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EXAMPLES OF SIMPLY-CONNECTED SYMPLECTIC NON-KÄHLERIAN MANIFOLDS

DUSA McDUFF

1. Introduction

A symplectic manifold (M, σ) is a pair consisting of a $2n$-dimensional manifold M together with a closed 2-form σ which is nondegenerate (that is, σ^n never vanishes). The form σ determines two pieces of topological data: the cohomology class $a = [\sigma] \in H^2(M; \mathbf{R})$ and a homotopy class of reductions of the structural group of the tangent bundle of M to $U(n) \simeq \mathrm{Sp}(2n; \mathbf{R})$, and hence a homotopy class $[J]$ of almost complex structures on M. Gromov showed in his thesis that, if M is open, any such pair $(a, [J])$ may be realised by some symplectic form (see [3], [4]). If M is closed, a^n must be a generator of $H^{2n}(M; \mathbf{R})$ which is positive with respect to the orientation defined by $[J]$. But even with this condition, it is not known whether the corresponding statement is true. In fact, very few closed symplectic manifolds are known. Any Kähler manifold is symplectic. Thurston showed in [6] how to construct a non-Kähler closed symplectic manifold. His examples are nil-manifolds and so are not simply-connected. (A similar example was known to Serre. See [10], Problem 42. Thurston's construction is further developed in [9] and [3].)

In [3] Gromov points out that if the symplectic manifold (M, σ) is symplectically embedded in (X, ω), then one can "blow up" X along M to obtain a new symplectic manifold $(\tilde{X}, \tilde{\omega})$. In this note we use this technique together with a symplectic embedding theorem (see [5], [2], [7]) to construct some simply-connected, closed symplectic manifolds which are not Kähler.

Here is one such example. Let (M, σ) be Thurston's 4-dimensional symplectic, but non-Kähler manifold. It is the quotient $\mathbf{R}^4/\Gamma$, where Γ is the discrete affine group generated by the unit translations along the x_1, x_2, x_3-axes together with the transformation $(x_1, x_2, x_3, x_4) \mapsto (x_1 + x_2, x_2, x_3, x_4 + 1)$. Thus M is a T^2-bundle over T^2. Its symplectic form σ lifts to $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ on $\mathbf{R}^4$. Note that σ is integral, that is, $[\sigma] \in H^2(M; \mathbf{Z})$. Further,

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the Betti numbers $\beta_1(M)$ and $\beta_3(M)$ equal 3. Since these are odd, M has no Kähler structure. Another non-Kähler feature of M is that it does not satisfy the Lefschetz theorem, that is, multiplication by $[\sigma]$ does not induce an isomorphism from $H^1(M; \mathbf{R})$ to $H^3(M; \mathbf{R})$.

Tischler proves in [7] (see also Narasimham and Ramanan [5] and Gromov [2], [3]) that the complex projective space $\mathbf{C}P^n$, with its standard Kähler form ω_0, is a universal integral symplectic manifold. In other words, any manifold with integral symplectic form σ may be embedded in $\mathbf{C}P^n$ for suitably large n by a map f such that $f^*\omega_0 = \sigma$. In fact, Gromov proves in [3, 3.4.2] that if $\dim M = 2m$ one can take $n = 2m + 1$. Thus, we may embed Thurston's manifold (M, σ) in $(\mathbf{C}P^5, \omega_0)$. Let $(\tilde{X}, \tilde{\omega})$ be the symplectic manifold obtained by blowing up $\mathbf{C}P^5$ along M. Then we claim

Theorem. *$(\tilde{X}, \tilde{\omega})$ is a simply-connected, symplectic closed manifold with $\beta_3(\tilde{X}) = \beta_1(M) = 3$. Hence $\tilde{X}$ is not Kähler.*

We will also see that $\tilde{X}$ does not satisfy the Lefschetz theorem.

This theorem is proved as follows. In §2 we show how to blow up any manifold X along a codimension $2k$ submanifold M whose normal bundle $\nu(M)$ has structural group $U(k)$. The resulting manifold $\tilde{X}$ covers X in the sense that there is a smooth map $\varphi\colon \tilde{X} \to X$ which is a diffeomorphism outside $\varphi^{-1}M$ and which restricts over M to a fibration with fiber $\mathbf{C}P^{k-1}$.

Proposition 2.4. *If $\tilde{X}$ is the blow up of X along M, then $\pi_1\tilde{X} = \pi_1 X$. Further, the real cohomology $H^*\tilde{X}$ of $\tilde{X}$ fits into a short exact sequence of graded $\mathbf{R}$-modules*

$$0 \to H^*X \to H^*\tilde{X} \to A^* \to 0,$$

where A^* is a free module over $H^*(M)$ with one generator in each dimension $2i$, $1 \leqslant i \leqslant k-1$.

