J. DIFFERENTIAL GEOMETRY
19 (1984) 1–29

HILBERT STABILITY OF RANK-TWO BUNDLES ON CURVES

DAVID GIESEKER & IAN MORRISON

1. Let k be an algebraically closed field, and let d and g be two integers with $g \geqslant 2$ and $d \geqslant 1000g(g-1)$. Let $n = d + 2 - 2g$, and let W be a vector space of dimension n. G will denote the grassmannian of all codimension-two subspaces of W, and $\mathfrak{S}$ will denote the universal rank-two bundle on G. In this paper, a curve will be a connected one-dimensional projective scheme. Let C be a curve on G, i.e., C is a subscheme of G which is a curve, and consider $E = \mathfrak{S}_C = \mathfrak{S}_{|C}$. Let $P_C(m) = \chi((\det E)^{\otimes m})$ be the Hilbert polynomial of C where $\det E = \wedge^2 E$. We let $S_{g,d}$ be the set of all curves C on G with $P_C(m) = dm + 2 - 2g$. Thus $S_{g,d}$ is the set of all curves of genus g and degree d on G.

Now W is identified with $H^0(G, \mathfrak{S})$, so given $C \in S_{g,d}$, there is a natural map

$$\varphi_1 : W \to H^0(C, E).$$

We will identify W with $H^0(C, E)$ if φ_1 is an isomorphism. Thus we obtain a map

$$\varphi_2 : \wedge^2 W \to H^0\left(C, \wedge^2 E\right).$$

So for any positive integer m, we obtain a map

$$\varphi_3 : S^m\left(\wedge^2 W\right) \to H^0\left(C, (\det E)^{\otimes m}\right).$$

We may and do choose m so that φ_3 is onto, so that $h^0(C, (\det E)^{\otimes m}) = P_C(m)$ for any $C \in S_{g,d}$. Thus we finally obtain a map

$$\varphi_C^m : \bigwedge^{P_C(m)} S^m\left(\wedge^2 W\right) \to \bigwedge^{P_C(m)} H^0\left(C, (\det E)^{\otimes m}\right) \cong k.$$

We say $C \subseteq G$ is m-Hilbert stable (resp., m-Hilbert semistable) if φ_C^m is properly stable (resp., semistable) under the induced action of $SL(W)$ in the

Received January 10, 1983, and, in revised form, July 7, 1983. Research partially supported by NSF Grant MCS79-03171.

terminology of Mumford, i.e., φ_C^m has closed orbit and finite stabilizer (resp., 0 is not in the closure of the orbit of φ_C^m). We say C is Hilbert stable if it is m-Hilbert stable for $m \gg 0$. We say a pair (C, E) consisting of a curve C and vector bundle E of rank two is m-Hilbert stable if (C, E) occurs as an m-Hilbert stable curve in $S_{g,d}$.

Now if E is a rank-two bundle on a smooth curve C, and L is a subbundle of E of maximal degree, we define $l_E = \deg E - 2 \deg L$. Recall that E is stable if $l_E > 0$ and semistable if $l_E \geqslant 0$.

A curve C is nodal if C is reduced and has only nodes as singularities. Let ω_C denote the dualizing sheaf of such a curve. Recall C is stable (resp., semistable) if ω_C has positive degree (resp., nonnegative degree) on each component of C [5]. For each semistable curve, the sections of $\omega_C^{\otimes 3}$ define a map to P^{5g-5}, and the image of C is a stable curve denoted C_s. C_s is obtained from C by collapsing all components on which ω_C is trivial. These components are smooth rational curves meeting the rest of C in exactly two points. A semistable subcurve C' of C is a subcurve which is the inverse image of a node of C_s.

We fix g for the rest of the paper.

Theorem 1.1. *There is a D so that for each $d \geqslant D$, there is an M depending on d so that if $m \geqslant M$, and C is a smooth curve in $S_{g,d}$ with $W = H^0(C, E)$, then C is m-Hilbert stable (resp., semistable) if and only if $\mathcal{E}_C$ is stable (resp., semistable).*

Theorem 1.2. *For g and d given, there is an M so that if $m \geqslant M$ and $C \in S_{g,d}$ is m-Hilbert semistable, then C is semistable as a curve and $W = H^0(C, \mathcal{E}_C)$.*

The proof of Theorem 1.1 is given in §§2–5 and that of Theorem 1.2 in §§6–9.

Now in §10 we will suppose $C \in S_{g,d}$ is m-Hilbert stable for m sufficiently large, and study $E = \mathcal{E}_C$. First we will show that if Q is a quotient line bundle of E, then

$$\deg E \leqslant 2 \deg Q. \tag{1.3.1}$$

Now let C' be a semistable subcurve of C. E is said to be acceptable on C' if either

(1.3.2.1) C' has one component and so is isomorphic to $\mathbf{P}^1$, and $E_{C'}$ is $\mathcal{O} \oplus \mathcal{O}(1)$ or $\mathcal{O}(1) \oplus \mathcal{O}(1)$ or

(1.3.2.2) C' has two components C_1 and C_2, and E_{C_i} is isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$. Further, $E_{C'}$ has no quotient isomorphic to $\mathcal{O}_{C'}$.

We will show

(1.3.3) E is acceptable on each semistable subcurve of C.

Finally, let d be odd and suppose C_s is an irreducible curve with a node. Let $\tilde{C}$ be the normalization of C_s. Then $\tilde{C}$ maps to C as a component of C if $C \neq C_s$. Thus we may consider $\tilde{E}$, the pullback of E to $\tilde{C}$. Then we will show

(1.3.4) If $C = C_s$ and d is odd, then $l_{\tilde{E}} \geqslant -1$. If $C \neq C_s$ then $\tilde{E}$ is semistable.

We wish to thank Ed Griffin for pointing out an error in an earlier version of this paper.

2. Let C be a curve in $S_{g,d}$. We wish to apply the Hilbert-Mumford numerical criterion to φ_C^m. First, a weighted basis (X_i, r_i) of W is an ordered basis of W together with rational numbers r_i with $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_n$. If the r_i are integers, and their sum is zero, we call B standard. A standard weighted basis determines a one-parameter subgroup of $SL(W)$ via

$$X_i^{\lambda(\alpha)} = \alpha^{r_i} X_i.$$

Every $1 - PS$ occurs in this way. A weighted basis B of W gives rise to weighted bases on the representations of $SL(W)$ discussed above, as shown in the table.

REPRESENTATION	BASIS ELEMENT	WEIGHT
$\wedge^2 W$	$Y_I = X_{i_1} \wedge X_{i_2}$	$r_I = r_{i_1} + r_{i_2}$
$S^m \wedge^2 W$	$M_{\mathfrak{I}} = Y_{I_1} \cdots Y_{I_m}$	$r_{\mathfrak{I}} = \sum_{k=1}^{m} r_{I_k}$
$\wedge^{P(m)} S^m \wedge^2 W$	$M_{\mathfrak{I}_1} \wedge \cdots \wedge M_{\mathfrak{I}_{P(m)}}$	$\sum_{k=1}^{P(m)} r_{\mathfrak{I}_k}$

If B is standard, so is each of these bases, and each diagonalizes the action of λ_B on the corresponding representation. The coordinate corresponding to $M_1 \wedge \cdots \wedge M_{P(m)}$ does not vanish at φ_C^m if and only if the images under φ_C^m of $M_1, \cdots, M_{P(m)}$ in $H^0(C, \wedge^2 E^{\otimes m})$ form a basis there. We will call such a basis a B-base of $H^0(C, \wedge^2 E^{\otimes m})$, and denote by $w_B(m)$ or $w_B(m, C)$ the minimum weight of such a basis. Each B determines a weighted filtration $F_B = \{(V_i, r_i)\}$ on W by $V_i = \text{span}\{X_i, \cdots, X_n\}$. A useful observation is

Lemma 2.1. *If $F_B = F_{B'}$, then $w_B(m) = w_{B'}(m)$.*

Recall the Hilbert-Mumford numerical criterion: a point x of a representation V of a reductive algebraic group G has stable orbit if and only if, given any nontrivial $1 - PS$ λ of G and coordinates which diagonalize the action of λ on V, there is a coordinate not vanishing at x whose λ-weight is negative. The discussion above therefore gives

Theorem 2.2. *(C, E) is m-Hilbert stable (resp., semistable) if and only if for any nontrivial standard weighted basis B of W, $w_B(m) < 0$ (resp., $w_B(m) \leqslant 0$).*

Corollary 2.3. *(C, E) is m-Hilbert stable (resp., semistable) if for any nontrivial weighted basis B of W*

$$w_B(m) < (resp., \leqslant) \frac{2mh^0\left(C, \left(\bigwedge^2 E\right)^{\otimes m}\right)}{h^0(C, E)} \sum_{i=1}^{n} r_i.$$

Proof. Since both sides of the inequality are linear in the r_i jointly, it suffices to prove this when the r_i are integers. We then associate to B the standard weighted basis $B' = \{(X_i, s_i)\}$, where $s_i = nr_i - \sum_{j=1}^n r_j$. The B'-weight of a monomial of degree m in the exterior products $X_i \wedge X_j$ equals n times its B-weight minus $2m\sum_{j=1}^n r_i$. Since any B-basis contains $h^0(C, (\bigwedge^2 E)^{\otimes m})$ elements,

$$w'_B(m) = h^0(C, E) w_B(m) - 2mh^0\left(C, \left(\bigwedge^2 E\right)^{\otimes m}\right) \sum_{i=1}^{n} r_i.$$

The corollary now follows immediately from Theorem 2.2.

We will say C is m-stable with respect to a weighted basis B if the inequality of Corollary 2.3 holds for $w_B(m)$. From the linearity of this inequality in the $\{r_i\}$ jointly, we see that we are free to translate and rescale the weights so that $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_n = 0$ and $\sum_{i=1}^n r_i = 1$. We say a weighted basis B satisfying these conditions is normalized. Note also that if the r_i are integers, then each side of the inequality in Corollary 2.3 is represented for large m by a polynomial of degree two in m whose leading term is of the form $\frac{1}{2}em^2$ with e an integer (cf. [6]). We call e the normalized leading coefficient, written n.l.c., of this polynomial, and define e when the r_i are rational using the linearity of e in the r_i jointly.

