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VARIATIONAL PROBLEMS AND ELLIPTIC MONGE — AMPÈRE EQUATIONS

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In the present paper we study the n-dimensional variational problems connected with the Dirichlet problem for n-dimensional Monge-Ampère equation

$$(*) \qquad \det\|u_{ij}\| = f(x_1, x_2, \cdots, x_n)$$

with zero boundary condition and prove the existence and uniqueness of an absolute minimum for this problem. This minimum is a generalized solution of the equation $(*)$ belonging to the class of all general convex functions.

The technique of convex hypersurfaces and bodies used in the geometric theory of elliptic Monge-Ampère equations turns out to be also essential for the investigations of the variational problems considered below. Therefore we also included the brief exposition of some necessary concepts and results of this theory.

CONTENTS

1. Introduction

1.1. Statement of problems. This paper is devoted to proving an existence and uniqueness of the absolute minimum for the functionals whose Euler equation is given by the Monge-Ampère equation

$$(1.1) \qquad \det\|u_{ij}\| = f(x_1, x_2, \cdots, x_n).$$

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Such a functional is given by Courant and Hilbert [14] for the case of functions of two variables:

$$(1.2) \qquad E(u) = -\iint_G \left(u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx} + 6fu\right) dx\, dy,$$

where G is an open convex bounded domain in the xy-plane. Unfortunately the functional (1.2) does not give any idea concerning the functional spaces in which this variational problem belongs, also it does not suggest the generalization for functions of n-variables.

We found in [4], [6] another functional

$$(1.3) \qquad I(u) = -\iint_G \left(u\left(u_{xx}u_{yy} - u_{xy}^2\right) - 3fu\right) dx\, dy,$$

whose Euler equation is

$$(1.4) \qquad u_{xx}u_{yy} - u_{xy}^2 = f(x, y).$$

If ∂G is a C^2-curve and if $u(x, y) \in C^2(\overline{G})$ and satisfies the condition $u|_{\partial G} = 0$, then $E(u) = 2I(u)$.

The functional (1.3) is closely connected with the Monge-Ampère operator $u_{xx}u_{yy} - u_{xy}^2$ and admits a simple geometric interpretation by means of the tangential mapping constructed by the function u. This functional also admits a simple natural generalization to functions of n variables:

$$(1.5) \qquad I_n(u) = -\int_G \left[u \det\|u_{ij}\| - (n + I)f(x)u\right] dx,$$

where G is an open convex bounded domain in Euclidean space E^n. Let $x_1, x_2, \cdots, x_n$ be Cartesian coordinates in E^n and $dx = dx_1 dx_2, \cdots, dx_n$. The formal Euler equation for $I_n(u)$ is the equation

$$(1.6) \qquad \det\|u_{ij}\| = f(x_1, x_2, \cdots, x_n).$$

In the papers [4], [6] we studied the two-dimensional variational problem for the functional (1.3) and proved that the absolute minimum for this problem is a generalized solution of the equation (1.4) with the boundary condition

$$(1.7) \qquad u|_{\partial G} = 0.$$

But some points used in [4], [6] do not generalize to the n-dimensional case and new ideas and techniques are therefore required. In the present paper we study the n-dimensional variational problem for the functional (1.5) and prove the existence and uniqueness of an absolute minimum of this problem. This minimum is a generalized solution of the equation (1.1). The proof of this result is based on deeper ideas and geometric constructions than those used in

[4], [6]; these ideas and constructions are based on new estimates for the main functional, and on the concepts and properties of dual convex hypersurfaces in the special space. They uncover the fundamental geometric contents of this variational problem.

If the function f is sufficiently smooth and strictly positive and if the boundary of G is also sufficiently smooth and strictly convex (all principal normal curvatures of ∂G are positive), then from Pogorelov [17], Cheng and Yau [12], [13] and our results it follows that the generalized solution of the considered variational problem is also smooth.

1.2. Preliminary considerations. Since the Euler equation (1.6) for the functional $I_n(u)$ has the second order instead of the fourth one, then the variational problems for this functional are degenerate. Therefore there is only one boundary condition in variational problems for $I_n(u)$. (Note that there are two boundary conditions in nondegenerate variational problems of the second order.) This fact influences the semiboundedness, continuity and asymptotic behavior of $I_n(u)$ if $\|u\| \to \infty$. Here $\|u\|$ is the norm of the function u in the corresponding function set. We can see for example that $I_n(u)$ is nonbounded from any side if we consider it for all functions $u \in C^2(G) \cap C(\overline{G})$ and if $u|_{\partial G} = 0$. Therefore it is natural to construct the domain of definition for $I_n(u)$ by taking into account the properties of elliptic solutions of the corresponding boundary value problem for the Monge-Ampère equation (1.6). In this paper we consider elliptic solutions of the Dirichlet problem for the equation (1.6) with the zero boundary condition.

Evidently all C^2 elliptic solutions $u(x_1, \cdots, x_n)$ have the fixed sign second differential everywhere in G. Hence they are either nonpositive convex or nonnegative concave functions in G. Therefore the variational problem corresponding to the absolute minimum (maximum) of $I_n(u)$ should be considered in the class of nonpositive convex (nonnegative concave) functions. It is clear that it is sufficient to consider only the problem of finding the absolute minimum of $I_n(u)$. The functional $I_n(u)$ can be extended to the class of all nonnegative convex functions, vanishing on ∂G, by means of the technique of generalized elliptic solutions for Monge-Ampère equations (see §§2 and 6 of this paper). Thus the main variational problem for the functional $I_n(u)$ is reduced to the establishment of the absolute minimum of the extension of $I_n(u)$ to the class of all convex functions vanishing on ∂G.

But the extension of $I_n(u)$ turns out to be discontinuous. This fact appears because the second boundary condition for the comparison functions is excluded. Moreover the problem of finding the first variation of $I_n(u)$ is connected with the extension of $I_n(u)$ to the class of all nonpositive continuous functions vanishing on ∂G. There required more deep and refined ideas and techniques for the corresponding extension of the functional $I_n(u)$ and the proofs of the existence of the absolute minimum and the expression of the first variation for the extended functional $I_n(u)$ than we had briefly considered above. We present all these problems in §§2, 3 and 4. In §5 we consider a development and generalizations of the main results from §§2, 3 and 4. §6 is devoted to the presentation of the necessary information about the geometric theory of the Monge-Ampère equations.

2. The functional $I_H(u)$ and its properties.

In this section we construct the extension $I_H(u)$ of the functional $I_n(u)$ to the set of all nonpositive continuous functions vanishing on ∂G and establish the continuity of $I_H(u)$. Here H is any convex subdomain of G distant from ∂G on some positive number and G is a given convex bounded domain in E^n.

2.1. Normal mapping and R-curvature of convex functions. For a detailed exposition see [2], [6, §§16, 17, 20], [7] and [9].

(A) *Normal mapping of convex functions.* Let $x_1, x_2, \cdots, x_n, x_{n+1}$ be Cartesian coordinates in $(n+1)$-dimensional Euclidean space E^{n+1}. Let E^n be the hyperplane $x_{n+1}=0$ in E^{n+1}, and let G be an open convex bounded domain in E^n. We introduce the notations $x_{n+1}=z$; $x=(x_1, x_2, \cdots, x_n)$ is a point of E^n and $(x, z)=(x_1, x_2, \cdots, x_n; z)$ is a point of E^{n+1}; $z(x)$ is a function z: $G \to R$ with graph S_z; $W^+(G)$ is the set of all convex functions on G; $W^-(G)$ is the set of all concave functions on G. If $z(x) \in W^{\pm}(G)$, then S_z is called a *convex* (*concave*) *hypersurface*.