If (M, σ) is a symplectic submanifold of (X, ω), then its normal bundle $\nu(M)$ may be canonically embedded in the tangent bundle TX as the ω-orthogonal complement to $TM \subset TX$. Hence ω restricts to give a canonical symplectic (i.e., skew-symmetric and bilinear) form on $\nu(M)$. It follows that the structural group of $\nu(M)$ reduces to $U(k)$. Therefore the blow up $\tilde{X}$ of X along M is defined. In §3 we prove

Proposition 3.7. *If M is compact, the blow up $\tilde{X}$ of (X, ω) along (M, σ) carries a symplectic form $\tilde{\omega}$ which equals $\varphi^*\omega$ outside a neighborhood of $\varphi^{-1}M$.*

The theorem follows immediately from these two propositions. Notice that the construction in the theorem may be varied considerably. For example, we could start with any $2n$-dimensional Kähler manifold (M, σ) and blow up a point. The resulting manifold X contains a copy of $\mathbf{C}P^{n-1}$. If we now blow up X along any symplectic submanifold of $\mathbf{C}P^{n-1}$ which has an odd Betti number, we will obtain a symplectic manifold $(\tilde{X}, \tilde{\omega})$ with an odd Betti number. Further $\tilde{X}$ will be simply-connected if M is.

2. The topology of the blow up

In this section we suppose that M is a compact submanifold in X of codimension $2k$ such that the structural group of its normal bundle $E \to M$ reduces to $U(k)$. We will show how to blow up X along M to obtain a manifold $\tilde{X}$, and will discuss the topology of $\tilde{X}$. We assume throughout that $k \geqslant 2$, since, when $k = 1$, $\tilde{X}$ equals X and all our results are trivial.

First, we construct $\tilde{X}$. By assumption, the normal bundle $E \to M$ may be identified with $P \times_{U(k)} \mathbf{C}^k \to M$ for some principal $U(k)$-bundle $P \to M$. Let $L \to \mathbf{C}P^{k-1}$ be the canonical line bundle over $\mathbf{C}P^{k-1}$. Since $U(k)$ acts on L by bundle automorphisms, we may form the fiber bundles $\tilde{\pi}$: $\tilde{E} \to M$ and p: $\tilde{M} \to M$ which are associated to $P \to M$ and have fibers L and $\mathbf{C}P^{k-1}$ respectively. Thus, $\tilde{E}$ is a complex line bundle over the projectivization $\tilde{M}$ of E, and there is the commutative diagram:

$$\begin{array}{ccccc} \tilde{E}_0 & \hookrightarrow & \tilde{E} & \xrightarrow{\mathbf{C}} & \tilde{M} \\ \cong\big\downarrow & & \varphi\big\downarrow & \tilde{F} \searrow^{\tilde{\pi}}_{\mathbf{C}^k} & p\big\downarrow \mathbf{C}P^{k-1} \\ E_0 & \hookrightarrow & E & \xrightarrow[\pi]{} & M \end{array} \tag{2.1}$$

Here E_0, $\tilde{E}_0$ are the nonzero vectors in E, $\tilde{E}$ and φ is induced by the obvious map $L \to \mathbf{C}^k$. Note that $\varphi|\tilde{E}_0$ is a diffeomorphism. The space $\tilde{E}$ is called the blow up of E along M. It fibers over M with fiber $\tilde{F} \cong L \cong \overline{\mathbf{C}P^k} - \text{pt}$, where $\overline{\mathbf{C}P^k}$ is $\mathbf{C}P^k$ with the opposite orientation. Let V be a subdisc bundle of E which is diffeomorphic to a closed tubular neighbourhood W of M in X, and set $\tilde{V} = \varphi^{-1}V$.

Definition 2.2. The blow up $\tilde{X}$ of X along M is the smooth manifold

$$\overline{X - W} \cup_{\partial \tilde{V}} \tilde{V},$$

where $\partial \tilde{V}$ is identified with ∂W in the obvious way.

Our first lemma will be needed in §3. Consider the fibration $\tilde{\pi}$: $\tilde{E} \to M$. Each fiber $\tilde{F}$ is a line bundle over $\mathbf{C}P^{k-1}$. Let $a_{\tilde{F}} \in H^2(\tilde{F})$ be the class represented by the pull-back of the Kähler form ω_0 on $\mathbf{C}P^{k-1}$. Then the real cohomology $H^*(\tilde{F})$ of $\tilde{F}$ is generated as an algebra by $a_{\tilde{F}}$, with the relation $a_{\tilde{F}}^k = 0$.

