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FINITE VOLUME AND FUNDAMENTAL GROUP ON MANIFOLDS OF NEGATIVE CURVATURE

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1. Introduction

Let V be a complete Riemannian manifold of dimension n and sectional curvature $K \leqslant 0$. Then V is a $K(\pi, 1)$-manifold with $\pi = \pi_1(V)$ [8, p. 103] and hence determined up to homotopy by the fundamental group. In particular, the homology $H_*(V)$ of V is isomorphic to the group homology $H_*(\pi_1(V))$ (see [1]). Therefore V is compact if and only if $H_n(\pi_1(V), \mathbf{Z}_2) = \mathbf{Z}_2$. Hence the compactness of V can be read off from $\pi_1(V)$.

We give a similar characterization for the condition of finite volume:

Theorem. *Let V be a complete Riemannian manifold of dimension $n \geqslant 3$ with curvature $-b^2 \leqslant K \leqslant -a^2 < 0$. Then the volume of V is finite if and only if*:

(1) *$\pi_1(V)$ contains only finitely many conjugation classes of maximal almost nilpotent subgroups of rank $n - 1$.*

(2) *If Δ is the amalgamated product of $\pi_1(V)$ with itself on these subgroups, then $H_n(\Delta, \mathbf{Z}_2) = \mathbf{Z}_2$.*

For a full definition of Δ we refer to §4.

For $n = 2$, the statement is wrong: Let V be a noncompact surface with constant negative curvature and finite volume. It is known that V has an end E diffeomorphic to $S^1 \times (0, \infty)$ with a warped product metric $f^2\, ds^2 + dt^2$. The curvature is given by $-f''/f$ and the volume of E by $2\pi\int_0^\infty f\, dt$. Using a suitable function $\bar{f}$ we can deform E to an expanding end, such that the new end has bounded negative curvature but infinite volume.

The first part of our proof (§3) leads to a description of the ends of finite volume in terms of the fundamental group. This part is based on the investigations of Heintze [6], Gromov [5] and Eberlein [3]. A topological argument then finishes the proof (§4).

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grateful to Mikhael Gromov who proposed the result and pointed out essential ideas for the proof.

2. Notation and basic results

(Compare [3], [4].) Let X be a Hadamard manifold, i.e., a complete simply connected Riemannian manifold with curvature $K \leqslant 0$, let $d(\ ,\ )$ be the distance function on X and let $\bar{X} = X \cup X(\infty)$ be the Eberlein-O'Neill compactification. For $x \in X$ and $z \in X(\infty)$ let $HS(x, z)$ be the horosphere at z which contains x and $HB(x, z)$ the corresponding (open) horoball. For an isometry γ of X we define the convex displacement function d_γ: $x \to d(x, \gamma x)$. γ is called elliptic (hyperbolic, parabolic), if d_γ has zero minimum (positive minimum, no minimum). An isometry γ can be extended to a homeomorphism of $\bar{X}$. If X has curvature $K \leqslant -a^2 < 0$, a nonelliptic isometry γ can be characterized by the fixed points $\mathrm{Fix}(\gamma)$ on $X(\infty)$: a hyperbolic isometry fixes exactly two points of $X(\infty)$ and translates the unique geodesic joining these points. A parabolic isometry γ has exactly one fixed point $z \in X(\infty)$ and leaves the horospheres $HS(x, z)$ invariant.

For a complete manifold V of negative curvature let X be the Riemannian universal covering, π: $X \to V$ the projection. Then $V = X/\Gamma$, where Γ is a freely acting, discrete group of isometries on X, $\Gamma \simeq \pi_1(V)$. We define the Γ-invariant function d_Γ: $X \to (0, \infty)$ by $d_\Gamma(x) := \min_{\gamma \in \Gamma - \mathrm{id}} d_\gamma(x)$. Then $d_\Gamma(x) = 2 \operatorname{Inj Rad}(\pi(x))$, where Inj Rad is the injectivity radius. $\operatorname{Inj Rad}(p) \geqslant \varepsilon$ and $K \leqslant 0$ imply that the volume of the distance ball $B_\varepsilon(p)$ is larger than the volume of the ε-ball in euclidean space. Therefore $\mathrm{vol}(V) < \infty$ implies that the set $\{\operatorname{Inj Rad} \geqslant \varepsilon\}$ is compact for all $\varepsilon > 0$.