Pick some arbitrary convex function $z(x) \in W^+(G)$,[1] and let α be a supporting hyperplane to S_z, with equation

$$Z - z^0 = \sum_{i=1}^{n} p_i^0 (X_i - x_i^0),$$

where $(x_1^0, \cdots, x_n^0; z^0) \in S_z \cap \alpha$, and (X, Z) is an arbitrary point of α.

The point $\chi_z(\alpha) = p_0 = (p_1^0, p_2^0, \cdots, p_n^0) \in \mathbf{R}^n$ is called the *normal image of the supporting hyperplane* α.

We construct the set $\chi_z(x_0) = \bigcup_\alpha \chi_z(\alpha)$, where α runs through all supporting hyperplanes to S_z at the point $(x_0, z(x_0)) \in S_z$. The set $\chi_z(x_0)$ is called the *normal image of the point* x_0 (relative to the function $z(x)$). It is clearly a closed

[1] Analogous for concave functions.

convex subset of R^n. Finally we put for any subset $e \subset G$,

$$\chi_z(e) = \bigcup_{x_0 \in e} \chi_z(x_0)$$

and call it *the normal image of the subset* $e \subset G$ (with respect to the function $z(x)$).

The main properties of the normal mapping. These are as follows.

(a) The set $\chi_z(e)$ is a closed subset of R^n for each closed subset e of the domain G; the set $\chi_z(e)$ is a Lebesgue measurable subset of R^n for each Borel subset $e \subset G$.

(b) Let $z_1(x)$ and $z_2(x)$ be convex functions, coinciding on ∂G, and $z_1(x) \leqslant z_2(x)$ for all $x \in G$. Then

$$\chi_{z_2}(G) \subset \chi_{z_1}(G). \tag{2.1}$$

(c) If $z(x) \in W^+(G) \cap C^2(G)$, then the normal mapping can be considered as a mapping of points, namely, the tangential mapping $\chi_z(x) = \operatorname{grad} z(x)$.

(B) *R-curvature.* Let $R(p) > 0$ be a locally summable function on R^n. The function of sets

$$\omega(R, z, e) = \int_{\chi_z(e)} R(p)\, dp, \qquad e \subset G, \tag{2.2}$$

is nonnegative and completely additive on the ring of Borel subsets of the convex domain G for all convex functions $z(x) \in W^+(G)$. This function is called *the R-curvature of the convex function* $z(x)$.

If $R(p) = 1$, then the 1-curvature of $z(x) \in W^+(G)$ is called the measure (or area) of the normal mapping and simply denoted by $\omega(z, e)$. If $R(p) = (1 + |p|^2)^{-(n+2)/n}$ then the corresponding R-curvature coincides with the area of the Gauss mapping of the hypersurface S_z. We set

$$A(R) = \int_{R^n} R(p)\, dp. \tag{2.3}$$

The properties of R-curvature.

(a) It is clear that $A(R) > 0$; note the case $A(R) = +\infty$ is not excluded.

(b) The inequality $\omega(R, z, G) \leqslant A(R)$ holds for all convex functions $z(x) \in W^+(G)$.

(c) If $z(x) \in W^+(G) \cap C^2(G)$, then

$$\omega(R, z, e) = \int_e \det\|z_{ij}\| R(\operatorname{grad} z)\, dx. \tag{2.4}$$

(d) *Weak convergence of R-curvatures.* If the sequence of convex functions $z_n(x) \in W^+(G)$ converges to the convex function $z(x) \in W^+(G)$ in all points

$x \in G$, then

$$\lim_{n\to\infty} \int_G \varphi(x)\omega(R, z, de) = \int_G \varphi(x)\omega(R, z, de), \tag{2.5}$$

where $\varphi(x)$ is any continuous function in G vanishing outside of some compact subset M distant from ∂G on a positive number.

(e) *Estimates of convex functions.* Let the following conditions be fulfilled: If $V(\omega_0) = \{z(x)\}$ is the set of all functions $z(x) \in W^+(G)$ satisfying the conditions

(2.6) (e.1) $$-\infty < m = \text{const} \leqslant z\,|_{\partial G} \leqslant M = \text{const} < +\infty,$$

(2.7) (e.2) $$\omega(R, z, G) \leqslant \omega_0 = \text{const} < A(R),$$

then the inequalities

$$m - T_R(\omega_0)d(G) \leqslant z(x) \leqslant M \tag{2.8}$$

hold for all points $x \in G$.

Here T_R: $[0, A(R)) \to R$ denotes the inverse for the function

$$g_R(r) = \int_{|p|\leqslant r} R(p)\,dp,$$

and $d(G)$ is the diameter of G.

Example. If $R(p) = 1$, then $g_R(r) = \mu_n r^n$, where μ_n is the volume of the unit n-ball. Therefore $T_R(g) = T_1(g) = (g/\mu_n)^{1/n}$.

2.2. The operator F_H and its properties. Let G be an open convex bounded domain in E^n and H be a convex subdomain of G distant from G on the positive distance h_H. Let $\overline{H}$ and $\overline{G}$ be the closures of H and G. Denote by $C_0^-(\overline{G})$ the closed subset of the space $C(\overline{G})$ consisting of all nonpositive continuous functions vanishing on ∂G. The operator F_H considered in this subsection maps the set $C_0^-(\overline{G})$ in the special class of convex functions which will be introduced below. This operator will be used for the extension of the functional $I_n(u)$ to the set $C_0^-(\overline{G})$ (see §2.3).

Now consider the construction of F_H and its properties. Let

$$v(x) = \begin{cases} u(x) & \text{if } x \in \overline{H}, \\ 0 & \text{if } x \in \overline{G}\setminus\overline{H} \end{cases} \tag{2.9}$$

for any function $u(x) \in C_0^-(\overline{G})$, i.e. $v(x) = u(x)\varphi_{\overline{H}}(x)$, where $\varphi_{\overline{H}}(x)$ is the characteristic function of the set $\overline{H}$. If S_v is the graph of $v(x)$,[2] then the boundary of $\overline{\text{Co}}\{S_v\}$ consists of two parts $\overline{G}$ and the graph of S_w of some

[2] All the graphs of functions are considered in the space E^{n+1} (see §2.1).

convex function $w(x)$; clearly

$$w(x) \in W^{+}(\overline{G}) \cap C_0^{-}(\overline{G}).$$

We say in this case the function $w(x)$ spans $u(x)$ on the set $\overline{H}$ from below, and denote by F_H the operator mapping any function $u(x) \in C_0^{-}(\overline{G})$ in the corresponding convex function $w(x)$.

The properties of F_H.

(1) Every supporting hyperplane β to S_w passing through the point $(x_0, w(x_0))$, where $x_0 \in G\setminus\overline{H}$, contains at least some line segment AB such that $AB \subset \beta \cap S_w$, $A \in \partial G$ and $B' \in \overline{H}$, where B' is the projection of the point $B \in S_w \cap \beta$ on the hyperplane E^n.