Lemma 2.3. *There is a unique class $a \in H^2(\tilde{E})$ which restricts to $a_{\tilde{F}}$ on each fiber $\tilde{F}$ and to 0 on $H^2(\tilde{E}_0)$. Further $H^*(\tilde{E})$ is a free module over $H^*(M)$ with generators $1, a, \cdots, a^{k-1}$.*

Proof. The first Chern class of the dual of the tautological line bundle over $\tilde{E}$ clearly has the properties required of a. The rest of the lemma then follows from the Leray-Hirsch theorem.

Proposition 2.4. *$\pi_1\tilde{X} \cong \pi_1 X$. Further, there is a short exact sequence*

$$0 \to H^*(X) \to H^*(\tilde{X}) \to A^* \to 0,$$

where $A^* \cong$ coker$(H^*M$ in $H^*\tilde{E})$ is a free module over H^*M with generators $a, \cdots, a^{k-1}$.

Proof. Since $\tilde{X} = \overline{X - W} \cup_{\partial\tilde{V}} \tilde{V}$, we have

$$\pi_1\tilde{X} \cong \tilde{\pi}_1(\overline{X - W}) *_{\pi_1(\partial\tilde{V})} \pi_1\tilde{V}.$$

Now, $\pi_1(\overline{X - W}) = \pi_1 X$ by general position, since W retracts onto M and M has codimension $\geqslant 4$ in X. Further, because $\tilde{V}$ and $\partial\tilde{V}$ both fiber over M with simply-connected fibers, it is easy to see that the inclusion $\partial\tilde{V} \hookrightarrow \tilde{V}$ induces an isomorphism on π_1. Hence $\pi_1\tilde{X} \cong \pi_1 X$.

Next, observe that the map $\varphi: \tilde{X} \to X$ has degree 1 and so induces an isomorphism on H^{2n}, where dim $X = 2n$. Since for every nonzero $a \in H^*(X)$ there is $b \in H^*(X)$ such that $a \cup b$ generates H^{2n}, it follows that φ^*: $H^*(X) \to H^*(\tilde{X})$ is injective. Therefore, there is a short exact sequence

$$0 \to H^*(X) \to H^*(\tilde{X}) \to H^{*+1}(X, \tilde{X}) \to 0,$$

where the last group is interpreted as the relative cohomology of the pair $(C_\varphi, \tilde{X})$, and where C_φ is the mapping cylinder of φ. In order to calculate $H^{*+1}(X, \tilde{X})$, consider the commutative diagram:

$$\begin{array}{ccccc} \tilde{M} & \hookrightarrow & \tilde{V} & \hookrightarrow & \tilde{X} \\ \downarrow & & \downarrow & & \downarrow \varphi \\ M & \hookrightarrow & V & \hookrightarrow & X \end{array}$$

Since φ takes $\tilde{X} - \tilde{M}$ diffeomorphically onto $X - M$, it follows by excision that $H^*(X, \tilde{X}) \cong H^*(V, \tilde{V})$. By Lemma 2.3, $H^*(V) \cong H^*(M)$ injects into $H^*(\tilde{V})$, and $H^*(\tilde{V}) \cong H^*(\tilde{M})$ is generated as an $H^*(V)$-module by the elements $1, a, \cdots, a^{k-1}$. Therefore, there are short exact sequences

$$0 \to H^i(V) \to H^i(\tilde{V}) \to H^{i+1}(V, \tilde{V}) \to 0,$$

and $H^{*+1}(V, \tilde{V}) \cong H^{*+1}(X, \tilde{X})$ is additively isomorphic to $A^* = H^*(M) \otimes (\oplus_{i=1}^{k-1} \mathbf{Z}a^i)$. q.e.d.

We will say that the Lefschetz theorem holds for a symplectic manifold (X, ω) of dimension $2n$ if, for all $i = 1, \cdots, n-1$, multiplication by a^{n-i}

induces an isomorphism from $H^i(X)$ to $H^{2n-i}(X)$, where $a = [\omega] \in H^2(X)$. It is well known that this theorem holds for compact Kähler manifolds. On the other hand, one can easily check that it fails in the case of Thurston's 4-manifold. The next result implies that it also fails for the manifold $(\tilde{X}, \tilde{\omega})$ constructed by blowing up $\mathbf{C}P^n$ along Thurston's manifold. As we will see in §3, the form $\tilde{\omega}$, when restricted to $\tilde{V}$, represents the cohomology class $b + \varepsilon a \in H^2(\tilde{V})$, where $b = [\sigma] \in H^2(M)$, $\varepsilon > 0$, and we identify $H^*(\tilde{V})$ with $H^*(M) \otimes \mathbf{Z}[1, \cdots, a^{k-1}]$ as in Lemma 2.3.

