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CLOSED GEODESICS ON MANIFOLDS WITH INFINITE ABELIAN FUNDAMENTAL GROUP

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While many results are known about closed geodesics on compact manifolds with finite fundamental group, there are few for manifolds with π_1 infinite ([2], [6]), with the exception of manifolds admitting metrics of nonpositive curvature. We prove that for every compact Riemannian manifold M with fundamental group $\pi_1 = \mathbf{Z}$ the number of geometrically distinct closed geodesics of length $\leqslant l$ grows at least like the prime numbers. In particular, there are infinitely many. Geodesics are called geometrically distinct if their images on M are different.

Manifolds with $\pi_1 = \mathbf{Z}$ can be found, e.g., by taking a bundle over S^1 with simply connected fiber or by attaching a handle $I \times S^{n-1}$ to a simply connected manifold of dimension $n \geqslant 3$. There are no examples in dimension 2.

Every nontrivial free homotopy class of closed curves on a compact manifold contains a curve of minimal length which is a closed geodesic. If $\pi_1 = \mathbf{Z}$ there are infinitely many free homotopy classes but the corresponding minimal length geodesics can be geometrically indistinct. If $\pi_1(M)$ is abelian and Rank $\pi_1 \otimes \mathbf{Q} \geqslant 2$, then the fundamental group is enough to ensure the existence of infinitely many closed geodesics: if $s, t \in \pi_1$ are independent and of infinite order, minimal length curves in the classes st^m will be geometrically distinct. While those in the classes t^m may not be distinct, the following generalization of our theorem is true:

Suppose $\pi_1(M)$ is abelian and $t \in \pi_1$ has infinite order. Then the number of geometrically distinct closed geodesics of length $\leqslant l$ in the classes t^m, $m \geqslant 1$, grows at least like the prime numbers.

The case where π_1 is infinite abelian but $\pi_1 \neq \mathbf{Z}$ is easier and is already known to some experts. We will include a proof for this case at the end of the paper.

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A covering space argument shows that the above conclusion also holds if π_1 has an infinite abelian subgroup of finite index. However, all our arguments seem to break down when π_1 is more nonabelian.

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Theorem. *Let M be a compact Riemannian manifold of dimension $\geqslant 2$ with $\pi_1(M) = \mathbf{Z}$. If $n(l)$ is the number of geometrically distinct closed geodesics of length $\leqslant l$, then*

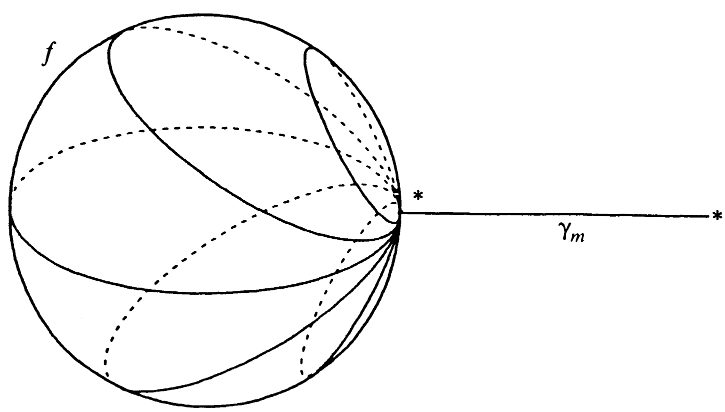
$$\liminf_{l \to \infty} n(l) \frac{\log l}{l} > 0.$$

Proof. Ω and Λ will denote the loop space and the free loop space of M. There are several ways to make Λ accessible to Lusternik-Schnirelmann theory; we can take e.g. H^1-curves [8] or piecewise geodesics [9]. The mth iterate of $\gamma \in \Lambda$ will be denoted by γ^m: $\gamma^m(\theta) = \gamma(m\theta)$. Let t be a generator of $\pi_1(M)$ and Λ_m the t^m-component of Λ. Since M is not homotopy equivalent to a circle, $\pi_n(M) \neq 0$ for some (minimal) $n > 1$.