An end of V is a function E that assigns to each compact subset K of V a connected component $E(K)$ of $V - K$ with the condition that $E(K) \supset E(K')$ if $K \subset K'$. An open set $U \subset V$ is a neighborhood of an end E if $E(K) \subset U$ for some compact subset K. An end E has finite volume if there is a neighborhood U of E with $\mathrm{vol}(U) < \infty$.

For the proof of our theorem, we can assume (by scaling the metric) that V satisfies the curvature condition $-1 \leqslant K \leqslant -a^2$, where a is positive. This enables us to use the Margulis lemma in the following form.

Margulis Lemma. *There is a number $\mu = \mu(n) > 0$, depending only on n, with the following property: let X be an n-dimensional Hadamard manifold with curvature $-1 \leqslant K \leqslant 0$, let Γ be a discrete group of isometries on X, $x \in X$, and let $\Gamma_\mu(x)$ be the subgroup of Γ generated by the elements $\gamma \in \Gamma$ with $d_\gamma(x) \leqslant \mu$. Then $\Gamma_\mu(x)$ is almost nilpotent, that is, $\Gamma_\mu(x)$ contains a nilpotent subgroup of finite index.*

For a proof see [11, p. 5.51], [2, p. 27], [5], [10].

Lemma 1. *Let X be a Hadamard manifold with curvature $K \leqslant -a^2$ and let Γ be a freely acting, discrete and almost nilpotent group of isometries on X. Then* $\mathrm{Fix}(\gamma_1) = \mathrm{Fix}(\gamma_2)$ *for all* $\gamma_1, \gamma_2 \in \Gamma - \mathrm{id}$. *Hence the elements of* $\Gamma - \mathrm{id}$ *are either all parabolic with a common fixed point $z \in X(\infty)$, or all hyperbolic with common axis c. In the second case Γ is infinite cyclic.*

For a proof see [3, Lemma 3.1b].

3. Ends of finite volume

The main result of this section is the following description of the ends of finite volume.

Proposition. *Let $V = X/\Gamma$ satisfy $-1 \leqslant K \leqslant -a^2$, $0 < r \leqslant \mu$.*

(1) *If E is an end of finite volume, then there is a unique connected component $U_r(E)$ of* $\{\mathrm{Inj\,Rad} < r/2\}$ *such that $U_r(E)$ is a neighborhood of E. The volume of $U_r(E)$ is finite. For two different ends E and E^* of finite volume, the neighborhoods $U_r(E)$ and $U_r(E^*)$ are disjoint.*

(2) *If* $n = \dim V \geqslant 3$, *then the ends of finite volume correspond one-to-one to the conjugation classes of the maximal almost nilpotent subgroups of rank $n-1$ in Γ.*

(3) *The ends of finite volume have disjoint neighborhoods U diffeomorphic to $B \times (0, \infty)$, where B is a compact codimension* 1 *submanifold of V.*

Before we will prove this result, we need some preparations. Our manifold V was represented as $V = X/\Gamma$. Now we look for a similar description for subsets $U \subset V$ as $U = W/\Gamma_W$, where $W \subset X$ is precisely invariant, i.e. for any $\gamma \in \Gamma$ either $\gamma W = W$ or $\gamma W \cap W = \varnothing$, and Γ_W is the subgroup $\{\gamma \in \Gamma | \gamma W = W\}$.