Corollary 2.4. *Fix g, d and a real number $\varepsilon > 0$. Then we can choose an integer M (depending only on g, d and ε) so that the statement below is verified:*

If B is a normalized weighted basis of W and

$$n.l.c.\ w_B(m, C) \leqslant \tfrac{4d}{n} - \varepsilon r_1,$$

$C \in S_{g,d}$, then for all $m \geqslant M$, C is m-stable with respect to B.

Proof. This can be established by techniques similar to the proof of Proposition 1.2 of [1].

Now if L is a subbundle of E with degree $\frac{1}{2}\deg E$ and $W \cong H^0(C, E)$, we can consider the normalized basis which assigns weight 0 to every element of $H^0(C, L)$ and equal weight to every element of $W/H^0(L)$. such a weighted basis will be said to be special for C. In this situation, we have

Proposition 2.5. (i) *There is a D so that for each $d \geqslant D$, there is an $\varepsilon > 0$ so that if $C \in S_{g,d}$ is smooth with $W = H^0(C, E)$ and B is a normalized weighted basis of W which is not special for C, then*

$$n.l.c.\, w_B(m, C) \leqslant \tfrac{4d}{n} - \varepsilon(r_1 - r_n).$$

(ii) *There is an M so that if $m \geqslant M$ and B is a normalized special basis of $W = H^0(C, E)$, then*

$$w_B(m) = \frac{2mh^0\left(C, \left(\wedge^2 E\right)^{\otimes m}\right)}{h^0(C, E)}.$$

Actually in (i) we will fix $C \in S_{g,d}$ and B, and show

$$\text{n.l.c. } w_B(m) < \frac{4d}{d+1-g},$$

and leave the question of the uniformity of ε with respect to C, E and B to the reader.

This is the key step to Theorem 1.1. The proof occupies the next three sections:

3. For §§3, 4 and 5 we fix a smooth curve C of genus g and a vector bundle E on C. Let $l_E = d - 2d_L$ where L is a linesubbundle of E of maximal degree. If E is decomposable, $l_E \leqslant 0$ but can be arbitrarily negative. However

Proposition 3.1 (*Nagata* [7]). *If E is indecomposable, $2 - 2g \leqslant l_E \leqslant g$.*

If L is a sublinebundle of E, we let $M_L = E/L$ and write M for M_L if the context determines L. We say L is nice if both L and M both have degree at least $2g + 1$.

Lemma 3.2. *If L is a nice subbundle of an indecomposable E, and U is any complement to $H^0(C, L)$ in $H^0(C, E)$, then the following hold*:

(i) *The projection from E to M maps U isomorphically onto $H^0(C, M)$.*

(ii) *E is generated by $H^0(C, L)$ and U.*

(iii) *The map $\phi_{L,M}\colon H^0(C, L) \otimes H^0(C, M) \to H^0(C, L \otimes M)$ is surjective.*

(iv) *The map ϕ_2 takes $H^0(C, L) \wedge U$ onto $H^0(C, \wedge^2 E)$.*

Moreover if $\deg E \geqslant \max(5g + 1, 4g + 2 - l_E)$, *and E indecomposable, then E has a nice linesubbundle.*

Proof. For the last statement, note that since $\frac{1}{2}(\deg E - g) \geqslant 2g + 1$ and $l_E \leqslant g$, E must have a sublinebundle L of degree at least $2g + 1$. The quotient M_L has degree $\deg E - \deg L \geqslant \frac{1}{2}(\deg E + l_E) \geqslant 2g + 1$.

The long exact sequence associated to the composition series $0 \to L \to E \to M \to 0$ is $0 \to H^0(C, L) \to H^0(C, E) \to H^0(C, M) \to 0$ by the hypothesis on

L and M, which gives (i). If $P \in C$, let S be a section of L not vanishing at P, and let $\tilde{T}$ be a section in U whose image in $H^0(C, M)$ is nonzero at P. Then S and $\tilde{T}$ generate E at P, which gives (ii). Since L and M have degree at least $2g+1$, the surjectivity of $\phi_{L,M}$ follows from [5, Theorem 6, p. 52]. Now observe that $L \otimes M = \wedge^2 E$ and that if $S \in H^0(C, L)$, $T \in H^0(C, M)$ and $\tilde{T}$ is the section in U lying over T, then $\phi_2(S \wedge \tilde{T}) = \phi_{L,M}(S \otimes T)$; this yields (iv).

Now for §§3, 4 and 5, we suppose E is semistable and $W = H^0(C, E)$. We next recall a Proposition (3.2) which follows from results of [4] concerning stability of line bundles on C. While we will use some results on multiplicities to obtain Proposition 3.2, they do not appear in its statement and will not be used elsewhere. For definitions and a discussion of these multiplicities see [4]. Let $S = \{(S_i, \sigma_i)\}$ be a weighted basis of $H^0(C, L)$ where L is a very ample line bundle on C. Then for large m, $S^m H^0(C, L)$ maps onto $H^0(C, L^{\otimes m})$, and we define $w_S(m)$ to be the least weight of a basis of $H^0(C, L^{\otimes m})$ consisting of monomials in the S_i. We let $\tilde{L}$ be the pullback of L to $C \times \mathbf{A}^1$. If the σ_i are nonnegative integers decreasing to zero, we define an ideal sheaf $\mathcal{G}_S$ on $C \times \mathbf{A}^1$ by $\Gamma(\mathcal{G}_S \cdot \tilde{L}) = \langle S_i t^{\sigma_i} \rangle$, where t is a parameter on $\mathbf{A}^1$, and let $e_{\tilde{L}}(\mathcal{G}_S)$ be the multiplicity of this ideal sheaf with respect to $\tilde{L}$. Then n.l.c. $w_S(m) = e_{\tilde{L}}(\mathcal{G}_S)$ by Corollary 3.3 of [4]. If $S = \{(S_i, \sigma_i)\}$ and $T = \{(T_j, \tau_j)\}$ are weighted bases of $H^0(C, L)$ and $H^0(C, M)$ respectively with L and M both of degree at least $2g+1$, then we define $w_{(S,T)}(m)$ to be the least weight of a basis of $H^0(C, (L \otimes M)^{\otimes m})$ consisting of monomials in the tensors $S_i \otimes T_j$ (with weight $\sigma_i + \tau_j$). Such a basis exists by (iii) of Lemma 3.2. If S and T both have integer weights decreasing to zero, then Proposition 3.9 of [4] and Lemma 3.10 give respectively

$$\text{n.l.c.}\left(w_{(S,T)}(m)\right) = e_{\tilde{L}}(\mathcal{G}_S) + 2e\left([\tilde{L}, \mathcal{G}_S], [\tilde{M}, \mathcal{G}_T]\right) + e_{\tilde{M}}(\mathcal{G}_T),$$
$$e\left([\tilde{L}, \mathcal{G}_S], [\tilde{M}, \mathcal{G}_T]\right) \leq \tfrac{1}{2}\left(e_{\tilde{L}}(\mathcal{G}_S) + e_{\tilde{M}}(\mathcal{G}_T)\right).$$

Hence we obtain

Proposition 3.3. *Suppose* $S = \{(S_i, \sigma_i)\}$ *and* $T = \{(T_j, \tau_j)\}$ *are weighted bases of* $H^0(C, L)$ *and* $H^0(C, M)$ *respectively such that the weights* σ_i *and* τ_j *both decrease to zero and such that* L *and* M *both have degree at least* $2g+1$. *Then* n.l.c. $(w_{(S,T)}(m)) \leq 2$ n.l.c. $(w_S(m) + w_T(m))$.

Note that by the homogeneity of this inequality we can allow the σ_i and τ_j to be rational. We will combine Proposition 3.3 and Lemma 3.2 to obtain an upper bound for $w_B(m)$ for each nice linesubbundle L of E. Fix a normalized weighted basis $B = \{(X_i, \sigma_i)\}$ of $H^0(C, E)$ and a nice subbundle L of E.

Recall that the associated long exact sequence is

$$0 \to H^0(C, L) \to H^0(C, E) \to H^0(C, M) \to 0.$$

Choose a basis $Y = \{Y_1, \cdots, Y_n\}$ of $H^0(C, E)$ so that

$$(3.4) \qquad \begin{array}{ll} \text{(i)} & \operatorname{span}\{Y_i, \cdots, Y_n\} = V_i = \operatorname{span}\{X_i, \cdots, X_n\}, \\ \text{(ii)} & Y = S \cup \tilde{T} \quad \text{where } S \text{ is a basis of } H^0(C, L). \end{array}$$

Let $B' = \{(Y_i, r_i)\}$. By Lemma 2.1, $w_B(m) = w_{B'}(m)$ so that in estimating $w_B(m)$ we may assume that B satisfies condition (3.4)(ii). We do so henceforth without comment and say the basis B is adapted to L. By Lemma 3.2(i) the image T of $\tilde{T}$ in $H^0(C, M)$ forms a basis there. Let $S = \{S_1, \cdots, S_{n_L}\}$, $\tilde{T} = \{\tilde{T}_1, \cdots, \tilde{T}_{N_M}\}$ and $T = \{T_1, \cdots, T_{n_m}\}$ ordered in each case so that the weights of the corresponding elements of B decrease.