Lemmas 1 and 2 below give a $k \in \mathbf{N}$ such that for all $m \in N$ there exist nontrivial classes $\alpha_m \in \pi_{n-1}(\Lambda_{mk}, \gamma_m)$ which are in the image of $\pi_{n-1}(\Omega, \gamma_m)$. Here we choose as base points closed geodesics $\gamma_m \in \Lambda_{mk}$ with length

$$\kappa_m = \inf\{\text{length}(\gamma) | \gamma \in \Lambda_{mk}\}.$$

Note that $\kappa_m \leqslant m\kappa_1$. A representative v_m for α_m is constructed from a representative f: $S^n \to M$ of a (fixed) class α in $\pi_n(M)$ as follows (for the precise definition see Lemma 2): The curves in the image of v_m are the product of γ_m with curves which are the image under f of curves which "wrap around" S^n.



Let

$$\tau_m = \inf_{f \in \alpha_m} \sup\{\text{length}(\gamma) | \gamma \in \operatorname{Im} f\}.$$

By Lusternik-Schnirelmann theory there exists a closed geodesic $\delta_m \in \Lambda_{mk}$ with length τ_m. By the construction of α_m we have

$$(*) \qquad \kappa_m \leqslant \tau_m \leqslant \kappa_m + L$$

for a constant L independent of m.

If $\kappa_m = \tau_m$ there exist infinitely many geometrically distinct closed geodesics of length κ_m. Otherwise, α_m could be represented in an orbit of the natural S^1-action on Λ which would imply either $\alpha_m = 0$ or $n = 2$ and $e_* \alpha_m \neq 0$ where $e: \Lambda \to M$ is the evaluation map at the basepoint. But $e_* \alpha_m = 0$ since α_m is in the image of $\pi_{n-1}(\Omega, \gamma_m)$.

Hence we assume $\tau_m > \kappa_m$. We may also assume that there exists $\varepsilon > 0$ such that Λ_k contains no closed geodesics with length in $(\kappa_1, \kappa_1 + \varepsilon]$. Let p be a prime number with $p\varepsilon > L$. The multiplicity of $\gamma \in \Lambda$, $\gamma \neq$ const, is the largest integer j so that $\gamma = \bar{\gamma}^j$ for some $\bar{\gamma} \in \Lambda$. We finish the proof by showing that γ_p and δ_p cannot both have multiplicity $> k$. Otherwise $\gamma_p = \bar{\gamma}^p$ and $\delta_p = \bar{\delta}^p$ for some $\bar{\gamma}, \bar{\delta} \in \Lambda_k$. Since $\tau_p > \kappa_p$ and $\kappa_p \leqslant p\kappa_1$, we would have length$(\bar{\gamma}) = \kappa_1$ and length$(\bar{\delta}) > \kappa_1 + \varepsilon$. Hence

$$\tau_p > \kappa_p + p\varepsilon > \kappa_p + L,$$

contradicting $(*)$. Thus if $p(n)$ is the nth prime, at least n/k of the geodesics among the γ_p and δ_p with $p \leqslant p(n)$ are geometrically distinct.

Lemma 1. *Let t be a generator of $\pi_1(M) = \mathbf{Z}$. For some $k \in \mathbf{N}$ there exists $\alpha \in \pi_n(M)$, $n \geqslant 2$, so that $\alpha \notin (1 - t^k)\pi_n$.*

It follows that $\alpha \notin (1 - t^{mk})\pi_n = (1 - t^k)(1 + t^k + \cdots + t^{k(m-1)})\pi_n$ for all $m \in \mathbf{N}$.

Proof. First note that π_n is a finitely generated module over the group ring $\mathbf{Z}[\pi_1]$ if $\pi_j(M) = 0$ for $1 < j < n$. To see this triangulate M so that the simplices are evenly covered by the universal covering $\tilde{M} \to M$. The pullback gives a triangulation of $\tilde{M}$ whose complex is finitely generated over $\mathbf{Z}[\pi_1]$. Since $\mathbf{Z}[\pi_1]$ is Noetherian, $H_n(\tilde{M})$ is a finitely generated $\mathbf{Z}[\pi_1]$-module. Note that the Hurewicz isomorphism $\pi_n(\tilde{M}) \to H_n(\tilde{M})$ commutes with the natural π_1-actions. Thus $\pi_n(M) \cong \pi_n(\tilde{M})$ is a finitely generated $\mathbf{Z}[\pi_1]$-module.