Proposition 2.5. *Let (M, σ) be a symplectic submanifold of $\mathbf{C}P^n$ of codimension $2k$ whose Chern classes all vanish. Further, suppose that $\tilde{\omega}$ is a symplectic form on the blow up $\tilde{X}$ of $\mathbf{C}P^n$ along M such that $[\tilde{\omega}|\tilde{V}] = b + \varepsilon a$. Then, if the Lefschetz theorem fails for M, it fails also for $\tilde{X}$.*

Proof. Using Proposition 2.4 and the fact that $X = \mathbf{C}P^n$, one sees easily that the restriction map $H^i(\tilde{X}) \to H^i(\tilde{V})$ is injective for $i \leqslant 2(n-k) = \dim M$, and has kernel generated by a suitable power of $\tilde{b}$ if $i > 2(n-k)$, where $\tilde{b} \in H^2(\tilde{X})$ restricts to $b \in H^2(\tilde{V})$. By Lemma 2.3, there is a relation of the form

$$a^k = u_k + u_{k-1}a + \cdots + u_1 a^{k-1}$$

in $H^*(\tilde{V})$, where $u_i \in H^{2i}(M)$. Because the construction of E and a is universal, the u_i must be characteristic classes of E. In fact, $u_i = -c_i(E)$. But $c_i(E) = c_i(T\mathbf{C}P^n|M) = \binom{n+1}{i} b^i$ because the Chern classes of M vanish. Therefore, each u_i is a multiple of b^i, and we have

$$\text{(i)} \qquad a^k = \sum_{j=0}^{k-1} \mu_j a^j b^{k-j}.$$

It follows that

$$\text{(ii)} \qquad \tilde{a}^k = \sum_{j=0}^{k-1} \nu_j \tilde{a}^j \tilde{b}^{k-j},$$

where $\tilde{a} \in H^2(\tilde{X})$ restricts to $a \in H^2(\tilde{V})$. Note also that, because $b^{n-k+1} = 0$, there is a relation

$$\text{(iii)} \qquad \tilde{a}\tilde{b}^{n-k+1} = \nu \tilde{b}^{n-k+2}.$$

By hypothesis there is a nonzero element $v \in H^i(M)$ for some $i < n-k$ such that $vb^{n-k-i} = 0$. By (i) this implies that

$$av(b + \varepsilon a)^{n-i-2} = \sum_{j=0}^{k-1} \lambda_j v a^j b^{n-j-i-1} = 0.$$

Therefore, if $\tilde{w} \in H^{i+2}(X)$ restricts to av, we have $\tilde{w}(\tilde{b} + \varepsilon\tilde{a})^{n-i-2} = 0$, unless $2n - i - 2$ is an even number $> 2(n-k)$. In the latter case, $\tilde{w}(\tilde{b} + \varepsilon\tilde{a})^{n-i-2}$ is a multiple of $\tilde{b}^{n-m}$, where $2m = i + 2 < 2k$. Observe, however, that when $m < k$ the elements $\tilde{a}^j\tilde{b}^{m-j}, j = 0,\cdots,m$, are linearly independent in $H^{2m}(\tilde{X})$, and the equation

$$(\tilde{b} + \varepsilon\tilde{a})^{n-2m}\left(\sum_{j=0}^{m} x_j\tilde{a}^j\tilde{b}^{m-j}\right) = \nu\tilde{b}^{n-m}$$

in the unknowns $x_0,\cdots,x_m$ always has a nonzero solution either for $\nu = 0$ or for $\nu = 1$. For by (ii), (iii) the left-hand side may be written as a linear combination of the $m+1$ terms $\tilde{b}^{n-m}$ and $\tilde{a}^j\tilde{b}^{n-m-j}$, $k - m \leqslant j \leqslant k-1$. Thus we have two systems of $m+1$ equations in $m+1$ unknowns, one of which must have a nonzero solution. Because $\tilde{w}$ cannot equal $\sum_{j=0}^m x_j\tilde{a}^j\tilde{b}^{m-j}$, it follows in either case that multiplication by $(\tilde{b} + \varepsilon\tilde{a})^{n-2m}$ is not injective on $H^{2m}(\tilde{X})$.

3. The symplectic structure of the blow up

In this section we construct the symplectic form $\tilde{\omega}$ on $\tilde{X}$, using a procedure outlined by Gromov in [3]. Since there is a canonical almost complex structure on the fibers of $\tilde{\pi}: \tilde{E} \to M$, it is easy to construct $\tilde{\omega}$ there. However, we must also control $\tilde{\omega}$ in the transverse directions, and for this we need the regularity Lemma 3.6.

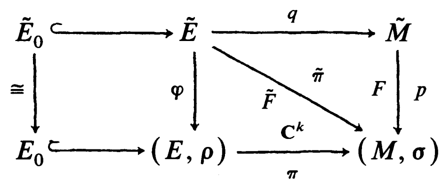
The normal bundle $E \to M$ of (M, σ) in (X, ω) may be identified with the ω-orthogonal complement of TM in $TX|M$. Thus ω restricts to give a canonical translation invariant symplectic form on each fiber of E. We will denote the zero section of $E \to M$ by Z, and will assume throughout that M is compact.

