the minimal resolution

$$\begin{aligned} &0 \to V^{(k,1^r)} \otimes S(V)(-r) \\ &\quad \to \cdots \to V^{(k,1)} \otimes S(V)(-1) \to S^k V \otimes S(V) \to T_k(V) \to 0, \end{aligned} \tag{1.a.10}$$

where $\dim V = r + 1$.

In the context of complex manifolds, if we have

$$\begin{cases} X & \text{a complex manifold,} \\ L \to X & \text{a holomorphic vector bundle,} \\ W \subseteq H^0(X, L) & \text{a base-point free linear system,} \end{cases}$$

then (1.a.10) becomes the exact sequence of bundles

(1.a.11) $0 \to W^{(k,1^r)} \otimes L^{-r} \to \cdots \to W^{(k,1)} \otimes L^{-1} \to S^k W \to L^k \to 0,$

where $\dim W = r + 1$. For $X = \mathbf{P}^r$, $L = H =$ the hyperplane bundle and $W = H^0(\mathbf{P}^r, H)$, (1.a.10) says that taking H^0 of each term of the sequence (1.a.11) tensored by H^l gives an exact sequence for all $l \geq 0$.

(b) **Change of base.** Let B, V be as in (1.a.1) and let $W \subset V$ be a linear subspace. Then we may also regard B as in $S(W)$-module and ask:

What is the relation between the $\mathscr{K}_{p,q}(B, W)$ and the $\mathscr{K}_{p,q}(B, V)$?

Proposition 1.b.1 (*Spectral sequence for change of base*). *For each $l \in \mathbf{Z}$, there is a spectral sequence with*

(1.b.2) $$E_1^{p,q} = \Lambda^{l-p}(V/W) \otimes \mathscr{K}_{-q,p+q}(B, W),$$

(1.b.3) $$E_\infty^{p,q} = \mathrm{Gr}^p\big(\mathscr{K}_{l-p-q,p+q}(B, V)\big).$$

Proof. Consider the Koszul complex

(1.b.4) $$\begin{array}{ccccccc} \cdots \to \Lambda^{l-q+1}V \otimes B_{q-1} & \to & \Lambda^{l-q}V \otimes B_q & \to & \Lambda^{l-q-1}V \otimes B_{q+1} & \to \cdots \\ \| & & \| & & \| & \\ K^{q-1} & & K^q & & K^{q+1} & \end{array}$$

with the filtration

(1.b.5) $$F^p(K^q) = \mathrm{im}(\Lambda^{p-q}W \otimes \Lambda^{l-p}V) \otimes B_q.$$

So

(1.b.6) $$\mathrm{Gr}^p(K^q) = \Lambda^{p-q}W \otimes \Lambda^{l-p}(V/W) \otimes B_q.$$

For this filtered complex

(1.b.7) $$E_1^{p,q} = \Lambda^{l-p}(V/W) \otimes \mathscr{K}_{-q,p+q}(B, W),$$

(1.b.8) $$E_\infty^{p,q} = \mathrm{Gr}^p\big(\mathscr{K}_{l-p-q,p+q}(B, V)\big).$$

Corollary 1.b.9. *For any p, q, we have*

(1.b.10) $$\mathscr{K}_{p,q}(B, V) = 0 \quad \text{if}\ \mathscr{K}_{p',q}(B, W) = 0\ \text{for all}\ p' \leq p.$$

Proof. Take $l = p + q$. Then

$$\mathrm{Gr}^{p'}\big(\mathscr{K}_{p,q}(B, V)\big) = E_\infty^{p',q-p'}, \qquad q \leq p' \leq p + q.$$

On the other hand

$$E_1^{p',q-p'} = \Lambda^{p+q-p'}(V/W) \otimes \mathscr{K}_{p'-q,q}(B, W) = 0 \quad \text{for}\ q \leq p' \leq p + q.$$

So $E_\infty^{p',q-p'} = 0$ and we are done.