The proof follows directly from the definition of S_w and the well-known properties of a convex hull.

(2) The equality

$$\text{mes}\,\chi_w(w, G\setminus\overline{H}) = 0 \tag{2.10}$$

holds for every function $w(x) = F_H(u(x))$.

Let β be any supporting hyperplane of S_w contracting with S_w in the point $(x_0, w(x_0))$, where $x_0 \in G\setminus\overline{H}$. Then from property (1) it follows that β is a singular supporting hyperplane of S_w. Since the spherical image of all supporting hyperplanes of every convex hypersurface has the zero measure (see [11, §4]), then

$$\text{mes}\,\chi_w(w, G\setminus\overline{H}) = 0. \tag{2.11}$$

Now we denote by $W_H^{+}(\overline{G})$ the set $F_H(C_0^{-}(\overline{G}))$. It is evident that

$$W_H^{+}(\overline{G}) \subset W^{+}(\overline{G}) \cap C_0^{-}(\overline{G})$$

and the set $W^{+}(\overline{G}) \subset C_0^{-}(\overline{G}) \cap W_H^{+}(\overline{G})$ is not empty.

(3) The equality

$$\chi_w(G) = \chi_w(\overline{H}) \tag{2.12}$$

holds for all functions $w(x) \in W_H^{+}(\overline{G})$.

The proof follows directly from property (1).

(4) The equality

$$w(x) = F_H(w(x)) \tag{2.13}$$

holds if and only if $w(x) \in W_H^{+}(\overline{G})$.

Proof. If the function $w(x) \in C_0^{-}(G)$ satisfies equation (2.13), then from the definition of the operator F_H it follows that $w(x) \in W_H^{+}(\overline{G})$. The converse assertion follows from the properties of the convex hull.

(5) The set $W_H^+(\bar{G})$ is a closed subset of the space $C(\bar{G})$.

Proof. Let $w(x)$ be the limit of functions $w_1(x), w_2(x), \cdots, w_m(x), \cdots$ belonging to $W_H^+(\bar{G})$ in the space $C(\bar{G})$. Evidently $w(x)$ is a convex function belonging to $C_0^-(\bar{G})$. The considered property will be proved if we establish the equality

$$w(x) = F_H(w(x)) \tag{2.14}$$

(see property (4)). Let

$$v(x) = w(x)\varphi_{\bar{H}}(x), \qquad v_m(x) = w_m(x)\varphi_{\bar{H}}(x). \tag{2.15}$$

The restrictions $v_m(x)$ and $v(x)$ on the convex compact set $\bar{H}$ are convex functions and $\lim_{m\to\infty} v_m(x) = v(x)$ for all $x \in \bar{H}$ and $v_m(x) = v(x) = 0$ for all $x \in \bar{G}\setminus\bar{H}$. Therefore

$$\overline{\mathrm{Co}}\,\{S_w\} = \lim_{m\to\infty} \overline{\mathrm{Co}}\,\{S_{w_m}\} = \lim_{m\to\infty} \overline{\mathrm{Co}}\,\{S_{v_m}\} = \overline{\mathrm{Co}}\,\{S_v\}$$

because the equality

$$\overline{\mathrm{Co}}\,\{S_{w_m}\} = \overline{\mathrm{Co}}\,\{S_{v_m}\} \tag{2.16}$$

follows from the condition that $w_m(x) \in W_H^+(\bar{G})$ for all positive integers m. Property (5) is proved since (2.16) is equivalent to equality (2.14).

(6) The set $\chi_w(G)$ is contained in the n-dimensional ball $|p| \leqslant \|w(x)\|/h_H$ for all functions $w(x) \in W_H^+(\bar{G})$.

Proof. Let α be the supporting hyperplane of the graph S_w of any function $w(x) \in W_H^+(\bar{G})$. Then there exists the point $x_0 \in \bar{H}$ such that the point $(x_0, w(x_0))$ belongs to α. Note that $\mathrm{dist}(x_0, \partial G)$ is not less than $h_H = \mathrm{dist}(\bar{H}, \partial G) > 0$.

Let K_{x_0} be the convex cone with the vertex $(x_0, w(x_0))$ and the base $U(x_0, h_H)$, where $U(x_0, h_H)$ is the closed n-ball with the center x_0 and the radius h_H. Let k_{x_0} be the convex function defining K_{x_0}. Then

$$\chi_w(\alpha) \subset \chi_{k_{x_0}}(U(x_0, h_H)).$$

The set $\chi_{k_{x_0}}(U(x_0, h_H))$ is the n-dimensional ball with the center $0(0, 0, \cdots, 0)$ and the radius $\rho = w(x)/h_H$. Therefore $\chi_w(G)$ is contained in the n-dimensional ball $|p| \leqslant \|w(x)\|/h_H$ in R^n for all functions $w(x) \in W_H^+(G)$.

(7) From (6) it follows directly that any function $w(x)$ belonging to $W_H^+(G)$ satisfies the Lipschitz condition with the constant $\|w(x)\|/h_H$, i.e.

$$|w(x+q) - w(x)| \leqslant \frac{\|w(x)\|}{h_H}\,|q|,$$

where x and $x + q$ are any points of G.

(8) The operator F_H: $C_0^-(\bar{G}) \to W_H^+(\bar{G})$ is continuous.

Proof. Let the functions $u_n(x) \in C_0^-(\bar{G})$ converge uniformly to the function $u(x) \in C_0^-(\bar{G})$. Take any number $\varepsilon > 0$ and consider two functions

$$v_\varepsilon^{(1)}(x) = \begin{cases} 0 & \text{if } x \in \bar{G}\setminus\bar{H} \text{ or if } u(x) \geqslant -\varepsilon, \\ u(x) + \varepsilon & \text{if } u(x) < -\varepsilon \end{cases}$$

and

$$v_\varepsilon^{(2)}(x) = \begin{cases} 0 & \text{if } x \in \bar{G}\setminus\bar{H}, \\ u(x) - \varepsilon & \text{if } x \in \bar{H}. \end{cases}$$

Let $v(x) = u(x)\varphi_{\bar{H}}(x)$ and $v_n(x) = u_n(x)\varphi_{\bar{H}}(x)$ be the functions considered by definition of the operator F_H (note that $\varphi_{\bar{H}}(x)$ is the characteristic function of the set $\bar{H}$). Then $v_\varepsilon^{(2)}(x) = v(x) - \varepsilon$ for all $x \in \bar{H}$ and

$$v_\varepsilon^{(1)}(x) = \begin{cases} v(x) + \varepsilon & \text{if } u(x) < -\varepsilon \\ 0 & \text{if } x \in \bar{H} \text{ and } u(x) \geqslant -\varepsilon \end{cases}$$

also for all $x \in \bar{H}$.

Since $v_n(x)$ uniformly converge to $v(x)$ in G, then there exists the natural number N such that

$$v_\varepsilon^{(2)}(x) \leqslant v_n(x) \leqslant v_\varepsilon^{(1)}(x)$$

for all $n \geqslant N$ and $x \in \bar{G}$. From the definition of the operator F_H it follows that

$$F_H\big(v_\varepsilon^{(2)}(x)\big) \leqslant w_n(x) = F_H(u_n(x)) \leqslant F_H\big(v_\varepsilon^{(1)}(x)\big)$$

for all $n \geqslant N$ and $x \in \bar{G}$. Since

$$\lim_{\varepsilon\to 0} F_H\big(v_\varepsilon^{(2)}(x)\big) = \lim_{\varepsilon\to 0} F_H\big(v_\varepsilon^{(1)}(x)\big) = w(x),$$

then

$$F_H(u(x)) = w(x) = \lim_{n\to\infty} w_n(x) = \lim_{n\to\infty} F(u_n(x)).$$

Property 8 is proved.