Lemma 3.1. *There is a closed 2-form ρ on E which restricts to this canonical form on each fiber and to σ on $Z \equiv M$. We may assume further that TZ is ρ-orthogonal to the tangent space of each fiber.*

Proof. Let $\{U_i\}$ be an open cover of M such that each $E|U_i$ is trivial. For each i, let β_i be a 1-form on $\pi^{-1}U_i$ which is 0 on $Z \cap \pi^{-1}U_i$ and is such that $d\beta_i$ restricts to the canonical form on each fiber of E. Then take $\rho = \pi^*\sigma + \sum d(\lambda_i\beta_i)$, where $\{\lambda_i\}$ is a partition of unity subordinate to $\{U_i\}$. q.e.d.

Since ρ is nondegenerate on Z, it is nondegenerate on a neighborhood of Z. Further, by construction, there is a bundle isomorphism $TE|Z \to TX|M$ which takes ρ to ω. Hence, by [8, Theorem 4.1], a neighborhood W of M in X is symplectically isomorphic to a neighborhood V of Z in E. We may assume that V is a compact disc bundle over M so that $\tilde{X} = \overline{X - M} \cup_{\partial\tilde{V}} \tilde{V}$ as in Definition 2.2. Our aim is to define a symplectic form $\tilde{\rho}$ on $\tilde{V}$ which equals $\varphi^*\rho$ near $\partial\tilde{V}$.

Recall from (2.1) that we have a commutative diagram:



We write Z for the zero section of π, and $\tilde{Z}$ for the zero section $\varphi^{-1}(Z) \equiv \tilde{M}$ of q. The typical fiber of p, respectively $\tilde{\pi}$, will be called F, respectively $\tilde{F}$, so that $\tilde{F}$ is the canonical line bundle over $F \cong \mathbf{C}P^{k-1}$. Observe, further, that because the action of $U(k)$ on $\mathbf{C}P^{k-1}$ preserves the Kähler form ω_0, each fiber F carries a canonical form which we also call ω_0.

Lemma 3.2. *There is a closed 2-form α on $\tilde{M}$ which restricts to ω_0 on each fiber F of p and pulls back to an exact form on $\tilde{E}_0$.*

Proof. Let $b = (q^*)^{-1}(a)$, where $a \in H^2(\tilde{E})$ is the class mentioned in Lemma 2.3. Clearly, it suffices to construct a closed 2-form α on $\tilde{M}$ which represents the class b and restricts to ω_0 on each F. Such α exists by Thurston's argument in [6]. Namely, one takes any representative β of b, and then puts $\alpha = \beta + d(\Sigma(\lambda_i \circ p)\gamma_i)$, where $\{\lambda_i\}$ is a partition of unity on M and the γ_i are 1-forms defined on the sets $p^{-1}(U_i)$, chosen so that for every fiber F over U_i the restriction of $\beta + d\gamma_i$ to F equals ω_0. For more details see [1].

Lemma 3.3 [6]. *There is an $\varepsilon_0 > 0$ such that the form $p^*\sigma + \varepsilon\alpha$ is nondegenerate on $\tilde{M}$ whenever $0 < \varepsilon \leqslant \varepsilon_0$.*

Proof. For each $x \in \tilde{M}$, let

$$W_x = \{v \in T_x\tilde{M}\colon \alpha(v, t) = 0, \forall t \in T_xF\}$$

be the α-orthogonal complement to the tangent space T_xF of the fiber through x. Because α is nondegenerate on each fiber F, $T_x\tilde{M}$ splits as the direct sum $T_xF \oplus W_x$. Hence, the differential $p_*\colon T\tilde{M} \to TM$ is injective on each W_x, and $p^*\sigma | W_x$ is nondegenerate. By the compactness of M, there is an $\varepsilon_0 > 0$ such that $p^*\sigma + \varepsilon\alpha$ is nondegenerate on each W_x for all $\varepsilon \leqslant \varepsilon_0$. Since $p^*\sigma = 0$ on T_xF, the subspaces T_xF and W_x are orthogonal with respect to $p^*\sigma + \varepsilon\alpha$. It follows easily that $p^*\sigma + \varepsilon\alpha$ is nondegenerate on $\tilde{M}$ for $0 < \varepsilon \leqslant \varepsilon_0$.