2. Koszul cohomology of powers of the hyperplane bundle on projective space

Consider

$$(2.1)\qquad \begin{cases} H \to \mathbf{P}^r & \text{the hyperplane bundle,} \\ V = H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1)). \end{cases}$$

Theorem 2.2. *For $k, d \in \mathbf{Z}$ with $d \geqslant 1$,*

$$(2.3)\qquad \mathcal{K}_{p,q}(\mathbf{P}^r, H^k, H^d) = 0 \quad \text{if } k + (q-1)d \geqslant p.$$

Proof. Let

$$(2.4)\qquad B^m = \bigoplus_{q \in \mathbf{Z}} H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(m + qd)).$$

From the exact sequence (1.a.11) we have the exact sequence of sheaves for any $l \in \mathbf{Z}$

$$(2.5)\qquad \begin{aligned} 0 &\to V^{(d,1^r)} \otimes \mathcal{O}_{\mathbf{P}^r}(l-r) \to \cdots \to V^{(d,1)} \otimes \mathcal{O}_{\mathbf{P}^r}(l-1) \\ &\to S^d V \otimes \mathcal{O}_{\mathbf{P}^r}(l) \to \mathcal{O}_{\mathbf{P}^r}(d+l) \to 0. \end{aligned}$$

By the spectral sequence for hypercohomology and the Bott vanishing theorem for $H^i(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(l))$, we obtain an exact sequence

$$(2.6)\qquad \begin{aligned} 0 &\to V^{(d,1^r)} \otimes S^{l-r}V \to \cdots \to V^{(d,1)} \otimes S^{l-1}V \\ &\to \ker(S^d V \otimes S^l V \to S^{d+l} V) \to 0. \end{aligned}$$

Let

$$(2.7)\qquad R = \bigoplus_{q \in \mathbf{Z}} R_q = \bigoplus_{q \in \mathbf{Z}} \ker(S^d V \otimes S^{k+d(q-1)} V \to S^{k+dq} V).$$

We have exact sequences of $S(S^d V)$-modules

$$(2.8)\qquad 0 \to R \to S^d V \otimes B^k(-1) \to B^k \to B^k_{-[k/d]} \to 0,$$

$$(2.9)\qquad 0 \to V^{(d,1^r)} \otimes B^{k-r}(-1) \to \cdots \to V^{(d,1)} \otimes B^{k-1}(-1) \to R \to 0.$$

By the general spectral sequence (1.d.3) of [2], (2.8) gives rise to a spectral sequence abutting to zero. For $q > -[k/d]$, the differentials coming in to $\mathcal{K}_{p,q}(B^k, S^d V)$ are

$$(2.10)\qquad S^d V \otimes \mathcal{K}_{p,q-1}(B^k, S^d V) \underset{0}{\overset{d_1}{\to}} \mathcal{K}_{p,q}(B^k, S^d V),$$

$$(2.11)\qquad \ker\left(d_1 : \mathcal{K}_{p-q,q+1}(R, S^d V) \to S^d V \otimes \mathcal{K}_{p-1,q+1}(B^k, S^d V)\right) \overset{d_2}{\to} \mathcal{K}_{p,q}(B^k, S^d V).$$

Note the map (2.10) is zero by (1.b.11) of [2]. For $q > -[k/d]$, there are no nonzero differentials emerging from $\mathcal{K}_{p,q}(B^k, S^dV)$. Thus, we have the implication:

(2.12)

$$\text{for } q > -[k/d], \text{ if } \mathcal{K}_{p-1,q+1}(R, S^dV) = 0, \text{ then } \mathcal{K}_{p,q}(B^k, S^dV) = 0.$$

Now, using the spectral sequence abutting to zero which arises from (2.9) using the general spectral sequence (1.d.3) of [2], we get that $\mathcal{K}_{p-1,q+1}(R, S^dV)$ is an E_1 term of this sequence. All the differentials emerging from this term have target zero, while the d_l's coming in have sources which are quotients of subspaces of

$$\mathcal{K}_{p-l,q+l-1}(B^{k-l}, S^dV) \otimes V^{(d,1^l)}, \qquad r \geqslant l \geqslant 1. \tag{2.13}$$

Our hypothesis $k + (q-1)d \geqslant p$ implies $q > -[k/d]$ and also implies, for all $l \geqslant 1$,

$$(k-l) + (q+l-2)d \geqslant p - l. \tag{2.14}$$

If we do an induction on p, (2.14) implies that the groups in (2.13) all vanish, and hence

$$\mathcal{K}_{p-1,q+1}(R, S^dV) = 0$$

which we have seen implies

$$\mathcal{K}_{p,q}(B^k, S^dV) = 0$$

or equivalently

$$\mathcal{K}_{p,q}(\mathbf{P}^r, H^k, H^d) = 0,$$

as desired. To complete our induction, it remains to check the case $p = 0$. We must check that

$$S^dV \otimes S^{k+d(q-1)}V \to S^{k+dq}V \tag{2.15}$$

is surjective if $k + d(q-1) \geqslant 0$; however, this is an elementary property of polynomials.