2.3. The functional $I_H(u)$. Let H be a convex subdomain of a given convex bounded domain G in E^n such that $\operatorname{dist}(\bar{H}, \partial G) = h_H > 0$. Let $u(x) \in C_0^-(\bar{G})$ and $w(x)$ be the convex function constructed above by means of $u(x)$ (see §2.2).

Now we define the functional

$$\phi_H(u) = -\int_G u\omega(w, de) \tag{2.17}$$

on the set $C_0^-(\overline{G})$, where $\omega(w, e)$ is the measure of the normal mapping of the convex function $w(x) = F_H(u(x))$. From property (2) (see §2.1) it follows that

$$\omega(w, G\setminus \overline{H}) = 0 \tag{2.18}$$

for all functions $w(x) = F_H(u(x))$ (see §2.2, equality (2.10)).

Letting $\psi(e)$ be a nonnegative completely additive set function on the subsets of G and $\psi(G) < +\infty$, we define the new function of sets

$$\psi_H(e) = \psi(e \cap H). \tag{2.19}$$

Clearly, $\psi_H(e)$ is a nonnegative completely additive set function on the subsets of G and

$$\psi_H(G\setminus H) = 0. \tag{2.20}$$

We now introduce the functionals

$$\tau_H(u) = \int_G u\psi_H(de) \tag{2.21}$$

and

$$I_H(u) = \phi_H(u) + (n+1)\tau_H(u) \tag{2.22}$$

on the set $C_0^-(\overline{G})$.

The properties of the functionals ϕ_H, τ_H and I_H.

Theorem 1. *The inequalities and equalities*

$$\phi_H(w) = \phi_H(u), \tag{2.23}$$

$$\tau_H(w) \leqslant \tau_H(u), \tag{2.24}$$

$$I_H(w) \leqslant I_H(u) \tag{2.25}$$

hold for all functions $u(x) \in C_0^-(\overline{G})$ and convex functions $w(x) = F_H(u(x))$.

Remark. Let $\psi(e) \geqslant C_0 \operatorname{mes}(e)$ for every Borel subset $e \subset G$, where $C_0 =$ const > 0. Then the equality can hold in (2.24) and (2.25) if and only if the restriction of $u(x)$ on the n-dimensional convex body $\overline{H}$ is some convex function, i.e.,

$$u(x)\,|_{\overline{H}} = w(x)\,|_{\overline{H}}, \tag{2.26}$$

where $w(x) = F_H(u(x))$.

Proof. From the equalities (2.18) and (2.20) it follows that

$$\phi_H(u) = -\int_{\overline{H}} u(x)\omega(w(x), de), \tag{2.27}$$

$$\tau_H(u) = \int_{\overline{H}} u(x)\psi_H(de). \tag{2.28}$$

Since $u(x) \geqslant w(x)$ for all $x \in \overline{H}$, then from (2.28) we obtain

$$\tau_H(u) \geqslant \int_H w(x)\psi_H(de) = \int_G w(x)\psi_H(de) = \tau_H(w).$$

It is clear that

$$\tau_H(u(x)) = \tau_H(w(x)) \tag{2.29}$$

if $u(x)$ and $w(x)$ coincide on the set $\overline{H}$. The condition $\psi(e) \geqslant C_0 \operatorname{mes}(e)$ (see Remark) and equality (2.29) yield conversely $u(x) = w(x)$ for all $x \in \overline{H}$.

We denote by H_u the set of points $x \in \overline{H}$, where $u(x) = w(x)$ and by S_{H_u} the part of the graph of $u(x)$ for $x \in H_u$. Every supporting hyperplane α of the graph of the function $w(x)$ has at least one common point with the set S_{H_u}. Therefore $\chi_w(w, G\setminus H_u)$ consists only of the images of singular supporting hyperplanes to the graph of $w(x)$. Thus from property (2) (see §2.2) it follows that $\omega(w, G\setminus H_u) = 0$. Hence

$$\begin{aligned}\Phi_H(u) &= \int_G u\omega(w, de) = \int_{H_u} u\omega(w, de) = \int_{H_u} w\omega(w, de)\\ &= \int_G w\omega(w, de) = \phi_H(w).\end{aligned}$$

The inequality (2.25) now follows directly from (2.23) and (2.24). Theorem 1 is proved.

Theorem 2. *The functionals $\phi_H(e)$, $\tau_H(e)$ and $I_H(e)$ are continuous on the set $C_0^-(\overline{G})$.*

Let the functions $u_1(x), u_2(x), \cdots, u_n(x), \cdots$ belong to $C_0^-(\overline{G})$ and uniformly converge to the function $u(x) \in C_0^-(\overline{G})$. Then the set functions $\omega(w_n, e)$ converge weakly to the set function $\omega(w, e)$ (see §2.1), where $w_n = F_H(u_n)$ and $w = F_H(u)$ are convex functions belonging to $W_H^+(\overline{G})$. Now using the facts mentioned above and property (2) (see §2.2) we obtain the proof of Theorem 2 by the standard considerations.

Thus we can seek the functions realizing the absolute minimum of the continuous functional $I_H(u)$ only in the set of convex functions $W_H^+(\overline{G})$.

3. Variational problem for the functional $I_H(u)$

From §2 it follows that the absolute minimum of the functional $I_H(u)$ can be reached only for the convex functions $w(x) \in W_H^+(\overline{G})$. In this section we establish the existence of absolute minimum $w(x)$ for the functional $I_H(u)$, where $w(x) \in W_H^+(\overline{G})$ (see Theorem 5). This is the first main result of the present paper. The proof is based on the nontrivial bilateral estimates for the values of I_H considered only on all convex functions $w(x) \in W_H^+(\overline{G})$ (see Theorems 3 and 4).

3.1. Bilateral estimates for $I_H(u)$.

Lemma 1. *The inequality*

$$|w(x)| \geqslant \frac{h_H}{\operatorname{diam} G} \|w(x)\| \tag{3.1}$$

holds for every convex function $w(x) \in W_H^+(\overline{G})$.