Consider the diagram

$$\begin{array}{ccc} H^0(C, L) \otimes H^0(C, M) & \xrightarrow{\phi_{L,M}} & H^0(C, L \otimes M) \\ \psi \downarrow & & \| \\ \wedge^2 H^0(C, E) & \xrightarrow{\phi_E} & H^0\left(C, \wedge^2 E\right) \end{array}$$

where ψ is defined by $\psi(S_i \otimes T_j) = S_i \wedge \tilde{T}_j$. The diagram commutes, and the rows are surjective by (iii) and (iv) of Lemma 3.2. Define weights $\{s_i\}$ on S and $\{t_j\}$ on $\tilde{T}$ and T so that the weight of each basis element equals the weight of the corresponding element of B. Then defining the weight of $R_{ij} = S_i \otimes T_j$ to be $s_i + t_j$ makes ψ weight preserving. We obtain a commutative diagram

$$\begin{array}{ccc} S^m(H^0(C, L) \otimes H^0(C, M)) & \longrightarrow & H^0\left(C, (L \otimes M)^{\otimes m}\right) \\ S^m\psi \downarrow & & \| \\ S^m \wedge^2 H^0(C, E) & \longrightarrow & H^0\left(C, \left(\wedge^2 E\right)^{\otimes m}\right) \end{array}$$

with surjective rows and with $S^m\psi$ weight preserving. Thus $w_B(m)$ is at most the minimum weight of a basis of $H^0(C, (L \otimes M)^m)$ consisting of monomials of degree m in the R_{ij}. Let $w_L = s_{n_L}$ and $w_M = t_{n_M}$, and define new weights σ_i and τ_j by $\sigma_i = s_i - w_L$ and $\tau_j = t_j - w_M$. Observe that one of w_L and w_M equals r_n which is zero since B is normalized, and that both the σ_i's and the τ_j's decrease to zero by the choice of the orderings on S and T. Let $S = \{(S_i, \sigma_i)\}$ and $T = \{(T_j, \tau_j)\}$ denote these weighted bases. As the (σ, τ)-weight of any of the R_{ij} differs from its (s, t) weight by $w_L + w_M$, the (σ, τ)-weight of a basis of $H^0(C, (L \otimes M)^m)$ consisting of monomials on the R_{ij} differs from its (s, t)

weight by $mh^0(C,(L\otimes M)^m)(w_L+w_M)$. Hence

$$w_B(m) \leqslant mh^0\big(C,(L\otimes M)^{\otimes m}\big)(w_L+w_M)+w_{(S,T)}(m).$$

Applying Proposition 3.3 and taking leading coefficients gives

Theorem 3.5. *If L is a nice subbundle of E, and B is a normalized weighted basis of $H^0(C,E)$ adapted to L, then*

$$n.l.c.\,w_B(m) \leqslant 2d(w_L+w_M)+2\big(n.l.c.\,w_S(m)+n.l.c.\,w_T(m)\big).$$

In the situation of the theorem; especially in §5, we will continue to use the notation developed in the preceding discussion (e.g., S, σ_i, w_L) to denote the quantities defined there.

4. Fix a weighted basis $B=\{(X_i,r_i)\}$ with associated weighted filtration $F_B=\{(V_i,r_i)\}$. We will give an estimate for n.l.c. $w_B(m)$ in terms of the subbundles of E generated by the sections in V_i. This criterion is an analogue for the rank-two case of estimates given for invertible sheaves in [2] and [6].

Let E_i be the subsheaf of E generated by the sections in V_i, $d_i=\deg E_i$, $e_i=d-d_i$, and let $s=s_B$ be the greatest index such that rank $E_i=2$. If i and j are less than or equal to s, and $0\leqslant k\leqslant m$, let $W_{i,j,k,N}$ be the image in $H^0(C,(\wedge^2E)^{\otimes(m+1)N})$ of

$$S^N\big(S^{m-k}(\wedge^2V_i)\vee S^k(\wedge^2V_j)\vee\wedge^2H^0(C,E)\big).$$

If $i\leqslant s$, let $W_{i,n,k,N}$ be the image of

$$S^N\big(S^{m-k}(\wedge^2V_i)\vee S^k(V_i\wedge V_n)\vee\wedge^2H^0(C,E)\big).$$

Lemma 4.1. *There is an N_0 depending only on the genus g of C such that if $N\geqslant N_0$ and $m\gg 0$, then:*

(i) *for $i,j\leqslant s$,* $\dim W_{i,j,k,N}\geqslant N((m-k)d_i+kd_j)$,

(ii) *for $i<s$,* $\dim W_{i,n,k,N}\geqslant N(m-k)d_i$.

Proof. We give the proof of (i), that for (ii) being similar. Since E_i is generated by the sections in V_i, $\wedge^2E_i$ is generated by the sections in $\wedge^2V_i$. Hence the elements of $W_{i,j,k,l}$ generate $L_{i,j,k}=(\wedge^2E_i)^{m-k}\otimes(\wedge^2E_j)^k\otimes\wedge^2E$. Since $\wedge^2E$ is very ample on C, and $\wedge^2H^0(C,E)$ maps onto a very ample sublinear system of $\wedge^2E$, $W_{i,j,k,l}$ forms a very ample sublinear system of $L_{i,j,k}$ without base points. Thus for N large, the elements of $W_{i,j,k,N}$ generate $H^0(C,L_{i,j,k}^{\otimes N})$ which by Riemann-Roch has dimension $N((m-k)d_i+kd_j+d)-g+1$ from which the desired inequality is immediate. We omit the check that N can be chosen independent of C and E, which follows by arguments like those of Lemma 2.1 of [2].

Suppose a vector space V with a weighted filtration contains subspaces U_i satisfying:

(i) $V=U_l\supset U_{l-1}\supset\cdots\supset U_1$,

(ii) $\operatorname{codim} U_i = c_i$,

(iii) the weight of every element of U_i is at most w_i,

(iv) $w_l \geqslant w_{l-1} \geqslant \cdots \geqslant w_1$.

Then V has a basis of weight at most $\sum_{i=1}^{l-1}(w_{i+1} - w_i)c_i + w_1 \dim V$. Now pick a sequence of integers $1 = i_1 < i_2 < \cdots < i_{l-1} < i_l = n$, where $i_{l-1} \leqslant s$, and apply this remark to the filtration of $H^0(C, (\wedge^2 E)^{\otimes(m+1)N})$ by $W_{i_1,i_2,0,N} \supset W_{i_1,i_2,1,N} \supset \cdots \supset W_{i_1,i_2,m,N} = W_{i_2,i_3,0,N} \supset W_{i_2,i_3,1,N} \supset \cdots \supset W_{i_{l-1},i_l,m,N}$. The weight of any section in $W_{i,j,k,N}$ is bounded by $2N((m-k)r_i + kr_j + r_0)$ if $j \leqslant s$, and by $N(2(m-k)r_i + k(r_i + r_n) + 2r_0)$ if $j = n$. From Lemma 4.1, for $j \leqslant s$ we have

$$\begin{aligned}
\operatorname{codim} W_{i,j,k,N} &\leqslant (N(m+1)d - g + 1) - N\big((m-k)d_i + kd_j\big) \\
&\leqslant N\big(d + (m\text{-}k)e_i + ke_j\big), \\
\operatorname{codim} W_{i,n,k,N} &\leqslant (N(m+1)d - g + 1) - N(m-k)d_i) \\
&\leqslant N\big((m-k)e_i + (k+1)d\big).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
w_B((m+1)N) \leqslant{} & \sum_{j=1}^{l-2} \sum_{k=0}^{m} 2N(r_{i_j} - r_{i_{j+1}})\Big(N\big((m-k)e_{i_j} + ke_{i_{j+1}} + d\big)\Big) \\
& + \sum_{k=0}^{m} N(r_{i_{l-1}} - r_n)\Big(N\big((m-k)e_{i_{l-1}} + (k+1)d\big)\Big) \\
& + N\big(m(r_{i_{l-1}} + r_n) + 2r_0\big)((m+1)Nd - g + 1) \\
={} & \frac{(mN)^2}{2}\Bigg[2\sum_{j=1}^{l-2}(r_{i_j} - r_{i_{j+1}})(e_{i_j} + e_{i_{j+1}}) \\
& + (r_{i_{l-1}} - r_n)\big(e_{i_{l-1}} + d\big) + 2(r_{i_{l-1}} + r_n)d\Bigg] + O(1),
\end{aligned}$$

where in the $O(1)$ term we have collected all terms of order 1 in m. If we take B to be normalized so that $r_n = 0$, then by applying this to *all* subsequences of $(1, \cdots, n)$ simultaneously and taking leading coefficients we obtain

Theorem 4.2. *If B is a normalized weighted basis of $H^0(C, E)$, then*

$$\begin{aligned}
n.l.c.\, w_B(m) \leqslant \min_{(1=i_1<\cdots<i_{l-1}\leqslant s)} & \; 2\sum_{j=0}^{l-2}(r_{i_j} - r_{i_{j+1}})(e_{i_j} + e_{i_{j+1}}) \\
& + r_{i_{l-1}}\big(e_{i_{l-1}} + 3d\big).
\end{aligned}$$

5. In this section we fix a smooth curve C and a rank-two bundle E of degree d on C. Our aim is to establish Proposition 2.5 and thereby to prove

Theorem 5.1. *There is an M depending only on g so that if $d \geqslant M$ and E is stable (resp. semistable), then (C, E) is Hilbert stable (resp. semistable).*

Proof. We assume E is semistable. Let $\alpha = g - 1$, and let $k = 10^6\alpha^2$. We say a line bundle is good if $\deg L \geqslant k$. We divide the proof into two cases. In our first case, we assume

$$\operatorname{rk} E_i = 2 \quad \text{for } i < n - k. \tag{5.1.1}$$

We first estimate $h^1(E_i)$ for $i \leqslant n - k$. E_i has rank two and at least $2g + 2$ sections. Let L_1 be the sublinebundle of E so that $S_1 \in H^0(L)$, and let $L_2 = E/L_1$. Then both L_1 and L_2 have sections, and at least one has $g + 1$ sections. Hence $h^1(L_i) \leqslant \alpha + 1 = g$, and $h^1(L_1)$ or $h^1(L_2)$ is zero. Since $h^1(E_i) \leqslant h^1(L_1) + h^1(L_2)$, we see

$$h^1(E_i) \leqslant \alpha + 1 \quad \text{if } i \leqslant n - k. \tag{5.1.2}$$

Next we claim

$$h^1(E_i) = 0 \quad \text{if } i < \tfrac{1}{2}n - 3\alpha. \tag{5.1.3}$$

Indeed, if $h^1(E_i) \neq 0$, then $E_i^{-1} \otimes \Omega^1$ has a section, and so E_i has a quotient of degree at most $2g - 2$. Thus E_i and hence E would have a subbundle of degree $d_i - 2\alpha$. Since E is semistable,

$$d \geqslant 2(d_i - 2\alpha). \tag{5.1.4}$$

But

$$\begin{aligned} d_i &= h^0(E_i) + 2\alpha - h^1(E_i) \\ &\geqslant (n - i + 1) + 2\alpha - \alpha - 1 \geqslant n - i + 1. \end{aligned} \tag{5.1.5}$$

Since $i < \frac{1}{2}n - 3\alpha$, we have

$$d_i \geqslant \tfrac{n}{2} + 3g - 2,$$

and by (5.1.4),

$$d \geqslant 2(\tfrac{n}{2} + g) = n + 2g,$$

which contradicts the fact that $d = n + 2\alpha$. Thus (5.1.3) is established.