Remark. Since $\mathrm{GL}(V)$ acts equivariantly on the Koszul complex

$$\cdots \to \Lambda^{p+1}(S^dV) \otimes S^{d(q-1)+k}V \to \Lambda^p(S^dV) \otimes S^{dq+k}V$$
$$\to \Lambda^{p-1}(S^dV) \otimes S^{d(q+1)+k}V \to \cdots$$

one ought eventually to have a formula

$$\mathcal{K}_{p,q}(\mathbf{P}^r, H^k, H^d) = \bigoplus_{i=1}^{N} V^{\rho_i},$$

where $V = H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1))$ and the ρ_i are representations of $\mathrm{GL}(V)$. The ρ_i will depend on p, q, k and d; however, they are independent of r (except that certain representations will have dimension zero if r is small). It would be interesting to have these formulas, of which Theorem 2.2 would be a special case.

In applications, it is important to be able to compute the $\mathscr{K}_{p,q}$'s when we use a linear subsystem of S^dV.

Theorem 2.16. *Let $W \subseteq H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(d))$ be a base-point free linear system. Then*

$$(2.17)\quad \mathscr{K}_{p,q}(\mathbf{P}^r, H^k, H^d, W) = 0 \quad \text{if } k + (q-1)d \geqslant p + \dim(S^dV/W),$$

provided $\dim(S^dV/W) \leqslant d + 1$.

Proof. By the Duality Theorem (2.c.6) of [2], for any fixed W, d and k, the theorem is true for all p when q is sufficiently large. For a fixed k, d, W, say (2.17) fails for $p = p_0$, $q = q_0$, but is true for every p for all higher q. By Proposition 1.b.1, there is a spectral sequence with (taking $l = p_0 + q_0 + \dim(S^dV/W)$)

$$(2.18)\quad E_1^{a,b} = \Lambda^{p_0+q_0+\dim(S^dV/W)-a}(S^dV/W) \otimes \mathscr{K}_{-b,a+b}(\mathbf{P}^r, H^k, H^d, W),$$

$$(2.19)\quad E_\infty^{a,b} = \mathrm{Gr}^a\left(\mathscr{K}_{p_0+q_0+\dim(S^dV/W)-a-b,a+b}(\mathbf{P}^r, H^k, H^d)\right).$$

So

$$(2.20)\quad E_1^{p_0+q_0,-p_0} = \Lambda^{\dim(S^dV/W)}(S^dV/W) \otimes \mathscr{K}_{p_0,q_0}(\mathbf{P}^r, H^k, H^d, W).$$

Further,

$$(2.21)\quad E_1^{a,b} = 0 \quad \text{for } a < p_0 + q_0$$

as the exterior power on the right-hand side of (2.18) vanishes. Another consequence of (2.18) is that for any $m \in \mathbf{Z}$,

$$(2.22)\quad \begin{aligned} &E_1^{p_0+q_0+m+1,-p_0-m} \\ &= \Lambda^{\dim(S^dV/W)-m-1}(S^dV/W) \otimes \mathscr{K}_{p_0+m,q_0+1}(\mathbf{P}^r, H^k, H^d, W). \end{aligned}$$

By hypothesis,

$$k + (q_0 - 1)d \geqslant p_0 + \dim(S^dV/W)$$

from which it follows that

$$(2.23)\quad k + q_0 d \geqslant \dim(S^dV/W) + p_0 + m,$$

whenever $m \leqslant d$. Thus using our reverse induction on q, the Koszul group on the right-hand side of (2.22) vanishes for $m \leqslant d$, while the exterior product on the right-hand side of (2.22) vanishes for $m \geqslant \dim(S^dV/W)$. Under the hypothesis $\dim(S^dV/W) \leqslant d+1$, this exhausts all possible m. So

$$E_1^{p_0+q_0+m+1,-p_0-m} = 0 \quad \text{for all } m \geqslant 0. \tag{2.24}$$

This implies that all differentials emerging from $E_1^{p_0+q_0,-p_0}$ are zero, while (2.21) implies that all differentials with target $E_1^{p_0+q_0,-p_0}$ are zero. Thus

$$\begin{aligned} E_1^{p_0+q_0,-p_0} &\simeq E_\infty^{p_0+q_0,-p_0} \\ &\simeq \mathrm{Gr}^{p_0+q_0}\left(\mathcal{K}_{p_0+\dim(S^dV/W),q_0}(\mathbf{P}^r, H^k, H^d)\right). \end{aligned} \tag{2.25}$$

By Theorem 2.2 this vanishes. Comparing this with (2.20), we conclude that

$$\mathcal{K}_{p_0,q_0}(\mathbf{P}^r, H^k, H^d, W) = 0$$

completing our reverse induction.