Lemma 2. *Let Γ be a discrete group of isometries acting on a Hadamard manifold X. Let $r > 0$ and let $W \subset X$ be a connected component of $\{d_\Gamma < r\}$. Then*:

(1) *W is precisely invariant.*

(2) *If $\gamma \in \Gamma$, $x \in W$ and $d_\gamma(x) < r$, then $\gamma \in \Gamma_W$.*

Proof. (1) Because d_Γ is Γ-invariant, γW is also a connected component of $\{d_\Gamma < r\}$ for all $\gamma \in \Gamma$. Thus $\gamma W \cap W \neq \varnothing$ implies $\gamma W = W$.

(2) $d_\gamma(x) = d_\gamma(\gamma x) < r$. The convexity of d_γ now implies $d_\gamma < r$ hence $d_\Gamma < r$ on the geodesic from x to γx. Thus both x and γx are in W. By (1), $\gamma \in \Gamma_W$. q.e.d.

Let U be a component (i.e., a connected component) of $\{\mathrm{Inj\,Rad} < r/2\}$ and W be a component of $\pi^{-1}(U) \subset X$. Then W is a component of $\{d_\Gamma < r\}$

and, by Lemma 2, $U = W/\Gamma_W$. With regard to the Margulis Lemma we will study components U of $\{\text{Inj Rad} < r/2\}$ and the corresponding components W of $\{d_\Gamma < r\}$, where r is smaller than the constant μ of the Margulis Lemma.

Lemma 3. *Let V be complete, $-1 \leqslant K \leqslant -a^2$, $0 < r \leqslant \mu$. Let $U \subset V$ be a component of $\{\text{Inj Rad} < r/2\}$ in V, W a component of $\pi^{-1}(U)$ in X and $\Gamma_W = \{\gamma \in \Gamma | \gamma W = W\}$.*

(1) *Either there is a unique geodesic c in X, such that Γ_W is the infinite cyclic group $\Gamma_W = \Gamma_c := \{\gamma \in \Gamma | \gamma$ has axis $c\}$ or Γ_W is a group of parabolic isometries and there is a unique $z \in X(\infty)$ with $\Gamma_W = \Gamma_z := \{\gamma \in \Gamma | \gamma(z) = z\}$. W is bounded in the first and unbounded in the second case.*

(2) $W = \{d_{\Gamma_W} < r\}$.

(3) *If W_1 and W_2 are distinct components of $\{d_\Gamma < r\}$, then Γ_{W_1} and Γ_{W_2} intersect only in the identity.*

Proof. (1) Using Lemma 1 it is easy to prove (see [3, Lemma 3.1c]): if $x, y \in W$, $d_\alpha(x)$, $d_\beta(y) < r$ for nontrivial $\alpha, \beta \in \Gamma$, then $\text{Fix}(\alpha) = \text{Fix}(\beta)$. Thus for $A := \{\gamma \in \Gamma - \text{id} |$ there exists $x \in W$ with $d_\gamma(x) < r\}$, the classification of isometries yields: either all $\alpha \in A$ are hyperbolic with a unique common axis c, or all $\alpha \in A$ are parabolic with a unique common fixed point z. If $\gamma \in \Gamma_W - \text{id}$, $x \in W$, then $\gamma x \in W$ and there is an $\alpha \in A$ with $r > d_\alpha(\gamma x) = d_{\gamma^{-1}\alpha\gamma}(x)$. Hence $\gamma^{-1}\alpha\gamma \in A$.

If $\alpha \in A$ is hyperbolic with axis c, then $\gamma^{-1}c$ is the axis of $\gamma^{-1}\alpha\gamma \in A$ and hence $\gamma^{-1}c = c$. Therefore γ leaves c invariant and γ is hyperbolic with axis c.

If $\alpha \in A$ is parabolic with fixed point $z \in X(\infty)$, the same argument shows that $\gamma z = z$. γ is also parabolic by [4, Proposition 6.8].