Proof. The inequality (1.3) holds trivially for the function $w(x) = 0$ in G. Therefore we assume that $\|w(x)\| > 0$. Let $w(x)$ be any function from $W_H^+(\overline{G})$. From property (1) (see §2.2) it follows that there exists the point $x_0 \in \overline{H}$ such that

$$\|w(x)\| = |w(x_0)|. \tag{3.2}$$

Now we consider the convex cone K with the vertex $(x_0, w(x_0))$ and the base ∂G. Let K be the graph of the convex function $k(x)$. Then

$$w(x) \leqslant k(x) \leqslant 0 \tag{3.3}$$

for all $x \in \overline{G}$ and

$$w(x)|_{\partial G} = k(x)|_{\partial G} = 0, \tag{3.4}$$

The equality

$$|k(x)| = \frac{|xx'|}{|x_0x'|} |k(x_0)| \tag{3.5}$$

holds for any point $x \in \overline{H}$, where x' is the point of intersection of the ray x_0x (with origin x_0) and ∂G, and $|xx'| = \operatorname{dist}(x, x')$, $|x_0x'| = \operatorname{dist}(x_0, x')$. Since

$$|xx'| \geqslant h_H \tag{3.6}$$

and

$$|x_0x'| \leqslant \operatorname{diam} G, \tag{3.7}$$

then from (3.6), (3.7) and (3.8) we obtain

$$|k(x)| \geqslant \frac{h_H}{\operatorname{diam} G} |k(x_0)|. \tag{3.8}$$

But

$$|k(x_0)| = |w(x_0)| = \|w\|, \tag{3.9}$$

since the cone K has the vertex in the point $(x_0, w(x_0))$ and

$$\|w(x)\| = |w(x_0)|. \tag{3.10}$$

Now from (3.3), (3.8) and (3.9) we obtain the inequality (3.1). Lemma 1 is proved.

Lemma 2. *The inequality*

$$\|w(x)\| \leqslant \left[\frac{\omega(w, G)}{\mu_n}\right]^{1/n} \cdot \operatorname{diam} G \tag{3.11}$$

holds for every convex function $w(x) \in W^+(\bar{G}) \cap C_0^-(\bar{G})$, *where* μ_n *is the volume of the n-unit ball.*

This lemma is the special case of Lemma 2 of the paper [8] (see also [6], [9]).

Theorem 3. *The inequality*

$$I_H(w) \geqslant \frac{\mu_n h_H}{(\operatorname{diam} G)^{n+1}} \|w(x)\|^{n+1} - \psi_H(G)(n+1)\|w(x)\| \tag{3.12}$$

holds for any $w(x) \in W_H^+(\bar{G})$.

Proof. From Lemma 1 we obtain

$$\int_G [-w(x)]\, \omega(w, de) \geqslant \frac{h_H}{\operatorname{diam} G} \|w(x)\| \omega(w, \bar{H}). \tag{3.13}$$

But from property (3) (§2.2) it follows that $\omega(w, \bar{H}) = \omega(w, G)$. Now from Lemma 2 we obtain

$$\omega(w, G) \geqslant \frac{\mu_n}{(\operatorname{diam} G)^n} \|w(x)\|. \tag{3.14}$$

Thus the inequalities (3.13) and (3.14) lead to the inequality

$$\int_G [-w(x)]\, \omega(w, de) \geqslant \frac{\mu_n h_H}{(\operatorname{diam} G)^{n+1}} \|w(x)\|^{n+1}. \tag{3.15}$$

From (3.15) we obtain finally the inequality (3.12) for

$$I_H(w) = -\int_G w(x) \omega(w, de) + (n+1)\int_G w(x) \psi_H(de).$$

Theorem 3 is proved.

Theorem 4. *The inequality*

$$I_H(w) \leqslant \frac{\mu_n}{h_H} \|w\|^{n+1} - \frac{(n+1)h_H}{\operatorname{diam} G} \psi_H(G) \|w\| \tag{3.16}$$

holds for every convex function $w(x) \in W_H^+(\bar{G})$.

Proof. First we estimate from above the integral $\int_G \{-w(x)\} \omega(w, de)$. We have

$$0 \leqslant \int_G \{-w(x)\} \omega(w, de) \leqslant \|w\| \omega(w, G)$$

and from property (6) (§2.2) we obtain

$$0 \leqslant \int_G \{-w(x)\}\omega(w, de) \leqslant \frac{\mu_n}{h_H^n} \|w\|^{n+1}. \tag{3.17}$$

Now we estimate from below $\int_G |w(x)| \psi_H(de)$. Since $\psi_H(G \setminus \overline{H}) = 0$ then

$$\int_G |w(x)| \psi_H(de) = \int_{\overline{H}} |w(x)| \psi_H(de).$$

Now from Lemma 1 it follows that

$$\int_G |w(x)| \psi_E(de) \geqslant \frac{h_H}{\operatorname{diam} G} \|w(x)\| \psi_H(G). \tag{3.18}$$

Thus from (3.17) and (3.18) we finally obtain

$$\begin{aligned} I_H(w) &= -\int_G w\omega(w, de) + (n+1)\int_G w\psi_H(de) \\ &\leqslant \frac{\mu_n}{h_H^n} \|w\|^{n+1} - \frac{(n+1)h_H}{\operatorname{diam} G} \psi_H(G)\|w\|, \end{aligned}$$

because $w(x) \leqslant 0$ in G. Theorem 4 is proved.

3.2. Main theorem about the functional $I_H(u)$.

Let $U(H, m, M)$ denote the subset of functions $w(x) \in W_H^+(\overline{G})$ satisfying the condition

$$m \leqslant \|w(x)\| \leqslant M, \tag{3.19}$$

where $0 \leqslant m < M < +\infty$ are constants. If $m = 0$, the $U(H, 0, M)$ consists of functions $w(x) \in W_H^+(\overline{G})$ satisfying the inequality

$$\|w(x)\| \leqslant M. \tag{3.20}$$

Lemma 3. *Every set $U(H, m, M)$ is compact in $C(\overline{G})$.*

Proof. The set $U(H, m, M)$ is bounded and closed in $C(\overline{G})$ and any function $w(x) \in U(H, m, M)$ satisfies the Lipschitz condition of the degree one and constant $M\mu_n^{1/n}(h_H)^{-1}$. Thus $U(H, m, M)$ is compact in $C(\overline{G})$. Lemma 3 is proved.

Theorem 5. (*Main theorem about the absolute minimum of the functional $I_H(u)$*). *The function $I_H(u)$ has at least one absolute minimum and the function $w_0(x)$ belonging to $W_H^+(\overline{G})$ and realizing this minimum satisfies the inequalities,*

$m_0 \leqslant \|w_0(x)\| \leqslant M_0$, where

$$t^* = \left[\frac{h_H^{n+1}\psi_H(G)}{\mu_n(\operatorname{diam} G)}\right]^{1/n}, \quad m_0 = \Phi^{-1}(\varphi(t^*)),$$

$$M_0 = \max\left\{1, \left[\frac{(n+1)\psi_H(G)+1}{\mu_n h_H}(\operatorname{diam} G)^{n=1}\right]^{1/n}\right\}.$$

the functions $\varphi(t)$ and $\phi(t)$ will be defined below.

Proof. From Theorem 3 it follows that $\lim_{k\to\infty} I_H(w_k) = +\infty$ if $w_k(x) \in W_H^+(\bar{G})$ and $\|w_k(x)\| \to +\infty$.

Therefore we can find a positive number M_0 such that $I_H(w) > 1$ if $\|w(x)\| > M_0$. For example we can take M_0 to be the number M_0 mentioned in Theorem 5. Now from the expression of $I_H(u)$ and Theorem 4 we can see that $I_H(0) = 0$ and $I_H(w) < 0$ if $w \in W_H^+(\bar{G})$, $\|w\| > 0$ and $\|w\|$ is sufficiently small.

Therefore the functional $I_H(u)$ is bounded from below and $I_H(u)$ takes negative values.