3. Koszul cohomology of sufficiently ample bundles

Definition 3.1. We will say that a property holds for *sufficiently ample* line bundles on a variety X if there exists an analytic line bundle $L_0 \to X$ such that the property holds for all analytic line bundles $L \to X$ satisfying $L \geqslant L_0$, i.e. $L \otimes L_0^{-1}$ is an ample line bundle. We will denote this by $L \gg 0$.

Theorem 3.2. *Let X be a smooth complete algebraic variety. For any $p_0 \in \mathbf{Z}$ and any analytic vector bundle $E \to X$,*

$$\mathcal{K}_{p,q}(X, E, L) = 0 \quad \text{for all } p \leqslant p_0,\ q \geqslant 2 \tag{3.3}$$

for L sufficiently ample.

Proof. Let

$$X^i = \underbrace{X \times X \times \cdots \times X}_{i \text{ times}},$$

$$\Delta_{jk} = \left\{(x_1,\cdots,x_i) \in X^i \mid x_j = x_k\right\},$$

$\pi_j\colon X^i \to X$, $\pi_{jk}\colon X^i \to X^2$ be the canonical projections,
$\Delta \subset X^2$ be the diagonal,
$M_\Delta \to X^2$ be the associated line bundle,

$$M_{jk} = \pi_{jk}^*(M_\Delta), \qquad V = H^0(X, L)$$

$$B^0 = \bigoplus_q B_q^0 = \bigoplus_q H^0(X, E \otimes L^q), \tag{3.4}$$

$$\begin{aligned} B^i = \bigoplus_q B_q^i = \bigoplus_q H^0\big(&X^{i+1}, M_{12}^{-1} \otimes M_{23}^{-1} \otimes \cdots \otimes M_{i,i+1}^{-1} \\ &\otimes \pi_1^*(E \otimes L^q) \otimes \pi_2^*(L) \otimes \cdots \otimes \pi_{i+1}^*(L)\big). \end{aligned} \tag{3.5}$$

We regard the B^i as graded $S(V)$ modules. From the restriction sequence

$$0 \to M_{i,i+1}^{-1} \to \mathcal{O}_{X^{i+1}} \to \mathcal{O}_{\Delta_{i,i+1}} \to 0 \tag{3.6}$$

we have a long exact sequence

$$\begin{aligned} 0 &\to B^i_{q-1} \to B^{i-1}_{q-1} \otimes V \to B^{i-1}_q \\ &\to H^1\big(X^{i+1}, M_{12}^{-1} \otimes \cdots \otimes M_{i,i+1}^{-1} \otimes \pi_1^*(E \otimes L^{q-1}) \otimes \pi_2^*(L) \\ &\qquad \otimes \cdots \otimes \pi_{i+1}^*(L)\big) \to \cdots . \end{aligned} \tag{3.7}$$

If $q \geqslant 2$ and $L \gg 0$, then H^1 above is zero. Using the truncation notation of §1(a), we obtain an exact sequence of graded $S(V)$-modules

$$0 \to T_1(B^i)(-1) \to T_1(B^{i-1})(-1) \otimes V \to T_1(B^{i-1}) \to B_1^{i-1} \to 0. \tag{3.8}$$

From the spectral sequence (1.d.3) of [2] associated to this, using the remark (1.a.4) on Koszul cohomology of truncations,

$$\mathcal{K}_{p,q-1}(B^{i-1}, V) \otimes V \xrightarrow{0} \mathcal{K}_{p,q}(B^{i-1}, V) \to \mathcal{K}_{p-1,q}(B^i, V) \to \cdots , \tag{3.9}$$

where the first map is zero by (1.b.11) of [2]. There is thus a sequence of injective maps for $q \geqslant 2$:

$$\begin{aligned} &\mathcal{K}_{p,q}(B^0, V) \hookrightarrow \mathcal{K}_{p-1,q}(B^1, V) \hookrightarrow \cdots \\ &\hookrightarrow \mathcal{K}_{0,q}(B^p, V) \hookrightarrow \mathcal{K}_{-1,q}(B^{p+1}, V) = 0 \end{aligned} \tag{3.10}$$

The one thing we must be careful of is to use the hypothesis $L \gg 0$ only a finite number of times; this is all right if we restrict $p \leqslant p_0$. We conclude

$$\mathcal{K}_{p,q}(B^0, V) = 0 \quad \text{for } p \leqslant p_0,\ q \geqslant 2,$$

or equivalently

$$\mathcal{K}_{p,q}(X, E, L) = 0 \quad \text{for } p \leqslant p_0,\ q \geqslant 2.$$