We see from (5.1.5) that

$$e_i = d - d_i = d - \big(h^0(E_i) + 2\alpha - h^1(E_i)\big) \leqslant i - 1 + h^1(E_i),$$

since $n + 2\alpha = d$ and $h^0(E_i) \geqslant n - i + 1$.

Define ε_i and f_i by

$$\varepsilon_i = \begin{cases} \frac{2\alpha}{d}(i-1) & \text{if } i \leqslant \frac{n}{2} - 3\alpha, \\ \frac{2\alpha}{d}(i-1) - \frac{n}{d}(\alpha+1) & \text{if } \frac{n}{2} - 3\alpha < i \leqslant n-k, \end{cases}$$

(5.1.6) $$f_i = \tfrac{d}{n}(i - 1 - \varepsilon_i).$$

We have

(5.1.7) $$\left(\tfrac{d}{n}\left(i - 1 - \tfrac{2\alpha}{d}(i-1)\right)\right) = i - 1,$$

so

(5.1.8) $$f_i \geqslant (i-1) + h^1(E_i) \geqslant e_i,$$

by (5.1.2) and (5.1.3).

Define

$$P_B(I) = 2 \min_{(1=i_1<\cdots<i_{l-1}=I)} \sum_{j=1}^{l-2} (r_{i_j} - r_{i_{j+1}})(e_{i_j} + e_{i_{j+1}}),$$

$$P(I) = 2 \min_{(1=i_1<\cdots<i_{l-1}=I)} \sum_{j=1}^{l-2} (r_{i_j} - r_{i_{j+1}})(f_{i_j} + f_{i_{j+1}}).$$

Then $P(I) \geqslant P_B(I)$. Further define

$$Q(I) = \max_{2<i\leqslant I} \frac{f_i^2}{(i-1)f_i - \sum_{j=1}^{i-1} f_j}.$$

By Corollary 4.3 of [4],

$$P(I) \leqslant 2Q(I) \sum_{j=1}^{I} (r_j - r_I).$$

Thus

$$P_B(I) \leqslant 2Q(I) \sum_{j=1}^{I} (r_j - r_I).$$

Our next object is to estimate $Q(I)$. To this end, we define δ_i by

$$\delta_i = \frac{2d}{n} - \frac{f_i^2}{(i-1)f_i - \sum_{j<i} f_j} \quad \text{for } i \geqslant 2.$$

We wish to show $\delta_i \geqslant 1/2n$. If $i \leqslant n/2 - 3\alpha$, then $f_i = (i-1)$ and a direct computation shows that $\delta_i \geqslant 1/(2n)$. Assume $i > n/2 - 3\alpha$. First notice that we have

$$|f_i - i + 1| \leqslant \alpha + 1$$

from (5.1.6) and (5.1.7). Hence

$$\left|(i-1)f_i - \sum_{j<i} f_j - (i-1)^2 - \tfrac{1}{2}(i-1)(i-2)\right| \leqslant 2(\alpha+1)i.$$

So

$$(i-1)f_i - \sum f_j \leqslant \tfrac{1}{2}(i-1)i + 2(\alpha+1)i \leqslant \tfrac{1}{2}i(i+4\alpha+3).$$

We compute

$$\begin{aligned}\left(\frac{n}{d}\right)^2\left((i-1)f_i - \sum f_j\right)\delta_i &= 2\left((i-1)f_i - \sum f_j\right) - \frac{n}{d}\left(f_i^2\right)\frac{n}{d} \\ &= (i-1) + 2\sum \varepsilon_j - \varepsilon_i^2.\end{aligned}$$

We next claim that

$$2\sum_{j<i} \varepsilon_j - \varepsilon_i^2 > -18\alpha^2, \tag{5.1.10}$$

for $i > n/2 - 3\alpha$. Once (5.1.10) is established, we will have

$$\delta_i \geqslant \frac{(i-1-18\alpha^2)d^2}{\left((i-1)f_i - \sum f_j\right)n^2} \geqslant \frac{2(i-18\alpha^2-1)d^2}{i(i+4\alpha+3)n^2} \geqslant \frac{1}{2n}.$$

Thus

$$\delta_i \geqslant \frac{1}{2n}. \tag{5.1.11}$$

Since (5.1.11) holds for $i \leqslant n/2 - 3\alpha$, (5.1.11) holds in general.

We next establish our claim (5.1.10). Let J be the greatest integer in $n/2 - 3\alpha$. Then

$$\begin{aligned}d\sum_{j=1}^{i-1} \varepsilon_j &= 2\alpha \sum_{j=1}^{i-1} (j-1) - n(\alpha+1)(i-J-1) \\ &\geqslant \alpha\left((i-1)(i-2) - 2n(i-J-1)\right).\end{aligned}$$

The function $f(i) = (i-1)(i-2) - 2n(i-J-1)$ has its minimum when $2i - 3 = 2n$. Thus since $i \leqslant N - k$ and $k > 10^6\alpha^2$,

$$\begin{aligned}f(i) &\geqslant (n-k-1)(n-k-2) - 2n(n-k-\tfrac{n}{2}+3\alpha-1) \\ &= -(6\alpha+1)n \geqslant -7\alpha^2 n.\end{aligned}$$

Also, for $n/2 - 3\alpha < i \leqslant n - k$, $-2 \leqslant \varepsilon_i \leqslant 2\alpha$. So

$$2\sum^{i-1} \varepsilon_j - \varepsilon_i^2 \geqslant -18\alpha^2.$$

Thus if (C, E) is not stable with respect to B, we would have for each I

$$\frac{4d}{n} \leqslant Q(I)\left(\sum_{j=1}^{I}(r_j - r_I)\right) + r_I(e_I + 3d).$$

From (5.1.11), we see

$$\frac{f_i^2}{(i-1)f_i - \Sigma f_j^2} = \frac{2d}{n} - \delta_i \geqslant \left(\frac{2d}{n} - \frac{1}{2n}\right).$$

So

$$Q(I) \leqslant \frac{2d}{n} - \frac{1}{2n}.$$

Thus

$$\frac{4d}{n} \leqslant \left(\frac{4d-1}{n}\right)\left(\sum_{j\leqslant I} r_j - r_I\right) + r_I(f_I + 3d). \tag{5.1.12}$$

Next let $\beta(I) = 1 - \Sigma_{i=1}^{I} r_i$. Since $\Sigma r_i = 1$, we can write (5.1.12) as

$$r_I\left(f_I + 3d - \frac{4d}{n}I\right) \geqslant \frac{4d}{n}\beta(I) + \frac{1}{n}\sum_{j\leqslant I}(r_j - r_I).$$

Now

$$f_I = \frac{d}{n}((I-1) - \varepsilon_I), \qquad -\varepsilon_I \leqslant \frac{1}{d}(n + 6g^2) \leqslant 2.$$

So

$$f_I + 3d - \frac{4d}{n}I \leqslant \frac{d}{n}(3(n-I)+1).$$

Thus

$$r_I(3(n-I)+1) \geqslant 4\beta(I) + \frac{1}{d}\sum_{j\leqslant I}(r_j - r_I). \tag{5.1.13}$$

In particular,

$$r_I(3(n-I)+1) \geqslant 4\beta(I). \tag{5.1.14}$$

Let $J_l = n - 10^l k$ where $k = 10^6\alpha^2$.

We claim

$$(k+2)r_{J_0} \leqslant \frac{1}{100n}. \tag{5.1.15}$$

Indeed, note for any J,

$$\beta(n - 10J) \geqslant 9Jr_{n-J} + \beta(n-J).$$

From (5.1.14),

$$r_{n-10J} \geqslant \frac{4}{3}\frac{\beta(n-10J)}{10J+1} \geqslant \frac{4}{3}\frac{9J}{10J+1} r_{n-J} \geqslant \frac{12}{11} r_{n-J}.$$

So $r_{J_l} \geqslant (12/11)^l r_{J_0}$. Choose l so that $(12/11)^l \geqslant 300(k+2)$ and $J_l \geqslant 2n/3$. (Recall that we are assuming that d and n are large with respect to g and hence to k.)

$$1 \geqslant \sum_{j=1}^{[2n/3]} r_i \geqslant \frac{n}{2} r_{J_l} \geqslant \frac{n}{2}(300(k+2)) r_{J_0}.$$

Thus our claim (5.1.15) is established.

Next note that

$$\sum^{I}(r_i - r_I) = 1 - \beta(I) - Ir_I,$$

so (5.1.13) shows that

$$r_I\left(3(n-I) + 1 + \frac{I}{d}\right) \geqslant 4\beta(I) + \frac{1}{d}(1 - \beta(I)) \geqslant \frac{1}{d}.$$

Finally, we take $I = J_0$. Then

$$\frac{3k+2}{100n(k+2)} \geqslant \frac{1}{d},$$

which contradicts $d = n + 2\alpha$. Thus we have established Theorem 5.1 under assumption (5.1.1).

We may accordingly assume rk $E_{n-k} = 1$ and hence rk $E_i = 1$ for $i \geqslant n - k$.

Let L be the sublinebundle of E containing E_i for $i \geqslant n - k$. We may replace B by a basis adapted to L without affecting the hypothesis. If l is the greatest integer so that $S_l \in H^0(L)$, then $l \geqslant n/2$ since otherwise L would have more than $n/2$ sections, thus contradicting the semistability of E. Thus $w_M \geqslant 2/n$ with strict inequality if E is stable.