Now we consider the function

$$\varphi(t) = \frac{\mu_n}{h_H}t^{n+1} - \frac{(n+1)h_H}{\operatorname{diam} G}\psi_H(G)t$$

for $t \in [0, +\infty)$. This function has only two roots 0 and some positive number t_0 and takes negative values only inside the interval $(0, t_0)$. Let t^* be the point such that

$$\varphi(t^*) = \inf_{[0,t_0]} \varphi(t).$$

Then $\varphi'(t^*) = 0$ and $t^* = [h_H^{n+1}\psi_H(G)/\mu_n(\operatorname{diam} G)]^{1/n}$.

Now the function

$$\Phi(t) = \frac{\mu_n h_H}{(\operatorname{diam} G)^{n+I}}t^{n+1} - (n+1)\psi_H(G)t$$

has only one negative minimum at the point

$$t^{**} = \left[\frac{\psi_H(G)[\operatorname{diam} G]^{n+1}}{\mu_n h_H}\right]^{1/n}$$

(evidently t^{**} is the unique root of $\Phi'(t)$ and $\Phi(t^{**}) = \inf_{[0,\infty)}\Phi(t)$). Since

$$t^* = \left[\frac{h_H}{\operatorname{diam} G}\right]^{(n+2)/n} t^{**} < t^{**}$$

and $\Phi'(t) < 0$ on $[0, t^{**})$, we can set

$$m_0 = \Phi^{-1}(\varphi(t^*)).$$

Recall that $\operatorname{Inf}_{W_H^+(\bar{G})} = I_H(w)$ is a finite negative number. It is clear that

$$\operatorname*{Inf}_{W_H^+(\bar{G})} I_H(w) = \operatorname*{Inf}_{U(H, m_0, M_0)} I_H(w),$$

where m_0 and M_0 were defined above.

From Lemma 3 it follows that there exists at least one function $w_0(x) \in U(H, m_0, M_0)$ such that

$$I_H(w_0(x)) = \inf_{U(H, m_0, M_0)} I_H(w).$$

Theorem 5 is proved.

4. Dual convex hypersurfaces and Euler's equation

From Theorem 5 (see §3.2) it follows that the absolute minimum of the functional $I_H(u)$ is achieved on some convex function $w_0(x) \in W_H^+(\bar{G})$. In the present section we establish that $w_0(x)$ is the general solution of the Dirichlet problem

$$\omega(w, e) = \psi_H(e), \qquad w|_{\partial G} = 0$$

(see Theorem 10). This is the second main result of this paper and its proof is based on the special formula for the first variation of the functional $I_H(u)$. The fundamental technique used by establishment of this formula is the dual convex hypersurfaces $P_H(u)$ constructed by means of the function $v(x) = u(x)\phi_{\bar{H}}(x)$ where $u(x)$ is any nonpositive continuous function in $\bar{G}$ and $\phi_{\bar{H}}(x)$ is the characteristic function of the closed subdomain $\bar{H}$ of domain G such that

$$\operatorname{dist}(\bar{H}, \partial G) = h_H > 0.$$

The dual hypersurfaces $P_H(u)$ generate some solid cones and the values of the functional $\Phi_H(u) = \int_G u\omega(w, de)$ are exactly the volumes of these bodies. The mentioned relationship permits us to use methods and results of the Minkowski mixed volumes theory by the investigation of the first variation for the functional $I_H(u)$.

4.1. Special map on the hemisphere. Let $\bar{G}$ be an open convex domain in E^n. Let $\tilde{R}^{n+i} = (p_1, p_2, \cdots, p_{n+1})$ be an $(n+1)$-dimensional Euclidean space and S_-^n be the unit n-hemisphere:

$$p_{n+1} < 0, \quad p_1^2 + p_2^2 + \cdots + p_{n+1}^2 = 1 \tag{4.1}$$

in $\tilde{R}^{n+1}$. We consider the map $\gamma\colon S_-^n \to E^n$ defined by

$$x_1 = \frac{p_1}{|p_{n+1}|},\ x_2 = \frac{p_2}{|p_{n+1}|},\cdots,x_n = \frac{p_n}{|p_{n+1}|}, \tag{4.2}$$

where $x = (x_1,\cdots,x_n) \in E^n$, $p = (p_1,\cdots,p_{n+1}) \in S_-^n$, $x = \gamma(p)$. We can also consider γ as a diffeomorphism between the smooth manifolds S_-^n and E^n with natural differential structures. Then the diffeomorphism $\gamma^{-1}\colon E^n \to S_-^n$ maps any point $x = (x_1, x_2,\cdots,x_n) \in E^n$ to the point

$$p = \gamma^{-1}(x) = \left(\frac{x_1}{q}, \frac{x_2}{q},\cdots,\frac{x_n}{q}, -\frac{1}{q}\right), \tag{4.3}$$

where $q = (1 + x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. We denote γ^{-1} by γ_1.

The set $\bar{G}^* = \gamma_1(\bar{G})$ is a closed convex domain in S_-^n, where $\bar{G}$ is the closure of G and

$$\operatorname{dist}(\partial S_-^n, \bar{G}^*) = \delta_0 > 0 \tag{4.4}$$

in the intrinsic spherical meaning.

4.2. Dual convex hypersurfaces. Let $u(x)$ be any continuous nonpositive function in $\bar{G}$ satisfying the condition $u|_{\partial G} = 0$.

The function $u(x)$ defines the new function $u^*(p)$ in $\bar{G}^*$ by the formula

$$u^*(p) = (1 - p_1^2 - \cdots - p_n^2)^{1/2} u(\gamma(p)) \tag{4.5}$$

for $p = (p_1,\cdots,p_n, p_{n+1}) \in \bar{G}^* \subset S_-^n$, where $x = \gamma(p)$ (see (4.2)). Conversely if we define

$$\tilde{u}^*(x) = u^*(\gamma_1(x)), \tag{4.6}$$

where $p = \gamma_1(x)$, then

$$\tilde{u}^*(x) = \frac{1}{(1 + x_1^2 + \cdots + x_n^2)^{1/2}} u(x). \tag{4.7}$$

We denote by H and $\bar{H}$ an open convex subdomain of G and its closure and assume that $\operatorname{dist}(\bar{H}, \partial G) = h_H > 0$. Then $H^* = \gamma_1(H)$ and its closure $\bar{H}^* = \gamma_1(\bar{H})$ are respectively open and closed spherical convex domains and the intrinsic distance h_{H^*} between $\bar{H}^*$ and ∂G^* is positive. Clearly h_{H^*} depends only on h_H.

The inequality

$$(p, z) \leqslant u^*(p) \tag{4.8}$$

defines the closed half-space $U_p \subset R^{n+1}$ for each fixed vector $p \in \bar{G}^*$ and any vector $z \in \tilde{R}^{n+1}$, satisfying the inequality (4.8). The set

$$Q_H(u) = \bigcap_{p \in \bar{H}^*} U_p \tag{4.9}$$

is a closed infinite convex body in $\tilde{R}^{n+1}$. The sets

$$K(\partial G^*) = \bigcap_{q \in \partial G^*} V_q, \quad K(\bar{G}^*) = \bigcap_{q \in \bar{G}^*} V_q \tag{4.10}$$

are one and the same solid convex cone in $\tilde{R}^{n+1}$ with vertex $\tilde{0}(0, 0, \cdots, 0)$, where V_q is the closed half-space

$$(q, z) \leqslant 0 \tag{4.11}$$

for any fixed $q \in \partial G^*$ (or $\bar{G}^*$) and any vector $z \in \tilde{R}^{n+1}$.