Hence we have proved that the elements of Γ_W are either all hyperbolic with axis c ($\Gamma_W \subset \Gamma_c$) or all parabolic with fixed point z ($\Gamma_W \subset \Gamma_z$). In the first case c is contained in W and hence $\Gamma_c \subset \Gamma_W$. The discreteness of Γ then implies that Γ_c is infinite cyclic. In the second case let g: $[0, \infty) \to X$ be a geodesic ray with $g(0) \in W$ and $g(\infty) = z$. Because $K \leqslant -a^2 < 0$, $d_\gamma(g(t)) \to 0$ for all $\gamma \in \Gamma_z$ as t goes to ∞. Hence g is contained in W and, by Lemma 2(2), $\Gamma_z \subset \Gamma_W$.

If U is bounded, then Inj Rad assumes a minimum in $p \in U$. Let $x \in W$ with $\pi(x) = p$ and $d_\Gamma(x) = d_\gamma(x)$ for some $\gamma \in \Gamma_W$. If γ is parabolic, then there is a nearby y with $d_\gamma(y) < d_\Gamma(x)$, hence $\text{Inj Rad}(\pi(y)) < \text{Inj Rad}(\pi(x))$, a contradiction.

On the other hand let Γ_W be an infinite cyclic group of isometries with common axis c. Then the curvature assumption implies that $d_{\Gamma_W}(y) > r$ for all $y \in X$ with $d(y, c) > R$ for a suitable R. Therefore $d(q, \pi(c)) < R$ for all $q \in U$ and U is bounded.

(2) By Lemma 2(2), $W \subset \{d_{\Gamma_W} < r\}$. Now it is easy to see that for

a geodesic c or a point $z \in X(\infty)$, the sets $\{d_{\Gamma_c} < r\}$ and $\{d_{\Gamma_z} < r\}$ are connected. Therefore $W = \{d_{\Gamma_W} < r\}$.

(3) Let $\gamma \in \Gamma_{W_1} \cap \Gamma_{W_2}$ be a nontrivial element. If γ is hyperbolic with axis c, then $\Gamma_{W_1} = \Gamma_c = \Gamma_{W_2}$ and if γ is parabolic with fixed point z, then $\Gamma_{W_1} = \Gamma_z = \Gamma_{W_2}$. By (2), $\Gamma_{W_1} = \Gamma_{W_2}$ implies $W_1 = W_2$.

Lemma 4. *Let $V = X/\Gamma$ satisfy $-1 \leqslant K \leqslant -a^2$, $0 < r \leqslant \mu$. Let $U \subset V$ be an unbounded component of $\{\text{Inj Rad} < r/2\}$, and let W be a component of $\pi^{-1}(U)$ with Γ_W as above. Then the volume of U is finite if and only if Γ_W is an almost nilpotent group of rank $n - 1$.*

Remark. The rank of an almost nilpotent group is the rank of a nilpotent subgroup of finite index. For the definition of rank and other facts about nilpotent groups compare Chapter II of [9].

Proof. We divide the proof into three steps:

(a) If $\text{vol}(U) < \infty$, then Γ_z is almost nilpotent and operates with compact quotient on the horospheres $HS(x, z)$:

The proof of Lemma 3.1g of [3] shows that Γ_z operates with compact quotient on the horospheres and therefore Γ_z is finitely generated. Let $\gamma_1, \cdots, \gamma_m$ be a system of generators. $K \leqslant -a^2$ implies that there is a point $g(t_0)$ with $d_{\gamma_i}(g(t_0)) \leqslant r$. By the Margulis Lemma, Γ_z is almost nilpotent with nilpotent subgroup N of finite index. Then N also operates with compact quotient on the horospheres.

(b) rank $N = n - 1$: N is nilpotent, finitely generated and without torsion. By a theorem of Malcev N is isomorphic to a lattice in a simply connected nilpotent Lie group A with $\dim A = \text{rank}\, N =: m$ [9, Theorem II.2.18]. Because every lattice in a nilpotent Lie group has a compact quotient and A is homeomorphic to $\mathbf{R}^m$, N operates with compact quotient on $\mathbf{R}^m$. Because N operates also on a horosphere, hence on $\mathbf{R}^{n-1}$ with compact quotient, we conclude $m = n - 1$ by comparing the homology groups of these $K(\pi, 1)$-manifolds.