Recall from Theorem 3.5 that n.l.c. $w_B(m) \leqslant 2(w_L)d +$ 2n.l.c.$(w_S(m) + w_T(m))$. Since L is good, d_L and d_M are greater than K, and it follows from Corollary 4.6 of [4] that n.l.c. $w_S(m) \leqslant 2\sum_{i=1}^{n_L}\sigma_i$ and n.l.c. $w_T(m) \leqslant 2\sum_{j=1}^{n_M}\tau_j$. Note that

$$1 = \sum_{i=1}^{n} r_i = n_M w_M + \sum_{i=1}^{n_L}\sigma_i + \sum_{j=1}^{n_M}\tau_j.$$

If E is stable we obtain

$$\text{n.l.c. } w_B(m) \leqslant 2w_M d + 4\left(\sum_{i=1}^{n_L} \sigma_i + \sum_{i=1}^{n_M} \tau_j\right)$$
$$= 2w_M d + 4(1 - n_M w_M) < \frac{4d}{n} - 2w_M(2n_M - n).$$

If E is semistable, then $n_M \geqslant n/2$, hence $w_M \leqslant 2/n$. Unless $n_M w_M = 1$, this implies

$$\text{n.l.c. } w_B(m) < \frac{4d}{n} - 2dw_M(2n_M - n) \leqslant \frac{4d}{n},$$

so that (C, E) is stable with respect to B. If $n_M w_M = 1$, this argument only shows that n.l.c. $w_B(m) \leqslant 4d/n$ which does not suffice to prove (C, E) semistable with respect to B. However, in this case all the σ_i's and τ_j's must be zero. Hence every section $R_{i_j} = S_i \otimes T_j$ has weight w_M. But then

$$w_B(m) \leqslant mh^0\left(C, L \otimes M)^{\otimes m}\right) w_M \leqslant \frac{2mh^0\left(C, \left(\wedge^2 E\right)^{\otimes m}\right)}{h^0(C, E)},$$

since $w_M \leqslant 1/n_M \leqslant 2/n$. This completes the proof of Proposition 2.5.

Now Theorem 5.1 follows from Corollary 2.3. In fact, if E is unstable, L is the destabilizing line subbundle, and B is any standard basis whose filtration is $W \supset H^0(C, L) \supset \{0\}$, then φ_3 kills all elements of nonpositive weight, hence so does each φ_C^m. Therefore $w_B(m) > 0$, and (C, E) is Hilbert unstable. Hence Theorem 1.1 is proved.

6. We continue to suppose that $d \geqslant 1000g(g - 1)$. Our object is to prove

Proposition 6.1. *There is an M (depending on d) so that if $m \geqslant M$, and φ_C^m is semistable for $C \in S_{g,d}$, then C is semistable as a curve.*

We begin with a few general definitions. Let $\mathfrak{F}$ be a coherent sheaf on a scheme, and let $W \subseteq H^0(X, \mathfrak{F})$ be a subspace so that $\mathfrak{F}$ is generated at each point by sections in W.

Definition 6.2. A weighted filtration on $\mathfrak{F}$

$$B = \begin{pmatrix} \mathfrak{F}_k & \mathfrak{F}_{k-1} \cdots \mathfrak{F}_1 \\ r_k & r_{k-1} \cdots r_1 \end{pmatrix}$$

is a sequence of subsheaves

$$\mathfrak{F}_k \subseteq \mathfrak{F}_{k-1} \subseteq \cdots \subseteq \mathfrak{F}_1 = \mathfrak{F}$$

and rational numbers r_i, $r_k \leqslant r_{k-1} \leqslant \cdots \leqslant r_1$. (*Note*: In the rest of this paper, filtrations will increase from left to right.)

If

$$B' = \begin{pmatrix} \mathcal{F}'_i \\ r'_i \end{pmatrix}$$

is another weighted filtration on $\mathcal{F}$, and if it happens that $\mathcal{F}_i \subseteq \mathcal{F}'_i$ whenever $r_i \leqslant r'_i$, we say B' dominates B.

Let π: $Y \to X$ be a map. Given a weighted filtration $B = \binom{\mathcal{G}_i}{r_i}$ on $\pi^*(\mathcal{F})$, there is an induced filtration $B' = \binom{W_i}{r_i}$ on W, where

$$W_i = \{s \in W \,|\, \pi^*(s) \in H^0(Y, \mathcal{G}_i)\}.$$

Conversely, given a weighted filtration on W, there is an induced filtration on $\pi^*(\mathcal{F})$, where $\mathcal{G}_i$ is the subsheaf of $\pi^*(\mathcal{F})$ generated by W_i.

The weight of a filtration $\binom{W_i}{r_i} = B$ on W is $\Sigma \dim(W_i/W_{i-1})r_i = w(B)$.

Now let φ: $\mathcal{F} \to \mathcal{G}$ be a map of coherent sheaves. The weighted filtration

$$\begin{pmatrix} \ker \varphi & \mathcal{F} \\ 0 & 1 \end{pmatrix}$$

will be denoted

$$[\mathcal{F} \to \mathcal{G}]. \tag{6.2.1}$$

Now let L be a line bundle on a curve C, and let $V \subseteq H^0(C, L)$ be a very ample linear system. Let $\binom{V_i}{r_i} = B$ be a weighted filtration on V. Choose a compatible weighted basis $\{(X_j, \rho_j)\}$ of V, and let $w_B(m, C)$ be the minimum weight of a basis of $H^0(C, L^{\otimes m})$. Then $w_B(m, C)$ is a polynomial in m for $m \gg 0$.

Now suppose that C is a curve on G and that $\binom{W_i}{r_i}$ is a weighted filtration on W. There is an induced weighted filtration B' on the image V of $\bigwedge^2 W$ in $H^0(C, \det \mathcal{E}_C)$. If V is very ample, we define $w_B(m, C) = w_{B'}(m, C)$.

For the remainder of this section, we consider a curve C, a very ample linear system $V \subseteq H^0(C, L)$ and a weighted filtration $B = \binom{V_i}{r_i}$. Our aim is to give two useful estimates for n.l.c. $w_B(m, C)$.

Lemma 6.4. *Suppose $C_i \subseteq C$ are subcurves of C, and the natural map $\varphi : \mathcal{O}_C \to \oplus \mathcal{O}_{C_i}$ has kernel and cokernel of finite length. Then*

$$n.l.c.\, w_B(m, C) \geqslant \sum_i n.l.c.\, w_B(m, C_i).$$

Proof. Let q be the maximum of the lengths of the kernel and cokernel of φ. Then for $m \gg 0$, the kernel and cokernel of

$$\varphi_m : H^0(C, L^{\otimes m}) \to \oplus H^0(C_i, L^{\otimes m})$$

have dimension $\leqslant q$. Given a basis $P_1, \cdots, P_t$ of $H^0(C, L^{\otimes m})$, we can suitably reorder the P_i and partition $P_1, \cdots, P_{t-q}$ into sets $Q_i \subseteq \{P_1, \cdots, P_{t-q}\}$ so that Q_i gives an independent set in $H^0(C_i, L^{\otimes m})$. Thus

$$w_B(m, C) - mr_1 q \geqslant \sum w_B(m, C_i) - mr_k q.$$

Taking normalized leading coefficients yields the lemma.

Now suppose C is irreducible. Let $\pi : \tilde{C} \to C$ be the normalization of C_{red}; and let $\mathcal{I} \subseteq \mathcal{O}_C$ be the ideal of C_{red}. Let l be the length of the local ring of the generic point of C. Suppose R is an effective divisor on $\tilde{C}$. Let $B = \binom{V_i}{r_i}$ be a weighted filtration and let p be an integer and suppose the r_i are integers.

Proposition 6.5. *Suppose that V_j maps to zero in $H^0(\tilde{C}, \tilde{L})$ for $j > p$ and that V_i maps to $H^0(\tilde{C}, \tilde{L}((-r_1 + r_i)R))$. If* $\deg L \geqslant (r_1 - r_p)\deg R$, *then we have*

$$n.l.c.\, w_B(m, C) \geqslant (r_1 - r_p)^2 \deg R + 2lr_p \deg \tilde{L}.$$

Proof. First, replace C by the subscheme defined by $\mathcal{I}^l$. Since $\mathcal{I}^l$ is supported at a finite number of points, neither the hypothesis nor conclusion of the theorem are changed.

Let B' be the weighted filtration

$$\begin{pmatrix} V_p & \cdots & V_1 \\ r_p & \cdots & r_1 \end{pmatrix},$$

that is, we change the weights of the V_i for $i \geqslant p$ from r_i to r_p. Now let $\{(X_i, \rho_i)\}$ be a basis of V compatible with B. Let M be a monomial in the X_i's which is nonzero in $H^0(C, L^{\otimes m})$. Then M can involve at most l of X_i's with $X_i \in V_p$, since $\mathcal{I}^l = 0$. Thus

$$\text{n.l.c. } w_B(m, C) = \text{n.l.c. } w_{B'}(m, C),$$

since the B and B' weights of a monomial differ by at most $l(r_p - r_k)$, where r_k is the lowest weight in B. Hence we may assume $B = B'$.

Next, notice that

$$h^0(C, L^{\otimes m}) = ml \deg_{\tilde{C}} \tilde{L} + O(1),$$

since $\mathcal{I}^{k-1}/\mathcal{I}^k$ is nonzero at the generic point of C_{red} for $k = 1, \cdots, l$. Consider a new weighted filtration

$$B' = \begin{pmatrix} V_i \\ r_i - r_p \end{pmatrix}.$$

Then

$$w_B(m, C) = w_{B'}(m, C) + mr_p h^0(C, L^{\otimes m})$$
$$= w_{B'}(m, C) + m^2 r_p l \deg \tilde{L} + O(m).$$

Hence it suffices to prove Proposition 6.5 for $r_p = 0$.

Since $r_i \geqslant 0$,

$$w_B(m, C) \geqslant w_B(m, C_{\mathrm{red}}),$$

so we may assume C is reduced. Now let M be any monomial in $V^{\otimes m}$ of weight Q. Then the image of M is in $H^0(\tilde{C}, \tilde{L}^{\otimes m}((Q - r_1 m)R))$. Thus there is a constant C_1 so that the image of an M of weight Q lies in a subspace of codimension at least $(r_1 m - Q)\deg R - C_1$ in $H^0(C, L^{\otimes m})$. Adding up the possible contributions for each weight Q, we see any basis must have weight at least

$$\sum_{Q=0}^{mr_1} [Q \deg R + O(1)] = r_1^2 \deg R \frac{m^2}{2} + O(m).$$

7. Let $C \in S_{g,d}$. We can find curves $C_i \subseteq C$ and integers l_i so that the following hold:

(7.1.1) Each C_i is irreducible.

(7.1.2) $\mathcal{I}_{C_i}^{l_i} = 0$, where $\mathcal{I}_{C_i}$ is the ideal of C_i in C.

(7.1.3) l_i is the length of the local ring of the generic point of C_i.

(7.1.4) The natural map $\mathcal{O}_C \to \oplus \mathcal{O}_{C_i}$ has kernel and cokernel of finite length.