4. An improvement of the Noether-Lefschetz Theorem for surfaces in $\mathbf{P}^3$

Let $U_d \subset \mathbf{P}^{\binom{d+3}{3}-1}$ be the open set of nonsingular surfaces of degree d in $\mathbf{P}^3$, and let $U_{d,k} \subset U_d$ be those surfaces which contain a curve of degree k which is not a complete intersection. (By a complete intersection C of a nonsingular surface S in $\mathbf{P}^3$ we mean that C is not in the linear system cut out on S by surfaces in $\mathbf{P}^3$ of any degree.) We answer a question raised in [1] by showing

Theorem 4.1.

(4.2) $$\operatorname{codim} U_{d,k} \geqslant d-3.$$

Note. The classical Noether-Lefschetz Theorem states that $\operatorname{codim} U_{d,k} > 0$.

Remark. What we show is actually somewhat stronger than (4.2). Let $S_0 \in U_d$ and $\gamma \in H^{1,1}_{\text{prim}}(S_0, \mathbf{Z})$. Let V be a small open neighborhood of S_0 in U_d, chosen so that for $S \in V$, there is a natural isomorphism

(4.3) $$H^2(S,\mathbf{Z}) \underset{\alpha_S}{\overset{\approx}{\to}} H^2(S_0,\mathbf{Z}).$$

Let

(4.4) $$U_d(\gamma) = \{S \in V \mid \alpha_S^{-1}(\gamma) \text{ has type } (1,1)\}.$$

Then we will show that

(4.5) $$\operatorname{codim} T_{S_0,\text{Zar}}(U_d(\gamma)) \geqslant d-3,$$

which implies (4.2).

It remains an open question whether, if equality holds in (4.2) for some component of $U_{d,k}$, this component consists of surfaces having a line, i.e. the curves of degree k on a generic S in this component of $U_{d,k}$ are residual to a multiple of a line under a power of the hyperplane series.

Proof of Theorem (4.1). Let $S \in U_d$ and $\gamma \in H^{1,1}_{\text{prim}}(S, \mathbf{Z})$. As in the discussion of [1], if

$$V = H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1)) \quad \text{and} \quad F \in S^dV, \qquad S = \operatorname{div} F,$$

and

$$J = \bigoplus_{k \geqslant 0} J_k$$

is the Jacobi ideal of F, then

(4.6) $$H^0(S, K_S) \simeq S^{d-4}V,$$

(4.7) $$H^1_{\text{prim}}(S, \Omega^1_S) \simeq S^{2d-4}V/J_{2d-4},$$

(4.8) $$H^2(S, \mathcal{O}_S) \simeq S^{3d-4}V/J_{3d-4}.$$

Furthermore, $T_{S,\text{Zar}}(U_d(\gamma))$ is the left annihilator of γ under the map

(4.9) $$S^dV \otimes S^{2d-4}V/J_{2d-4} \to S^{3d-4}V/J_{3d-4}.$$

Using the duality of Macauley's Theorem, if $W = T_{S,\text{Zar}}(U_d(\gamma)) \subset S^dV$, then the image of

(4.10) $$W \otimes S^{d-4}V \to S^{2d-4}V/J_{2d-4}$$

is orthogonal to γ. Thus, a fortiori, the map

$$W \otimes S^{d-4}V \to S^{2d-4}V \tag{4.11}$$

is not surjective, i.e., $\mathscr{K}_{0,1}(\mathbf{P}^3, (d-4)H, dH, W) \neq 0$. Since $J_d \subset W$ and F is nonsingular, we know that W is base-point free. We can now invoke Theorem 2.16 to conclude that, if codim W in S^dV is $\leqslant d-4$, then

$$\mathscr{K}_{0,1}\big(\mathbf{P}^3, (d-4)H, dH, W\big) = 0.$$

This would be a contradiction, so W has codimension $\geqslant d-3$. This proves (4.5) and hence Theorem 4.1.

Remark. If one does not stop to prove the vanishing of other Koszul groups, there exist much simpler ways to see that if (4.11) is not surjective and W is base-point free, then W has codimension $\geqslant d-3$ in S^dV.

Added in proof. By a variant of the argument given, using an induction on $\dim(S^dV/W)$, the hypothesis $\dim(S^dV/W) \leqslant d+1$ of Theorem 2.16 and therefore (0.4) can be dropped.

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