(c) If Γ_z contains a nilpotent subgroup N of finite index and rank $n - 1$, then N and hence Γ_z operate with compact quotient on the horospheres $HS(x, z)$ by inversion of the arguments of b. Because $d_{\Gamma_z}(g(t)) \to \infty$ as $t \to -\infty$, we conclude easily that there is a horoball $HB(x_0, z)$ with $W \subset HB(x_0, z)$, and thus $\text{vol}(U) \leqslant \text{vol}(HB(x_0, z)/\Gamma_z)$. We prove that the latter is finite: $HB(x_0, z)/\Gamma_z$ is diffeomorphic to $B \times (0, \infty)$, where the projection on $(0, \infty)$ is a riemannian submersion and $B_t = B \times \{t\}$ is the quotient of a horosphere. Because of the curvature condition, we control the stable Jacobi-fields (see [7]). This implies $\text{vol}(B_t) \leqslant ke^{-at}$ with a constant k. Hence

$$\text{vol}(HB(x_0, z)/\Gamma_z) \leqslant \int_0^\infty ke^{-at}\, dt < \infty.$$

Lemma 5. *Let* $V = X/\Gamma$ *satisfy* $-1 \leqslant K \leqslant -a^2$, $0 < r_1 \leqslant r_2 \leqslant \mu$. *Let* U_i *be components of* $\{\text{Inj Rad} < r_i/2\}$ *with* $U_1 \subset U_2$ *and let* W_i *be components of* $\pi^{-1}(U_i)$ *with* $W_1 \subset W_2$. *Then*:

(1) $\Gamma_{W_1} = \Gamma_{W_2}$.

(2) U_1 *is the only component of* $\{\text{Inj Rad} < r_1/2\}$ *which is contained in* U_2.

Proof. (1) $W_1 \subset W_2$ immediately implies $\Gamma_{W_1} \subset \Gamma_{W_2}$. Using Lemma 3(1) we conclude that either $\Gamma_{W_1} = \Gamma_c = \Gamma_{W_2}$ or $\Gamma_{W_1} = \Gamma_z = \Gamma_{W_2}$ for a geodesic c or a point $z \in X(\infty)$.

(2) is a consequence of (1) and Lemma 3(3).

Now we are able to prove our proposition.

Proof. (1) Because E has finite volume, there is a compact set $K \subset V$ with $\text{vol}(E(K)) < \infty$ and $\text{Inj Rad}_{|E(K)} < r/2$. Let $U_r(E)$ be the component of $\{\text{Inj Rad} < r/2\}$ which contains $E(K)$. If U' is another component of $\{\text{Inj Rad} < r/2\}$ which is a neighborhood of E, then $U' \cap U_r(E) \neq \varnothing$ and hence $U' = U_r(E)$.

We now prove that $\text{vol}(U_r(E)) < \infty$. Let K be as above. Then there is an r' with $0 < r' < r$ and $\text{Inj Rad}_{|K} > r'/2$. By construction $U_{r'}(E) \subset E(K) \subset U_r(E)$ and hence $\text{vol}(U_{r'}(E)) < \infty$. Let $W_{r'} \subset W_r$ be components of $\pi^{-1}(U_{r'}(E))$ and $\pi^{-1}(U_r(E))$. By Lemma 5, $\Gamma_{W_{r'}} = \Gamma_{W_r}$ and, by Lemma 4, the finiteness of the volume of $U_{r'}(E)$ implies $\text{vol}\,(U_r(E)) < \infty$.

If E, E^* are different ends of finite volume, there is a compact set $K \subset V$ with $E(K) \neq E^*(K)$ and hence $E(K)$ and $E^*(K)$ are disjoint. As above there is an r', $0 < r' < r$, with $U_{r'}(E) \subset E(K)$ and $U_{r'}(E^*) \subset E^*(K)$. By Lemma 5(2), $U_r(E)$ and $U_r(E^*)$ are distinct, hence disjoint.