Given a weighted filtration B on W, Lemma 6.4 shows that

$$\text{n.l.c. } w_B(m, C) \geqslant \sum \text{n.l.c. } w_B(m, C_i).$$

Now let $E = \mathcal{E} \otimes \mathcal{O}_C$, let $\tilde{C}_i$ be the normalization of $(C_i)_{\mathrm{red}}$, and let $\pi_i : \tilde{C}_i \to C$ be the induced map. Let $\tilde{E}_i = \pi_i^*(E)$ and let $d_i = \deg_{\tilde{C}_i} E_i$. Let B be a weighted filtration on W. If B_i is a weighted filtration on $\tilde{E}_i$, we say B dominates B_i if the filtration induced from B on $\tilde{E}_i$ dominates B_i.

Lemma 7.2. *Let R be an effective divisor on $\tilde{C}_i$, and let $k = \deg_{C_i} E - 2 \deg R$. Suppose B dominates*

$$\begin{pmatrix} \tilde{E}_i(-R) & E_i \\ 0 & 1 \end{pmatrix}.$$

If $k \geqslant 0$, then

(7.2.1) $$n.l.c.\, w_B(m, C_i) \geqslant 4 \deg R,$$

while if $k + \deg R \geqslant 0$ and $k < 0$, then

(7.2.2) $$n.l.c.\, w_B(m, C_i) \geqslant \deg R + 2 l_i d_i.$$

Proof. If $k \geqslant 0$, the filtration induced by W on $\bigwedge^2 E$ dominates

$$\begin{pmatrix} \bigwedge^2 \tilde{E}_i(-2R) & \bigwedge^2 \tilde{E}_i(-R) & \bigwedge^2 \tilde{E}_i \\ 0 & 1 & 2 \end{pmatrix}.$$

Applying Proposition 6.5 gives (7.2.1).

If $k + \deg R \geqslant 0$ and $k < 0$, the filtration induced by W on $\bigwedge^2 E$ dominates

$$\begin{pmatrix} \bigwedge^2 \tilde{E}_i(-R) & \bigwedge^2 \tilde{E}_i \\ 1 & 2 \end{pmatrix},$$

since $H^0(C, \bigwedge^2 \tilde{E}_i(-2R)) = 0$. Applying Proposition 6.5 gives

$$\text{n.l.c. } w_B(m, C_i) \geqslant \deg R + 2l_i d_i.$$

Lemma 7.3. *Let E' be a rank-two subsheaf of $\tilde{E}_i$ with* $\deg E' \geqslant 0$. *Suppose B dominates*

$$\begin{pmatrix} E' & \tilde{E}_i \\ 0 & 1 \end{pmatrix}.$$

Then

$$n.l.c.\, w_B(m, C_i) \geqslant d_i - \deg E'.$$

Proof. The filtration induced on $\bigwedge^2 \tilde{E}_i$ dominates

$$\begin{pmatrix} \bigwedge^2 E' & \bigwedge^2 \tilde{E}_i \\ 0 & 1 \end{pmatrix}.$$

Now $\bigwedge^2 E' = \bigwedge^2 \tilde{E}_i(-R)$, where $\deg R = d_i - \deg E'$. Proposition 6.4 applies.

Lemma 7.4. *Suppose that $0 \to M \to \tilde{E}_i \to L \to 0$ is exact with M and L invertible and that B dominates*

$$\begin{pmatrix} M(-R) & \tilde{E}_i \\ 0 & 1 \end{pmatrix}.$$

Then

$$n.l.c.\, w_B(m, C_i) \geqslant \deg R + 2l_i d_i,$$

if $\deg R \leqslant \deg \tilde{E}_i$.

Proof. The induced filtration on $\bigwedge^2 \tilde{E}_i$ dominates

$$\begin{pmatrix} \bigwedge^2 \tilde{E}_i(-R) & \bigwedge^2 \tilde{E}_i \\ 1 & 2 \end{pmatrix}.$$

Lemma 7.5. *If B dominates*

$$\begin{pmatrix} 0 & \tilde{E}_i \\ 0 & 1 \end{pmatrix},$$

then n.l.c. $w_B(m, C_i) \geqslant 4l_i d_i$.

Proof. Left to reader.

Now write $d/n = 1 + \varepsilon$. Since $n = d + 2(1-g)$ and $n \geqslant 1000g(g-1)$, we see $\varepsilon \leqslant 1/998g$. Let B be a weighted filtration on W. We will say B is destabilizing if

$$\text{n.l.c. } w_B(m, C) > 4(1+\varepsilon)w(B).$$

Throughout the rest of the section, *we will assume $C \in S_{g,d}$ has no destabilizing flags*. Our aim in this section is to establish that $l_i = 1$.

Lemma 7.6. *If $\tilde{E}_i$ has a trivial quotient $\tilde{E}_i \to \mathcal{O} \to 0$, then $l_i = 1$ and $d_i = 1$.*

Proof. We consider the filtration B induced on W by $[\tilde{E}_i \to \mathcal{O}]$ in the notation of (6.2.1).

Lemma 7.4 with $R = \varnothing$ gives

$$\text{(7.6.1)} \qquad \text{n.l.c. } w_B(m, C_i) \geqslant 2l_i d_i.$$

On the other hand, if there is a component C_j meeting C_i, Lemma 7.3 shows

$$\text{n.l.c. } w_B(m, C_j) \geqslant 1.$$

Hence from (7.6.1),

$$4(1+\varepsilon) > \text{n.l.c. } w_B(m, C) \geqslant \text{n.l.c. } w_N(m, C_i) \geqslant 2l_i d_i.$$

Hence $l_i d_i \leqslant 2$, so C_i must meet some C_j. Thus

$$(1+\varepsilon) \geqslant \tfrac{1}{2} l_i d_i + \tfrac{1}{4},$$

which shows $l_i d_i = 1$. The same method of proof shows

Corollary 7.6.2. *If $C' \subseteq C$ is a curve, and $E_{C'}$ has a trivial quotient, then C' has one component, and $E_{C'}$ has degree* 1.

Lemma 7.7. $l_i = 1$ *for all* i.

Proof. Suppose $l_i \geqslant 2$. Let B be the weighted filtration on W induced by

$$\begin{pmatrix} 0 & \tilde{E}_i \\ 0 & 1 \end{pmatrix}.$$

First, suppose B is the trivial filtration, i.e.,

$$B = \begin{pmatrix} 0 & W \\ 0 & 1 \end{pmatrix}.$$

Then the map from W to $H^0(\tilde{E}_i)$ is injective. Since $\Sigma l_j d_j = d$, we have $d_i \leqslant \frac{1}{2}d$. Hence

$$d + 2(1-g) \leqslant h^0(\tilde{E}_i) \leqslant \deg \tilde{E}_i + 2 \leqslant \tfrac{d}{2} + 2,$$

which is impossible.

The total weight of B is less than or equal to $h^0(\tilde{E}_i) \leqslant d_i + 2$. Hence

$$(7.7.1) \qquad (1+\varepsilon)(d_i + 2) \geqslant (1+\varepsilon)h^0(\tilde{E}_i) \geqslant l_i d_i + \tfrac{\delta}{4},$$

where $\delta = \Sigma_{j \neq i} w_B(m, C_j) \geqslant 0$. We reach a contradiction if $l_i \geqslant 3$ or $d_i \geqslant 3$. So we may assume $l_i = 2$ and $d_i \leqslant 2$.

Now $\deg_{C_i} \bigwedge^2 E \leqslant 4$, so C_i must meet another component C_j. Suppose $P \in \tilde{C}_j$ maps to $C_i \cap C_j$. Then the filtration on $\tilde{E}_j$ induced by B dominates

$$\begin{pmatrix} \tilde{E}_j(-P) & \tilde{E}_j \\ 0 & 1 \end{pmatrix}.$$

Applying (7.2.1) if $d_j \geqslant 2$, and (7.2.2) if $d_j = 1$, we see

$$\text{n.l.c. } w_B(m, C_j) \geqslant \begin{cases} 4 & \text{if } d_j \geqslant 2, \\ 3 & \text{if } d_j = 1. \end{cases}$$

Now if $d_j = 1$, then either C_i or C_j must meet another component C_k, and Lemma 7.3 shows that

$$\text{n.l.c. } w_B(m, C_k) \geqslant 1.$$

In either case, $\delta \geqslant 4$. This contradicts (7.7.1) if $l_i \geqslant 2$ and $d_i = 2$. If $l_i \geqslant 2$ and $d_i = 1$, then C_i is P^1, and hence E_{C_i} has a trivial quotient, contradicting Lemma 7.6. Thus $l_i = 1$ in all cases.

8. Our aim in this section is to show that C_{red} has only nodes as singularities. Let $C' \subseteq C_{\text{red}}$ be a curve.

Lemma 8.1. *If* $h^0(C', E) \leqslant \deg_{C'} E$, *then* $\deg_{C'}(E) \geqslant 20g$.

Proof. Suppose not. Then some component C_j of C must meet C' as we are assuming $d \geqslant 1000g(g-1)$. Consider the weighted filtration B given by $[E \to E_{C'}]$. Then

$$\begin{aligned} \text{n.l.c. } w_B(m, C) &\geqslant \text{n.l.c. } w_B(m, C') + \text{n.l.c. } w_B(m, C_j) \\ &\geqslant 4\deg_{C'}(E) + 1, \end{aligned}$$

by (7.5) and (7.3) respectively. But

$$w(B) = h^0(C', E) \geqslant \deg_{C'}(E),$$

$$\text{n.l.c. } w_B(m, C) \leqslant 4(1+\varepsilon)w(B).$$

Combining these gives

$$4(1+\varepsilon)\deg_{C'}(E) \geqslant 4\deg_{C'}(E) + 1,$$

which is impossible if $\deg_{C'}(E) < 20g$.

Lemma 8.2. *Let $C' \subseteq C_{\mathrm{red}}$ be a curve and let C'' be a component of C'. Then there is a short exact sequence*

$$0 \to L \to E_{C'} \to M \to 0,$$

where L and M are invertible, L and M have nonnegative degree on each component of C', and $\deg_{C''} L > 0$.