Lemma 3.4. *There is an $\varepsilon_1 > 0$ such that the form $\tilde{\rho}_\varepsilon = \varphi^*\rho + \varepsilon q^*\alpha$ is nondegenerate on $\tilde{V}$ whenever $0 < \varepsilon \leqslant \varepsilon_1$.*

Proof. By construction, $\tilde{\rho}_\varepsilon | \tilde{F} = \varphi^*\omega_1 + q^*\omega_0$, where ω_0, ω_1 are the standard Kähler forms on $\mathbf{C}P^{k-1}$ and $\mathbf{C}^k$ respectively. Therefore, if J is the standard almost complex structure on $\tilde{F}$, $\varphi^*\omega_1(v, Jv)$ and $q^*\omega_0(v, Jv)$ are $\geqslant 0$ for all $v \in T\tilde{F}$. Since the kernel of $\varphi_*\colon T\tilde{E} \to TE$ consists of vectors tangent to the fibers of the projection $\tilde{Z} \to M$ and since $q^*\omega_0$ is nondegenerate on these fibers, it follows easily that $\tilde{\rho}_\varepsilon|\tilde{F}$ is nondegenerate for all $\varepsilon > 0$. Hence, if g is a Riemannian metric on $\tilde{E}$ which restricts on each $\tilde{F}$ to the Kähler metric $g(v, w) = \tilde{\rho}_1(v, Jw)$, there is a constant $K > 0$ such that

$$\text{(i)} \qquad \max\{\tilde{\rho}_\varepsilon(t, t') \colon t' \in T_x\tilde{F}, \|t'\|_{\tilde{E}} = 1\} \geqslant K\varepsilon\|t\|_{\tilde{E}}$$

for all $t \in T_x\tilde{F}$, $x \in \tilde{V}$ and $\varepsilon > 0$, where $\|t\|_{\tilde{E}} = g(t, t)$. Further, because $\varphi^*\rho$ restricts to $p^*\sigma$ on $\tilde{Z} \equiv \tilde{M}$ and is nondegenerate on the fibers of the line bundle $\tilde{E} \to \tilde{M}$, it follows from Lemma 3.3 that $\tilde{\rho}_\varepsilon$ is nondegenerate on $T_x\tilde{E}$ for all $x \in \tilde{Z}$, provided that $0 < \varepsilon \leqslant \varepsilon_0$.

It remains to show that $\tilde{\rho}_\varepsilon$ is nondegenerate on $\tilde{V} - \tilde{Z}$. To do this, we adapt the argument of Lemma 3.3. For each $x \in \tilde{V} - \tilde{Z}$, define

$$\tilde{W}_x = \{v \in T_x\tilde{E} \colon \varphi^*\rho(v, t) = 0, \forall t \in T_x\tilde{F}\}.$$

Because $\varphi^*\rho$ is nondegenerate on $\tilde{V} - \tilde{Z}$, the tangent space $T_x\tilde{E}$ splits as the direct sum $T_x\tilde{F} \oplus \tilde{W}_x$. Further, we will prove below that the following result holds.

Lemma 3.5. *Given any Riemannian metric on E, there are constants $c_1, c_2 > 0$ such that for all $w \in \tilde{W}_x$ and $x \in \tilde{V} - \tilde{Z}$, we have*

$$c_1\|w\|_{\tilde{E}} \leqslant \|\varphi_*(w)\|_E \leqslant c_2\|w\|_{\tilde{E}}.$$

In other words, the subspaces $\tilde{W}_x$ do not get too close to $\ker \varphi_* \subset T_x\tilde{F}$ as x approaches $\tilde{Z}$.

Because ρ is nondegenerate on the compact set V, there is a constant $K' > 0$ such that, for all $v \in T_xV$ and $x \in V$, we have

$$\max\{\rho(v, v') \colon v' \in T_xV, \|v'\|_E = 1\} \geqslant K'\|v\|_E.$$

This implies, by Lemma 3.5, that there is a constant $c' > 0$ such that, for all $w \in \tilde{W}_x$ and $x \in \tilde{V} - \tilde{Z}$, we have

$$\max\{\varphi^*\rho(w, w') \colon w' \in \tilde{W}_x, \|w'\|_{\tilde{E}} = 1\} \geqslant c'\|w\|_{\tilde{E}}.$$

Hence, if ε_2 is sufficiently small, we have exactly the same kind of estimate for $\tilde{\rho}_\varepsilon$, $0 < \varepsilon \leqslant \varepsilon_2$, namely: there is $c > 0$ such that

$$\text{(ii)} \qquad \max\{\tilde{\rho}_\varepsilon(w, w') \colon w' \in \tilde{W}_x, \|w'\|_{\tilde{E}} = 1\} \geqslant c\|w\|_{\tilde{E}}$$

for all $w \in \tilde{W}_x$, $x \in \tilde{V} - \tilde{Z}$, and all $0 < \varepsilon \leqslant \varepsilon_2$.