(2) For an end E of finite volume let $U_r(E)$, W_r be as in (1). By Lemma 4, Γ_{W_r} is almost nilpotent of rank $n - 1$ and $\Gamma_{W_r} = \Gamma_z$ for some $z \in X(\infty)$. Γ_z is maximal almost nilpotent: if $\Gamma' \supset \Gamma_z$ is almost nilpotent, then, by Lemma 1, all $\gamma \in \Gamma'$ have a common fixed point in $X(\infty)$ and hence $\Gamma' \subset \Gamma_z$.

If $W_r' = \gamma W_r$ is another component of $\pi^{-1}(U_r(E))$, then $\Gamma_{W_r'} = \gamma \Gamma_{W_r} \gamma^{-1}$. Thus we assign to every end of finite volume a conjugation class of the maximal almost nilpotent subgroups of rank $n - 1$. We prove that this map is bijective:

(a) Different ends E and E^* have disjoint $U_r(E)$ and $U_r(E^*)$. If W_r and W_r^* are components of $\pi^{-1}(U_r(E))$ and $\pi^{-1}(U_r(E^*))$, then there is no $\gamma \in \Gamma$ with $\gamma W_r = W_r^*$. Therefore Γ_{W_r} and $\Gamma_{W_r^*}$ define different conjugation classes by Lemma 3(3).

(b) On the other hand let $\Delta \subset \Gamma$ be a maximal almost nilpotent subgroup of rank $n - 1 \geqslant 2$. Then Δ is not infinite cyclic and hence, by Lemma 1, Δ is a group of parabolic isometries with a common fixed point $z \in X(\infty)$. Thus

$\Delta \subset \Gamma_z$. By the arguments of Lemma 4, Δ operates with compact quotient on the horospheres $HS(x, z)$ and $\mathrm{vol}(HB(x, z)) < \infty$. Then Γ_z also operates with compact quotient on the horospheres and the argument of Lemma 4(a) proves that Γ_z is almost nilpotent. Hence $\Delta = \Gamma_z$ by maximality. Part (c) of that lemma shows that for suitable $x \in X$ the volume of $HB(x, z)/\Gamma_z$ is arbitrarily small, and hence also the injectivity radius on $\pi(HB(x, z))$ is small. For $0 < r \leqslant \mu$ let U_r be the component of $\{\mathrm{Inj\,Rad} < r/2\}$ which contains $\pi(HB(x, z))$ for suitable x. Let W_r be the component of $\pi^{-1}(U_r)$ containing $HB(x, r)$. Then $\Gamma_{W_r} = \Gamma_z$ and, by Lemma 4, $\mathrm{vol}(U_r) < \infty$. By definition $U_{r'} \subset U_r$ for $0 < r' \leqslant r \leqslant \mu$, and therefore one checks that the following function E defines an end of finite volume:

For compact $K \subset V$ let $E(K)$ be the component of $V - K$ which contains U_r, where r is chosen such that $\mathrm{Inj\,Rad}_{|K} > r/2$. By construction the conjugation class assigned by E is the class of Δ.

(3) The proof of (2) shows that an end E of finite volume has a neighborhood of the form $E(B) = HB(x, z)/\Gamma_z$ which is diffeomorphic to $B \times (0, \infty)$ with $B = HS(x, z)/\Gamma_z$. These neighborhoods are contained in $U_r(E)$, hence different ends have disjoint neighborhoods.

Remark. Part (1) implies the theorem, due to Heintze [6, p. 33], that a complete manifold V with $\mathrm{vol}(V) < \infty$ and $-1 \leqslant K \leqslant -a^2$ has only finitely many ends: the ends have disjoint neighborhoods $U_r(E)$. In $U_r(E)$ we will find an injectively imbedded $r/4$-ball, thus $\mathrm{vol}(U_r(E))$ is larger than a constant depending on r and n.