Proof. Let $P_1, \cdots, P_k$ be the singular points of C' and let $E' = E_{C'}$. Let Z_i be the common zeros of sections of E' which vanish at P_i. Then Z_i is a finite set, since if $Z_i \supseteq C_j$, the dimension of the image of $H^0(E')$ in $H^0(C_j, E')$ would be at most one. But $\wedge^2 E$ is very ample. By picking a point $P \in C''$ not in any Z_i, we can find a section s which vanishes at P, but not at any singular point. We then let L be the smallest subbundle of E containing S to establish our lemma.

Corollary 8.2.1. *Suppose every line bundle L in $E_{C'}$, which has positive total degree and nonnegative degree on each component of C', satisfies $h^0(C', L) \leqslant \deg_{C'} L$. Then $\deg_{C'} E \geqslant 20g$.*

Proof. We write

$$0 \to L \to E_{C'} \to M \to 0.$$

Since $E_{C'}$ is generated by global sections, M has nonnegative degree on each component of C'. If $\deg_{C'}(M) = 0$, $E_{C'}$ has a trivial quotient, so Corollary 7.6.2 shows C' is smooth and rational, and the hypothesis of Corollary 8.2.1 fails. Hence

$$h^0(C', L) \leqslant \deg_{C'}(L),$$
$$h^0(C', M) \leqslant \deg_{C'}(M).$$

So

$$h^0(C', E) \leqslant \deg_{C'}(E),$$

and Lemma 8.1 applies.

Lemma 8.3. *Let P be a point of C_i. Then the map $\pi_i \colon \tilde{C}_i \to C$ is unramified at P.*

Proof. Suppose not. Let $Q = \pi_i(P)$. Then every section of $\mathcal{O}_{C,Q}$ vanishing at Q vanishes at least twice at P. Thus the hypothesis of Corollary 8.2.1 is satisfied since $(C_i)_{\mathrm{red}}$ is singular. Hence $\deg_{C_i} E \geqslant 20$.

Now consider the filtration on W

$$B = \begin{pmatrix} W_3 & W_2 & W_1 \\ 0 & 1 & 3 \end{pmatrix}$$

induced by

$$\begin{pmatrix} \tilde{E}_i(-3P) & \tilde{E}_i(-2P) & \tilde{E}_i \\ 0 & 1 & 3 \end{pmatrix}.$$

Now $\dim W_1/W_2 \leqslant 2$ as the map from $\tilde{C}_i$ to C is ramified at P. Further $\dim W_2/W_3 \leqslant 2$. Hence $w(B) \leqslant 8$. On the other hand, the induced filtration on $\wedge^2 \tilde{E}_i$ is

$$\begin{pmatrix} (\wedge^2 \tilde{E}_i)((-6+k)P) \\ k \end{pmatrix}.$$

Proposition 6.5 shows that n.l.c. $w_B(m, C) \geqslant 36$. So $4(1+\varepsilon)8 \geqslant 36$, a contradiction.

Lemma 8.4. *C_{red} has no triple points.*

Proof. Suppose three distinct components, say C_1, C_2, C_3, meet at a point P. We let B be the weighted filtration on W induced by $[E \to E_P]$. Then $w(B) \leqslant 2$. Now (7.2.1) and (7.2.2) show that

$$\text{n.l.c. } w_B(m, C_i) \geqslant 3,$$

for $i = 1, 2, 3$ and n.l.c. $w_B(m, C_i) \geqslant 0$ for $i > 3$ and therefore

$$\text{n.l.c. } w_B(m, C) \geqslant 9$$

by (6.4). Hence $4(1+\varepsilon)2 > 9$, a contradiction.

Now if C_1 and C_2 meet at a singular point $P \in C_1$, then $\deg C_1 \geqslant 20$. Using (7.2.1) applied to C_1 and $R = \pi_1^{-1}(P)$, we see

$$\text{n.l.c. } w_B(m, C_1) \geqslant 8,$$

and we obtain a contradiction as before.

Similarly, C_1 cannot have a triple point.

Lemma 8.5. *C has no tacnodes.*

Proof. Suppose that C_1 and C_2 meet at P, and that the tangent lines of C_1 and C_2 are identical. Then the two weighted filtrations induced on W by

$$B_i = \begin{pmatrix} \tilde{E}_i(-2P) & \tilde{E}_i(-P) & \tilde{E}_i \\ 0 & 1 & 2 \end{pmatrix}$$

for $i = 1, 2$ are identical. Call this filtration B.

We may assume $d_1 \leqslant d_2$. Now if $d_1 = 1$, then C_1 is rational and $E_{C_1} \cong \mathcal{O} \oplus \mathcal{O}(1)$. Thus the map from $H^0(C_1, E(-P))$ to $E(-P) \otimes k_P$ is not surjective. So $w(B) \leqslant 5$ if $d_1 = 1$, and $w(B) \leqslant 6$ if $d_1 > 1$.

Now $C_1 \cup C_2$ satisfies the hypothesis of Lemma 8.1, so $d_1 + d_2 \geqslant 20g \geqslant 40$, and hence $d_2 \geqslant 4$. Applying Proposition 6.5, we see that

$$\text{n.l.c.}\, w_B(m, C_i) \geqslant 16,$$

if $d_i \geqslant 4$. On the other hand, if $d_1 \leqslant 4$, the filtration induced by W on $\wedge^2 \tilde{E}_1$ dominates

$$\begin{pmatrix} \wedge^2\tilde{E}_1(-d_1P) & \cdots & \wedge^2\tilde{E}_1(-P) & \wedge^2\tilde{E}_1 \\ 4 - d_1 & \cdots & 3 & 4 \end{pmatrix},$$

since $H^0(C_1, \wedge^2 E((-d_1 - 1)P)) = 0$. Applying Proposition 6.5,

$$\text{n.l.c.}\, w_B(m, C_1) \geqslant d_1^2 + 2(4 - d_1)d_1 \geqslant d_1(8 - d_1).$$

Thus if $d_1 = 1$, then

$$4(5)(1 + \varepsilon) \geqslant \text{n.l.c.}\, w_B(m, C) \geqslant 16 + 7 = 23,$$

a contradiction. If $d_1 \geqslant 2$, then

$$4(6)(1 + \varepsilon) \geqslant \text{n.l.c.}\, w_B(m, C) \geqslant 16 + 12 = 28,$$

a contradiction. So C_1 and C_2 cross transversally.

Finally, if C_1 has a tacnode, then $d_1 \geqslant 8$. A similar argument produces a contradiction once again.

We have established

Proposition 8.6. *C_{red} has only nodes as singularities.*

9. Our main aim in this section is to establish that C is semistable as a curve, and that the map $W \to H^0(C, E)$ is an isomorphism.

We begin with a version of Clifford's Theorem following Saint-Donat.

Lemma 9.1. *Let D be a reduced curve with only nodes, and let L be a line bundle on D generated by global sections. If $H^1(D, L) \neq 0$, there is a curve $C' \subseteq D$ so that*

$$h^0(C', L) \leqslant \tfrac{1}{2} \deg_{C'} L + 1.$$

Proof. Since $H^1(D, L) \neq 0$, $H^0(L^{-1} \otimes \omega_D) \neq 0$. So there is a nonzero $\varphi : L \to \omega_D$. We can find a curve $C' \subseteq D$ so that φ is not identically zero on each component of C', but φ vanishes at all points $C' \cap \overline{D - C'} = \{P_1, \cdots, P_k\}$. Since $\omega_{C'} = \omega_D(-P_1 \cdots -P_k)$, we actually obtain

$$\varphi : L_{C'} \to \omega_{C'}.$$

Choose a basis $s_1,\cdots,s_r$ of $\mathrm{Hom}(L_{C'}, \omega_{C'})$ so that $\varphi = s_1$. We can choose a basis $t_1 \cdots t_p$ of $H^0(L_{C'})$ so that t_1 does not vanish at the zeros of s_1 nor at any singular point of C'. Suppose

$$a_1\langle s_1, t_1\rangle + a_2\langle s_1, t_2\rangle + \cdots = b_2\langle s_2, t_1\rangle + b_3\langle s_3, t_1\rangle + \cdots,$$

where the pairing $\langle s, t\rangle$ is into $H^0(C', \omega_{C'})$. Then $\langle s_1, t\rangle = \langle s, t_1\rangle$, where $t \in H^0(C', L_{C'})$, and s is a linear combination of $s_2,\cdots,s_r$. Since t vanishes where t_1 does, t is a multiple of t_1. Hence s is a multiple of s_1, contradicting the independence of the s_i's. So

$$h^0(L_{C'}) + h^0(\omega_{C'} \otimes L_{C'}^{-1}) \leqslant g + 1,$$
$$h^0(L_{C'}) - h^0(\omega_{C'} \otimes L_{C'}^{-1}) \leqslant \deg_{C'}(L) + 1 - g.$$

Adding the above two inequalities thus gives the desired result.

Lemma 9.2. *Let C' be a proper subcurve of C_{red}. Then*

$$h^0(C', E) > \deg_{C'}(E) + 2(1 - g).$$

Proof. Suppose not. Let $d' = \deg_{C'}(E)$. Consider the filtration B induced on W by $[E \to E_{C'}]$. Since $\dim W = d + 2(1 - g) > d' + 2(1 - g) = w(B)$, B is a nontrivial filtration. Further,

$$\text{n.l.c. } w_B(m, C) \geqslant \text{n.l.c. } w_B(m, C') \geqslant 4d',$$

from Lemma 7.5. Thus

$$\frac{d}{d + 2(1 - g)} \cdot (d' + 2(1 - g)) \geqslant \frac{1}{4}\text{n.l.c. } w_B(m, C) \geqslant d'.$$

This contradicts $d' < d$.

Lemma 9.3. $H^1(C_{\mathrm{red}}, \wedge^2 E) = 0$.

Proof. Suppose not. Lemma 9.1 shows there is a curve $C' \subseteq C_{\mathrm{red}}$ with

$$h^0(C', \wedge^2 E) \leqslant \tfrac{1}{2}\deg_{C'} E + 1.$$

Thus C' is not rational, and therefore Lemma 8.1 shows $\deg_{C'}(E) \geqslant 20g$. On the other hand, E is generated by global sections, so we can find a nowhere vanishing section of E over C':

$$0 \to \mathcal{O}_{C'} \to E_{C'} \to (\wedge^2 E)_{C'} \to 0. \tag{9.3.1}$$

Hence

$$h^0(C, E) \leqslant \frac{\deg_{C'}(E)}{2} + 2 \leqslant \deg_{C'}(E) + 2 - 10g.$$

In particular,

$$h^0(C', E) < \deg_{C'}(E) + 2(1-g),$$

which contradicts Lemma 9.2.