Now let $\mathfrak{A} \subset \mathbf{Z}[\pi_1] = \mathbf{Z}[t, t^{-1}]$ be the annihilator of π_n, and let $\mathfrak{M}$ be a maximal ideal containing $\mathfrak{A}$. Then $\mathbf{Z}[\pi_1]/\mathfrak{M}$ is a finite field: If $\mathfrak{M} \cap \mathbf{Z} = (0)$ then $(1) \neq \mathbf{Q}\mathfrak{M} = (f)$ since $\mathbf{Q}[t]$ is a PID. If we choose f in $\mathbf{Z}[t]$ and primitive then $\mathfrak{M} \subseteq (f)$, a contradiction since $(f) \subsetneq (f, p) \neq (1)$ if $p \in \mathbf{Z}$ is prime to the coefficients of f. Hence $\mathfrak{M} \cap \mathbf{Z} = (p)$ and $\mathbf{Z}[\pi_1]/\mathfrak{M}$ is finite. For a less elementary proof using only the assumption that π_1 is abelian and finitely generated see [5, p. 353]. Since t is a unit in $\mathbf{Z}[\pi_1]$, $1 - t^k \in \mathfrak{M}$ for some $k \in \mathbf{N}$.

On the other hand $(1 - t^k)\pi_n = \pi_n$ implies that $1 - t^k$ is a unit in $\mathbf{Z}[\pi_1]/\mathfrak{A}$, see [1, Corollary 2.5, p. 21]. Hence $1 - t^k$ is a unit in $\mathbf{Z}[\pi_1]/\mathfrak{M}$, contradicting $1 - t^k \in \mathfrak{M}$. This proves Lemma 1.

Lemma 2. *Suppose $t^k \in \pi_1(M)$ and f: $(S^n, *) \to (M, *)$ satisfy $[f] \notin (1 - t^k)\pi_n$. Let γ: $(S^1, *) \to (M, *)$ represent t^k, and let σ represent a generator of $\pi_{n-1}(\Omega S^n, *)$. Then the composition v_γ: $(S^{n-1}, *) \to (\Lambda M, \gamma)$ given by*

$$v_\gamma\colon (S^{n-1}, *) \xrightarrow{\sigma} (\Omega S^n, *) \xrightarrow{\Omega f} (\Omega M, *) \xrightarrow{\gamma_*} (\Omega M, \gamma) \to (\Lambda M, \gamma)$$

represents a nontrivial class in $\pi_{n-1}(\Lambda M, \gamma)$.

The curves δ in the image of v_γ have

$$\operatorname{length}(\delta) \leqslant \operatorname{length}(\gamma) + \max\{\operatorname{length}(\delta') | \delta' \in \operatorname{Im}(\Omega f \circ \sigma)\}.$$

Proof. See [10, especially pp. 474–476], as a general reference. Lemma 2 essentially follows from [7, Lemma 1.5], except that in our case $A = t^k$: $\tilde{M} \to \tilde{M}$ has no fixed point and we need to take the basepoint $\gamma \in \Lambda M$. Lemma 2 can be proved directly as follows: Suppose

$$h\colon (S^{n-1} \times I, \{*\} \times I) \to (\Lambda, \gamma)$$

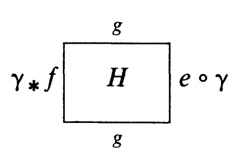
is a null homotopy of v_γ, i.e. $h_0 = v_\gamma$ and h_1: $S^{n-1} \to \{\gamma\}$. Let c: $\Lambda \times I \to \Lambda$ lift the natural S^1-action and let e: $\Lambda \to M$ be the evaluation map. The composition

$$H\colon S^{n-1} \times I \times I \xrightarrow{h \times \mathrm{id}} \Lambda \times I \xrightarrow{c} \Lambda \xrightarrow{e} M$$

gives a homotopy between

$$H_{|S^{n-1} \times \{0\} \times I} \quad \text{and} \quad H_{|S^{n-1} \times I \times \{0\} \cup S^{n-1} \times \{1\} \times I \cup S^{n-1} \times I \times \{1\}}.$$

The former represents the class $[\gamma_* f] = t^k[f]$, the latter that of $(1 - t^k)[g]$ where $g = H_{|S^{n-1} \times I \times \{0\}}$. (Note that all these restrictions of H factor through S^n.)