4. Finite volume and fundamental group

Let V be a complete Riemannian manifold of dimension $n \geqslant 3$, which satisfies $-1 \leqslant K \leqslant -a^2$. Using the result of Heintze remarked above, we see that the volume of V is finite if and only if V has only finitely many ends and every end has finite volume. This is equivalent to the conditions:

(1) V has only finitely many ends of finite volume, and

(2) V has no further ends.

According to the proposition, condition (1) is equivalent to the finiteness of the conjugation classes of the maximal almost nilpotent subgroups of rank $n - 1$ in $\pi_1(V)$.

We will prove that (2) also is equivalent to a condition on the fundamental group. Therefore let us assume that V has finitely many ends $E_0, \cdots, E_k$ of finite volume. By our proposition the ends E_i have disjoint neighborhoods diffeomorphic to $B_i \times (0, \infty)$. We identify $B_i \times (0, \infty)$ with subsets of V. Then

$M := V - \bigcup_{i=0}^{k}(B_i \times (0, \infty))$ is a manifold with $k + 1$ boundary components $B_0, \cdots, B_k$. It is easily checked that V has no further ends if and only if M is compact. Now we define a manifold W without boundary by glueing two copies M^1, M^2 of M canonically along their common boundary. Clearly M is compact if and only if W is compact. Therefore condition (2) is equivalent to:

(2*) W is compact.

To prove that (2*) is a condition on $\pi_1(V)$, we show:

(a) The fundamental group of W can be computed purely algebraically from $\pi_1(V)$.

(b) W is a $K(\pi, 1)$-manifold, hence W is compact if and only if $H_n(\pi_1(W), \mathbf{Z}_2) = \mathbf{Z}_2$.

Proof of (a). By the theorem of Zaidenman ([12], compare Steenrod's reviews, Part I, Amer. Math. Soc., 1968, p. 52) we can compute the fundamental group of W in the following way: we choose points $p_i \in B_i$, and by arcs from p_i to p_0 we define imbeddings ϕ_i^j: $\pi_1(B_i, p_i) \to \pi_1(M^j, p_0)$. Let F_k be the free group with k generators $\gamma_1, \cdots, \gamma_k$. Then $\pi_1(W)$ is isomorphic to the quotient of the free product $\pi_1(M^1, p_0) * \pi_1(M^2, p_0) * F_k$ divided by the normal subgroup generated by the elements $\phi_0^1(\alpha_0)\phi_0^2(\alpha_0)^{-1}$, $\phi_i^1(\alpha_i)\gamma_i\phi_i^2(\alpha_i)^{-1}\gamma_i^{-1}$, $1 \leqslant i \leqslant k$, where $\alpha_i \in \pi_1(B_i, p_i)$. This computation is purely algebraic, because by the construction of our proposition $\phi_i^j(\pi_1(B_i, p_i))$ is a maximal system of pairwise nonconjugate maximal almost nilpotent subgroups of rank $n - 1$: $\pi_1(W)$ is an amalgamated product with itself on the maximal almost nilpotent subgroups of rank $n - 1$.

Proof of (b). To prove that W is a $K(\pi, 1)$-manifold, we note:

(i) $B_i \subset M$ is, as a quotient of a horosphere, a $K(\pi, 1)$-manifold.

(ii) By construction, the inclusion $B_i \subset M$ induces an injection $\pi_1(B_i) \to \pi_1(M)$.

(iii) It is easy to see that the inclusions $M^1, M^2 \subset W$ induce injections $\pi_1(M^j) \to \pi_1(W)$.

Now W is a $K(\pi, 1)$-manifold by the following lemma, which is an easy consequence of Whitehead's theorem [1, p. 49].

Lemma 6. *Let W be a CW-complex which is the union of two connected subcomplexes M^1 and M^2 whose intersection consists of $k + 1$ components $B_0, \cdots, B_k$. Let M^1, M^2, $B_0, \cdots, B_k$ be $K(\pi, 1)$-spaces and the maps $\pi_1(B_i) \to \pi_1(W)$, $\pi_1(M^j) \to \pi_1(W)$, induced by the inclusions, be injective. Then W is a $K(\pi, 1)$-manifold.*

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