Lemma 9.4. $H^1(C_{\mathrm{red}}, E) = 0.$

Proof. Suppose not. Then there is a nonzero map $\varphi: E \to \omega_{C_{\mathrm{red}}}$. Using the techniques of the proof of Lemma 9.1, we can find a curve C' of C_{red} of genus g' and a map $\varphi: E \to \omega_{C'}$ which is nonzero on each component of C'. Note $g' \geqslant 2$, since otherwise E would have a trivial quotient. Then from (9.3.1),

$$h^0(C', E) \leqslant h^0\left(C', \wedge^2 E\right) + 1 \leqslant \deg_{C'}(E) + 1 - g' + 1,$$

since $H^1(C', \wedge^2 E) = 0$. We see $\deg_{C'}(E) \geqslant 20g$ from Lemma 8.1. Further $g' \leqslant 2g$, since otherwise

$$h^0(C', E) < \deg_{C'}(E) + 2(1-g),$$

contradicting Lemma 9.2.

Now consider the filtration induced on W by $[E \to \omega_{C'}]$. We have $h^0(C', \omega_{C'}) = g'$, so $\Sigma r_i \leqslant g'$. We also have

$$\text{n.l.c. } w_B(m, C) \geqslant 2\deg_{C'}(E),$$

from Lemma 7.4. So

$$4(2g) \geqslant 4g' \geqslant 4\sum r_i \geqslant 2\deg_{C'}(E) \geqslant 40g.$$

Hence we reach a contradiction.

Corollary 9.5. *C is reduced and $W = H^0(C, E)$.*

Proof. Consider $\mathscr{I}$, the ideal defining C_{red} in C. $\mathscr{I}$ is supported at a finite number of points. We claim

$$W \cap H^0(C, \mathscr{I} \cdot E) \neq 0. \tag{9.5.1}$$

Let g' be the genus of C_{red}, and l be the length of $\mathscr{I}$. Then $g' = g + l$. Thus if $l > 0$, then

$$H^0(C_{\mathrm{red}}, E) < \deg E + 2(1-g) = \dim W,$$

since $H^1(C_{\mathrm{red}}, E) = 0$. So (9.5.1) is established.

Now consider the filtration B induced on W by

$$\begin{pmatrix} E \cdot \mathscr{I} & E \\ 0 & 1 \end{pmatrix}.$$

Then $\Sigma r_i < \dim W$, but n.l.c. $w_B(m, C) = 4d$. We have again reached a contradiction.

Proposition 9.6. *C is semistable.*

Proof. Suppose $C = C' \cup C''$, where $C' \cap C''$ is a point P, and C'' is a chain of rational curves. The genus of C' is g, so

$$h^0(C', E) = \deg_{C'}(E) + 2(1 - g).$$

We have contradicted Lemma 9.2. So C is semistable.

10. Our purpose in this section is to establish some properties of E.

Proposition 10.1. *Let L be a quotient of E. Then* $2\deg_C L \geqslant \deg_C E$.

Proof. Let $M = \ker(E \to L)$. Consider the filtration B:

$$\begin{pmatrix} M & E \\ 0 & 1 \end{pmatrix}.$$

It is easy to see B is destabilizing if $2\deg L < \deg E$.

Now suppose $C' \subseteq C$ is a chain of rational curves $C_1 \cup \cdots \cup C_l$, where the C_i are nonsingular rational, and C_i meets only C_{i-1} and C_{i+1}. We further suppose that $C'' = \overline{C - C'}$ is connected, and that C'' meets C_1 at one point P and C_l at one point Q, and meets no other C_i.

Lemma 10.2. $\deg_{C'}(E) \leqslant 2$.

Proof. Suppose not. The genus of C'' is $g - 1$. Consider the filtration B induced on W by $[E \to E_{C''}]$. First, notice that since $3 \leqslant d' = \deg_{C'} E$, and E is generated by global sections over C', $H^0(C', E) > 4$. Hence the filtration B is nontrivial. We claim that

$$\text{n.l.c. } w_B(m, C') \geqslant 8. \tag{10.2.1}$$

Suppose (10.2.1) has been established. Let $d'' = d - d'$. Then $h^0(C'', E) = d'' + 2(2 - g)$, since C'' has genus $g - 1$. So

$$\frac{d}{d + 2(1 - g)}[d'' + 2(2 - g)] \geqslant d'' + 2.$$

After a short computation, we obtain $d' \leqslant 2$.

To establish (10.2.1), consider case one: $l = 1$. If we let $R = P + Q$, and apply (7.2.1) if $d' \geqslant 4$ and (7.2.2) if $d' = 3$, then we obtain (10.2.1). Next, consider case two: $d' = 3$. We claim that $H^0(C', \bigwedge^2 E(-2P - 2Q)) = 0$. Let s be such a nonzero section. We must have $\deg_{C_1} \bigwedge^2 E = 1$ or $\deg_{C_l} \bigwedge^2 E = 1$, since $d' = 3$. Say $\deg_{C_1} \bigwedge^2 E = 1$. Then s vanishes on C_1, and therefore on $C_1 \cap C_2$. If $l = 2$, s vanishes twice at Q and once at $C_1 \cap C_2$, and so s vanishes. If $l = 3$, then $\deg_{C_3}(\bigwedge^2 E) = 1$. So s vanishes on C_3 also. But then s

vanishes on C_2 as well, since $\deg_{C_2} E = 1$. Hence $H^0(C', \wedge^2 E(-2P - 2Q)) = 0$. So the filtration induced by B on $\wedge^2 E_{C'}$ is dominated by

$$\begin{pmatrix} E(-P-Q) & E \\ 1 & 2 \end{pmatrix}.$$

Applying Lemma 7.2, (10.2.1) holds, and $d' < 2$.

By applying cases one and two to subchains of C, we may assume that E does not have degree 3 on any subchain, and that $\deg_{C_i} E \leqslant 2$ for each i. It follows that the degree of E' on each C_i is two. But applying Lemma 7.2, we see

$$w_B(m, C_1) \geqslant 4, \qquad w_B(m, C_l) \geqslant 4.$$

Then using Lemma 6.4, (10.2.1) holds, and $d' \leqslant 2$.

Now suppose the stable model C_s of C is an irreducible curve with a node N. Let $\tilde{C}_0$ be the normalization of C_s, and $d' = \deg \tilde{E}_0$.

Lemma 10.3. *Assume d to be odd. Let L be a quotient of $\tilde{E}_0$. Then $2\deg L \geqslant d - 1$ if $d = d'$, and $\tilde{E}_0$ is semistable if $d \neq d'$.*

Proof. Suppose for some $\delta \geqslant 0$

$$2\deg L \leqslant d - 2 - \delta. \tag{10.3.1}$$

Then

$$h^0(L) \leqslant \tfrac{1}{2}d + 1 - g + \tfrac{1}{2}\delta. \tag{10.3.2}$$

Indeed, if $h^1(L) = 0$, (10.3.2) follows from Riemann-Roch. If $h^1(L) \neq 0$, then $h^0(L) \leqslant g - 1$. But $d' \geqslant 20g$ (Lemma 8.1). So (10.3.2) follows in any case.

Now consider the weighted filtration B on W induced by $[\tilde{E} \to L]$. First, suppose $C = C_s$, and let $P, Q \in \tilde{C}_0$ be the points corresponding to N. Now $\tilde{E}_P$ and $\tilde{E}_Q$ are identified with E_N. Under this identification, $L_P \neq L_Q$ as quotients. Indeed, if $L_P = L_Q$, then L descends to a line bundle on C. This possibility is ruled out by Proposition 10.1. Thus if $M = \ker(\tilde{E}_0 \to L)$, then B is dominated by the filtration induced by

$$B' = \begin{pmatrix} M(-P-Q) & \tilde{E}_0 \\ 0 & 1 \end{pmatrix}.$$

From Lemma 7.4 we see

$$\text{n.l.c. } w_B(m, C_0) \geqslant 2d + 2.$$

Combining these inequalities with n.l.c. $w_B(m, C) \leqslant 4dw(B)/n$, we obtain

$$\frac{d}{d + 2(1-g)}\left(\frac{d}{2} + 1 - g\right) \geqslant \frac{1}{4}(2d + 2). \tag{10.3.3}$$

A short computation shows (10.3.3) is impossible.

Next suppose that $d \neq d'$ and that $\tilde{E}_0$ is not semistable. Since $d - d' \leqslant 2$ and d is odd, we may assume there is an L satisfying (10.3.1) with $\delta = 1$. Now letting $C' = \overline{C - C_0}$, we see

$$\text{n.l.c. } w_B(m, C') \geqslant 2, \qquad \text{n.l.c. } w_B(m, C_0) \geqslant 2d'.$$

As above, this leads to

$$\text{(10.3.4)} \qquad \frac{d}{d + 2(1-g)}\left(\frac{d}{2} + 1 - g - \frac{1}{2}\right) \geqslant \frac{1}{4}(2d' + 2).$$

A short computation shows (10.3.4) cannot occur.

Thus we have established (1.3.1), (1.3.3) and (1.3.4).

References

[1] P. Deligne & D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. No. 36 (1969) 75–109.

[2] D. Gieseker, *Global moduli for surfaces of general type*, Invent. Math. **43** (1977) 233–282.

[3] K. Knopp, *Theorie und Anwendung der unendlichen Reichen*, Springer, Berlin, 1964.

[4] I. L. Morrison, *Projective stability of ruled surfaces*, Invent. Math. **56** (1980) 269–304.

[5] D. Mumford, *Varieties defined by quadratic equations*, Questions on Algebraic Varieties, (C.I.M.E., 1969), Edizioni Cremonese, Roma, 1970, 31–100.

[6] ——, *Stability of projective varieties*, Enseignement Math. **23** (1977) 39–110.

[7] M. Nagata, *On self-intersection number of a section on a ruled surface*, Nagoya Math. J. **37** (1970) 191–196.

[8] J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. **61** (1955) 197–278.

University of California, Los Angeles
Columbia University