To show that $\tilde{\rho}_\varepsilon$ is nondegenerate on $\tilde{V} - \tilde{Z}$, it suffices to show that, for all $x \in \tilde{V} - \tilde{Z}$ and nonzero $v \in T_x\tilde{E}$, there is $v' \in T_x\tilde{E}$ such that $\tilde{\rho}_\varepsilon(v, v') \neq 0$. Choose C so that

$$\text{(iii)} \qquad |\alpha(v, v')| \leqslant C\|v\|\|v'\|$$

for all $x \in \tilde{V}$ and $v, v' \in T_x\tilde{V}$. (Here and subsequently $\| \ \|$ denotes $\| \ \|_{\tilde{E}}$.) Further, let

$$\varepsilon_1 = \min\left(\varepsilon_0, \varepsilon_2, Kc/2C^2\right).$$

Then, given $v = t + w$ with $\|w\| < K\|t\|/C$, choose $t' \in T_x\tilde{F}$ with $\|t'\| = 1$ and so that $\tilde{\rho}_\varepsilon(t, t')$ is a maximum. Because $T_x\tilde{F}$ and $\tilde{W}_x$ are $\varphi^*\rho$-orthogonal, we will have

$$\begin{aligned} \left|\tilde{\rho}_\varepsilon(t + w, t')\right| &= \left|\tilde{\rho}_\varepsilon(t, t') + \varepsilon\alpha(w, t')\right| \\ &\geqslant K\varepsilon\|t\| - \varepsilon C\|w\| \quad \text{by (i), (ii)} \\ &\neq 0 \quad \text{since } \|w\| < K\|t\|/C. \end{aligned}$$

On the other hand, if $\|w\| \geqslant K\|t\|/C$, choose $w' \in \tilde{W}_x$ with $\|w'\| = 1$ and so that $\tilde{\rho}_\varepsilon(w, w')$ is a maximum. We then have

$$\begin{aligned} \left|\tilde{\rho}_\varepsilon(t + w, w')\right| &= \left|\tilde{\rho}_\varepsilon(w, w') + \varepsilon\alpha(t, w')\right| \\ &\geqslant c\|w\| - \varepsilon C\|t\| \quad \text{by (ii), (iii)} \\ &\neq 0 \quad \text{since } \|w\| \geqslant K\|t\|/C, \end{aligned}$$

provided that $\varepsilon \leqslant \varepsilon_1$. Therefore $\tilde{\rho}_\varepsilon$ is nondegenerate on $\tilde{V}$ for $0 < \varepsilon \leqslant \varepsilon_1$ as required.

Proof of Lemma 3.5. For $x \in E$, let W_x be the ρ-orthogonal complement of the tangent space to the fibers of π: $E \to M$. Then $\mathscr{W} = \bigcup W_x$ is a smooth subbundle of TE which, by the choice of ρ, equals TZ along the zero section Z of π. Also $\tilde{W}_{\tilde{x}} = \varphi^*(W_{\varphi(\tilde{x})})$ for $\tilde{x} \in \tilde{V} - \tilde{Z}$. Since the kernel of φ_*: $T\tilde{E} \to TE$ consists of the tangent spaces to the fibers of the projection $\tilde{Z} \to M$, it clearly suffices to prove the following regularity lemma.

Lemma 3.6. *Let $\mathscr{W} = \bigcup W_x$ be any C^1-smooth subbundle of TE which equals TZ along Z, and let $\tilde{W}_{\tilde{x}} = \varphi^*(W_{\varphi(\tilde{x})})$ for $\tilde{x} \in \tilde{E} - \tilde{Z}$. Then the closure in $T\tilde{E}$ of the set $\bigcup\{\tilde{W}_{\tilde{x}}: \tilde{x} \in \tilde{E} - \tilde{Z}\}$ contains no nonzero vectors which are tangent to the fibers of $\tilde{\pi}$: $\tilde{Z} \to M$.*

Proof. The blow up $\tilde{\mathbf{C}}^k$ of $\mathbf{C}^k$ at 0 sits inside $\mathbf{C}^k \times \mathbf{C}P^{k-1}$ as the submanifold $\{(z_1, \cdots, z_k; [w_1: \cdots :w_k]): z_i w_j = z_j w_i, \forall i, j\}$. Further φ: $\tilde{\mathbf{C}}^k \to \mathbf{C}^k$ is just given by the projection $\mathbf{C}^k \times \mathbf{C}P^{k-1} \to \mathbf{C}^k$. Thus the map

$$(z_1, w_2, \cdots, w_k) \mapsto (z_1, z_1w_2, \cdots, z_1w_k; [1: w_2: \cdots :w_k])$$