Now the sets

$$P_H(u) = \partial Q_H(u) \tag{4.12}$$

and

$$L(\partial G^*) = \partial K(\partial G^*) \tag{4.13}$$

are complete infinite convex hypersurfaces in R^{n+1} and the latter is a convex n-dimensional cone with the vertex $\tilde{0}(0, 0, \cdots, 0)$.

Theorem 6. *Let $w(x)$ be the convex function spanning $u(x) \in C_0^-(\bar{G})$ from below on the set $\bar{H}$. Then*

$$Q_H(u) = Q_H(w) \quad \textit{and} \quad P_H(u) = P_H(w). \tag{4.14}$$

Moreover the convex body $Q_H(u)$ and the convex hypersurface $P_H(u)$ have one and the same supporting function $w^*(p)$ defined on $\bar{H}^*$.

Proof. From definition of the function $w(x)$ it follows that $w(x) \leqslant u(x) \leqslant 0$ for any $x \in \bar{H}$. Therefore $w^*(p) \leqslant u^*(p) \leqslant 0$ for any $p \in \bar{H}^*$. Thus $W_p \subset U_p$ for any $p \in \bar{H}^*$, where W_p and U_p correspondingly are the closed half-spaces $(p, z) \leqslant w^*(p)$ and $(p, z) \leqslant u^*(p)$ for every fixed vector $p \in \bar{H}^*$ and any vector $z \in \hat{R}^{n+1}$. Therefore

$$Q_H(w) = \bigcap_{p \in \bar{H}^*} W_p \subset \bigcap_{p \in \bar{H}^*} U_p = Q_H(u). \tag{4.15}$$

From the theory of convex bodies it is well known that if M is an infinite closed convex body and $v^*(p) < 0$, $p \in \bar{H}^* \subset S_-^n$ the supporting function of M, then the function

$$v(x) = \left(1 + \sum_{i=1}^{n} x_i^2\right)^{1/2} v^*(\gamma(x))$$

is a negative convex function for $x \in \bar{H}$, where

$$\gamma(x) = \left(\frac{x_1}{q}, \frac{x_2}{q}, \cdots, \frac{x_n}{q}, -\frac{1}{q}\right) \in \bar{H}^*$$

and $q = (1 + \Sigma_{i=1}^{n} x_i^2)^{1/2}$. Now we apply this fact to the case $M = Q_H(u)$. Let $v^*(p)$ be the supporting function of the convex body $Q_H(u)$. Then clearly $0 \geqslant u^*(p) \geqslant v^*(p)$ for any $p \in \overline{H}^*$. Therefore we obtain for negative convex function $v(x)$ the inequality $0 \geqslant u(x) \geqslant v(x)$ for any $x \in \overline{H}$. From the definition of the convex function $w(x)$ spanning by $u(x)$ from below on $\overline{H}$ it follows that $u(x) \geqslant w(x) \geqslant v(x)$ for any $x \in \overline{H}$. Repeating our reasoning for the functions $w(x)$ and $v(x)$ we obtain

$$Q_H(w) \supset Q_H(v) = Q_H(u). \tag{4.16}$$

From (4.15) and (4.16) it follows that $Q_H(u) = Q_H(v)$, and hence $P_H(u) = P_H(v)$. Theorem 6 is proved.

Now we consider the new convex body

$$Q_G(w) = \bigcap_{p \in \overline{G}^*} W_p \tag{4.17}$$

for every function $w(x) \in W_H^+(\overline{G})$, where the closed half-space W_p was defined above in this section.

Theorem 7.

$$Q_G(w) = Q_H(w) \cap K(\partial G^*). \tag{4.18}$$

Proof. It follows from definitions of the sets $Q_G(w)$ and $Q_H(w)$ that

$$Q_G(w) = Q_H(w) \cap Q_{G \setminus H}(w), \tag{4.19}$$

where

$$Q_{G \setminus H}(w) = \bigcap_{p \in \overline{G}^* \setminus \overline{H}^*} W_p.$$

First of all we note that the asymptotic solid cone $K_h(w)$ to $Q_H(w)$ has the set $\overline{H}^*$ as a spherical image. If the vertex of $K_H(w)$ lies inside $Q_H(w)$, then the whole cone $K_H(w)$ lies inside $Q_H(w)$. Let $L_H(w)$ be the boundary of $K_H(w)$. We suppose that the vertex of $K_H(w)$ coincides with the nearest point of $P_H(w)$ to the origin $\tilde{0}$ of $\tilde{R}^{n+1}$.

Then the set

$$\lambda(w) = L_H(w) \cap L(\partial G^*)$$

is the $(n-1)$-dimensional hypersurface homeomorphic to $(n-1)$-sphere. Recall that $L(\partial G^*) = \partial K(G^*)$, where $K(\partial G^*)$ is the convex solid cone.

Evidently $\sup_{Z \in \lambda(w)} \{\mathrm{dist}\{0, Z\}\}$ can be estimated above by means of $\|w(x)\|$, $h_H = \mathrm{dist}\{\overline{H}, \partial G\}$ and $\delta_0 = \mathrm{dist}(\partial S_-^n, \overline{G}^*)$.

If

$$\nu(w) = P_H(w) \cap L(\partial G^*) \tag{4.20}$$

then $\nu(w)$ is also homeomorphic to $(n-1)$-sphere and $\nu(w)$ lies between $\lambda(w)$ and the origin of $\tilde{R}^{n+1}$. Thus

$$\sup_{Z\in\nu(w)}\{\operatorname{dist}\{\tilde{0}, Z\}\} \tag{4.21}$$

can also be estimated by means of $\|w(x)\|$, h_H and δ_0.

Now all supporting hyperplanes to the graph S_W of the convex function $w(x)$ of the points $(x, w(x))$ will be singular if x belongs to $\overline{G}\setminus\overline{H}$ (see the proof of Theorem 1, §2). Let α be such a supporting hyperplane: then $\alpha\cap S_w$ is some closed bounded convex k-dimensional body, where $1\leqslant k\leqslant n-1$. We denote by $\pi_\alpha\subset\overline{G}$ the closed k-dimensional convex body which is the projection of the set $\alpha\cap S_w$. Then π_α determines the singular point Y on $P_H(w)$ with k-dimensional set of supporting hyperplanes to $P_H(w)$, because $\pi_\alpha\cap\overline{H}\neq\varnothing$. The spherical image of this set of supporting hyperplanes coincides with $\gamma_1(\pi_\alpha)\subset S^n_-$. (The definition of the mapping γ_1 is in §4.1.) Since α passes through some point $(x_0, 0)$, where $x_0\in\partial G$, then Y_α belongs to the cone $L(\partial G^*)$ or more precisely $Y_\alpha\in\nu(w)$ (see (4.20)).

Clearly $\nu(w)=\bigcup_\alpha Y_\alpha$, where α runs through the set of all supporting hyperplanes to S_w having contact points $(x, w(x))$ with S_w, where $x\in\overline{G}\setminus\overline{H}$.

Therefore from (4.17), (4.18), (4.19) and the last considerations it follows that $Q_G(w)=Q_H(w)\cap K(\partial G^*)$. Theorem 7 is proved.