Thus $[f] = (1 - t^k)t^{-k}[g]$, contradicting our hypothesis on f and t^k. This proves Lemma 2.

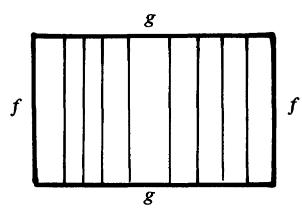
We add a proof of the more general statement mentioned in the introduction. Actually the preceding proof carries over literally unless M is a $K(\pi_1, 1)$ (so that M is covered by a homotopy torus). However the following proof works whenever $\pi_1(M)$ is infinite abelian and $\pi_1 \neq \mathbf{Z}$: We choose $t \in \pi_1$ of infinite order and $s \in \pi_1$ so that s and t are independent. We denote by Λ_m the t^m-component of Λ. Our statement follows from the arguments given in the proof of the theorem if we can find homotopy classes $\alpha_m \in \pi_1(\Lambda_m)$ such that:

(a) $e_*(\alpha_m) = s$,

(b) $\tau_m = \inf_{v \in \alpha_m} \sup\{\text{length}(\delta) | \delta \in \operatorname{Im} v\}$ satisfies $\tau_m \leqslant m\kappa_1 + L$, where L is a constant independent of m and $\kappa_1 = \inf\{\text{length}(\delta) | \delta \in \Lambda_1\}$.

In the proof of the theorem, (a) ensures that α_m cannot be represented in a neighborhood of a critical orbit. Note that the proof requires only (b) rather than the stronger property (*).

The existence of $\alpha_1 \in \pi_1(\Lambda_1)$ with (a) follows from the fact that s and t commute: if f and g represent t and s, a homotopy between $f \cdot g$ and $g \cdot f$ gives a circle of curves in Λ_1 whose basepoints trace out g.



Choose v: $S^1 \to \Lambda_1$ representing α_1 with $\kappa_1 = \text{length}(v(0))$. We let α_m be the class of v_m: $S^1 \to \Lambda_m$, where $v_m(\theta) = (v(\theta))^m$. To obtain (b) apply the homotopy defined in [**4**, Theorem 1] (for a figure see [3, p. 87]) to v^m. This yields $v_m \in \alpha_m$ and $L \in \mathbf{R}$ so that

$$\sup\{\text{length}(\delta) | \delta \in \operatorname{Im} v_m\} \leqslant m\kappa_1 + L.$$

References

[1] M. F. Atiyah & I. G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, MA, 1969.

[2] W. Ballman, G. Thorbergsson & W. Ziller, *Closed geodesics and the fundamental group*, Duke Math. J. **48** (1981) 585–588.

[3] V. Bangert, *Closed geodesics on complete surfaces*, Math. Ann. **251** (1981) 83–96.

[4] V. Bangert & W. Klingenberg, *Homology generated by iterated closed geodesics*, Topology, **22** (1983) 379–388.

[5] N. Bourbaki, *Commutative algebra*, Hermann, Paris; Addison-Wesley, Reading, MA, 1972.
[6] M. Gromov, *Three remarks on geodesic dynamics and the fundamental group*, preprint, State University of New York, Stony Brook, New York.
[7] K. Grove, *Isometry-invariant geodesics*, Topology **13** (1974) 281–292.
[8] W. Klingenberg, *Lectures on closed geodesics*, Springer, Berlin, 1978.
[9] J. Milnor, *Morse theory*, Ann. of Math. Studies 51, Princeton University Press, Princeton, NJ, 1969.
[10] G. W. Whitehead, *Elements of homotopy theory*, Springer, Berlin, 1978.

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