provides local coordinates for a neighborhood of the point $(0, \cdots, 0; [1:0: \cdots :0])$ in $\tilde{\mathbf{C}}^k$. It follows that $\tilde{E}$ may be covered by open sets in which φ: $\tilde{E} \to E$ takes the form

$$(y_1, \cdots, y_m; z_1, w_2, \cdots, w_k) \mapsto (y_1, \cdots, y_m; z_1, z_1w_2, \cdots, z_1w_k) \in U \times \mathbf{C}^k,$$

where $(y_1,\cdots,y_m) \in \mathbf{R}^m$ are coordinates on $U \subset M$. Using coordinates $(y;\zeta) = (y_1,\cdots,y_m;\zeta_1,\cdots,\zeta_k)$ on $U \times \mathbf{C}^k \subset E$, we may write the real subspace W_x as

$$W_x = \{v \in T_x\tilde{E} : d\zeta_\alpha(v) + \Sigma b_{i\alpha}(y,\zeta)\, dy_i(v) = 0\}$$

for suitable complex valued C^1-functions $b_{i\alpha}$ and for x near Z. Hence $\tilde{W}_{\tilde{x}} = \varphi^*(W_{\varphi(\tilde{x})})$ is given by the equations

$$dz_1 + \sum b_{i1}(\varphi(\tilde{x}))\, dy_i = 0,$$

$$w_\alpha\, dz_1 + z_1\, dw_\alpha + \sum b_{i\alpha}(\varphi(\tilde{x}))\, dy_i = 0, \qquad 2 \leqslant \alpha \leqslant k.$$

Because W_x approaches T_xZ as x approaches Z, the functions $b_{i\alpha}(y,\zeta)$ tend to zero as $\zeta \to 0$. Thus

$$b_{i\alpha}(\varphi(\tilde{x})) = b_{i\alpha}(y; z_1, z_1w_2,\cdots,z_1w_k) = O(|z_1|) \quad \text{as } z_1 \to 0.$$

Further, observe that as the point $\tilde{x}$ approaches $\tilde{Z}$, the coordinate $z_1(\tilde{x})$ tends to 0. It follows easily that the functions $dz_1(v)$ and $dw_\alpha(v)$ of the $dy_i(v)$, $v \in \tilde{W}_{\tilde{x}}$, are uniformly bounded in a neighborhood N of $\tilde{Z}$. This implies that the closure of $\bigcup \tilde{W}_{\tilde{x}}$ in $T\tilde{E}|N$ contains no nonzero vectors v with $dy_i(v) = 0$, $i = 1,\cdots,m$. The lemma follows.

We are finally ready to construct the symplectic form $\tilde{\omega}$ on the blow up.

Proposition 3.7. *If M is compact, the blow up $\tilde{X}$ of (X,ω) along (M,σ) carries a symplectic form $\tilde{\omega}$ which equals $\varphi^*\omega$ outside a neighborhood of $\varphi^{-1}M$.*

Proof. Since $q^*\alpha$ is exact on $\tilde{E}_0$ by Lemma 3.2, there is a 1-form β on $\tilde{E}_0$ such that $\alpha = d\beta$ on $\tilde{E}_0$. Let λ be a smooth function on $\tilde{Y}$ which equals 1 near $\tilde{Z}$ and 0 near the boundary of $\tilde{V}$. Then define $\tilde{\rho}$ by:

$$\tilde{\rho} = \begin{cases} \tilde{\rho}_\varepsilon & \text{on } \tilde{Z}, \\ \varphi^*\rho + \varepsilon d(\lambda\beta) & \text{on } \tilde{V} - \tilde{Z}. \end{cases}$$

Clearly $\tilde{\rho}$ is a smooth 2-form on $\tilde{V}$. Since $\varphi^*\rho$ is nondegenerate on $\tilde{V} - \tilde{Z}$, it follows easily from Lemma 3.4 that $\tilde{\rho}$ will be nondegenerate on $\tilde{V}$ if we choose ε sufficiently small. Therefore, because $\tilde{\rho} = \varphi^*\rho$ outside a neighborhood of $\tilde{Z}$ in Int $\tilde{V}$, we may define $\tilde{\omega}$ on $\tilde{X} = \overline{X - W} \cup_{\partial\tilde{V}} \tilde{V}$ by setting it equal to ω on $\overline{X - W}$ and to $\tilde{\rho}$ on $\tilde{V}$. q.e.d.

Note. By construction, $\tilde{\omega} = \tilde{\rho}_\varepsilon +$ (exact) on $\tilde{V}$. Also $[\tilde{\rho}_\varepsilon] = [\varphi^*\rho] + \varepsilon[q^*\alpha] = \tilde{\pi}^*[\sigma] + \varepsilon a$ by Lemma 3.2. Hence $[\tilde{\omega}|\tilde{V}] = b + \varepsilon a$ as in Proposition 2.5.

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