Thus the convex hypersurface

$$P_G(w)=\partial Q_G(w) \tag{4.22}$$

consists of two parts: the first one $S_H(w)$ lies inside the solid cone $k(\partial G^*)$ and the second one $T_{\partial G}(w)$ lies on the boundary $L(\partial G^*)$ of the cone $K(\partial G^*)$. Both hypersurfaces have one and the same boundary $\nu(w)\subset P_G(w)$. Let us agree to include $\nu(w)$ as a part of $S_H(w)$ and $T_{\partial G}(w)$ and consider both hypersurfaces as closed hypersurfaces with boundary.

We will call $S_H(w)$ the *dual convex hypersurface* (with respect to $\overline{H}\subset G$) of the convex function $w(x)\in W^+_H(\overline{G})$. The function

$$w^*(p)=\left(1-p_1^2-\cdots-p_n^2\right)^{1/2}w(\gamma(p)) \tag{4.23}$$

is the supporting function for $S_H(w)$ for any $p\in\overline{G}^*$.

4.3. Expression of the functional $I_H(u)$ by means of dual convex hypersurfaces. Let $w(x)$ be any convex function belonging to $W^+_H(\overline{G})$ and $S_H(w)$ be its dual convex hypersurface. We denote by $\sigma(S_H(w), e')$ the surface function of $S_H(w)$ (see [10], [11]). The surface function $\sigma(S_H(w), e')$ is defined as the completely additive nonnegative function on the ring of Borel's subsets e' of the domain $\overline{G}^*\subset S^n_-$ and the values of this function equal to the area of the sets $\tilde{e}\subset S_H(w)$ such that $\tilde{e}$ consist of all points of $S_H(w)$ having supporting hyperplanes with unit outside normals belonging to e'. From our considerations it follows that

$$\sigma\big(S_H(w), \bar{G}^* \setminus \bar{H}^*\big) = 0. \tag{4.24}$$

It is well known (see [11]) that $\{\sigma(S_H(w_k), e')\}$ converges weakly to $\sigma(S_H(w_0), e')$ if $\lim_{k\to\infty} \|w_k - w_0\| = 0$.

Let $V_H(w)$ be the volume of the part of the convex cone $K(\partial G^*)$ situated under the dual convex hypersurface $S_H(w)$.

Theorem 8. *The equality*

$$V_H(w) = -\frac{1}{n+1}\int_{\bar{H}} w(x)\omega(w, de) \tag{4.25}$$

holds for every convex function $w(x) \in W_H^+(G)$.

Proof. If $w(x) \in W_H^+(\bar{G})$, then for the volume $V_H(w)$ there is the formula

$$V_H(w) = -\frac{1}{(n+1)}\int_{\bar{G}^*} w^*(p)\sigma(S_H(w), de') \tag{4.26}$$

(see [10], [11]).

But the surface function $\sigma(S_H(w), e')$ has the representation

$$\sigma(S_H(w), e') = \int_e \left(1 + \sum_{i=1}^n x_i^2\right)^{1/2} \omega(w, de), \tag{4.27}$$

in the map γ (see §4.1). Formula (4.27) can first be proved for convex polyhedrons and extend for all class $W_H^+(\bar{G})$ of convex functions by approximation of polyhedrons.[3] From (4.26) and (4.27) it follows that

$$\begin{aligned} V_H(w) &= -\frac{1}{(n+1)}\int_{\bar{G}^*} w^*(\gamma_1(x))\left(1 + \sum_{i=1}^n x_I^2\right)^{1/2} \omega(w, de) \\ &= -\frac{1}{(n+1)}\int_{\bar{G}} w(x)\omega(w, de) = -\frac{1}{(n+1)}\int_{\bar{H}} w(x)\omega(w, de), \end{aligned}$$

because $\omega(w, \bar{G}\setminus\bar{H}) = 0$. Theorem 8 is proved.

Remark. Since any convex polyhedron can be approximated by C^2 convex hypersurfaces (function) with everywhere strictly principal normal curvatures, then it is sufficient to establish (4.27) only of such a class of hypersurfaces (functions).

From the Gauss theorem it follows that

$$\sigma(S_H(w), e') = \int_{e'} \frac{ds_p}{K(p)}, \tag{4.28}$$

[3] Of course we use the weak convergence of the surface functions.

where ds_p is the element of area on S^n_- and $K(p)$ is the Gauss curvature of $S_H(w)$ in the point of $S_H(w)$ with the outside unit normal p. We find

$$\int_{e'} \frac{ds_p}{K(p)} = \int_e \left(1 + \sum_{i=1}^n x_i^2\right)^{1/2} \det\|w_{x_i x_j}\| dx, \tag{4.29}$$

where $e' = \gamma^{-1}(e)$ (see §4.1 and also [11]). From (4.28) and (4.29) we obtain (4.27) for the C^2-convex functions (hypersurfaces) with strictly positive principal normal curvatures.

Theorem 9. *The functional $I_H(u)$ in the $\overline{C}_0(\overline{G})$ has the representation*

$$\begin{aligned} I_H(u) &= (n+1)\left[V_H(F_H(u)) + \int_{H^*} u^*(p)\psi_H^*(de')\right] \\ &= \left[-\int_{H_u^*} u^*(p)\sigma(S_H(w), de') + (n+1)\int_{H^*} u^*(p)\Psi_H^*(de')\right], \end{aligned} \tag{4.30}$$

where $w(x) = F_H(u(x))$ is the convex function spanning $u(x)$ from below on $\overline{H} \subset G$; $\Psi_H^*(e')$ is the nonnegative completely additive function of Borel subsets e' of G^* determined by the formula

$$\psi_H^*(e') = \int_{\gamma(e)} \left(1 + \sum_{i=1}^n x_i^2\right)^{1/2} \Psi_H(de); \tag{4.31}$$

$H_u^* = \gamma_1(H_u)$ and H_u is the closed subset of $\overline{H}$ where $u(x) = w(x)$.

Proof. It follows from the definition of $I_H(u)$ that

$$I_H(u) = -\int_G u\omega(w, de) + (n+1)\int_G u\Psi_H(de). \tag{4.32}$$

Now

$$\int_G u(x)\psi_H(de) = \int_H u^*(x)\left(1 + \sum_{i=1}^n x_i^2\right)^{1/2} \Psi_H(de)$$

since $\psi_H(G \setminus H) = 0$. Therefore

$$\int_G u(x)\Psi_H(de) = \int_{H^*} u^*(p)\Psi_H^*(de') \tag{4.33}$$

if we use (2.39) and (4.31).

From Theorems 1 and 8 we obtain

$$\begin{aligned} (n+1)V_H(F_H(u)) &= -\int_{\overline{H}} w(x)\omega(w, de) = -\int_{H_u} w(x)\omega(w, de) \\ &= -\int_{H_u} u(x)\omega(w, de) = -\int_{H_u^*} u^*(p)\sigma(S_H(w), de'). \end{aligned} \tag{4.34}$$

From Theorem 1 we obtain

$$-\int_G u(x)\omega(w, de) = -\int_{H_u} u(x)\omega(de). \tag{4.35}$$

Now it follows from (4.34) and (4.35) that

$$-\int_G u(x)\omega(w, de) = (n+1)V_H(F_H(u)) = \int_{H_u^*} u^*(p)\sigma(S_H(w), de'). \tag{4.36}$$

Thus from (4.32), (4.33) and (4.36) we obtain (4.30). Theorem 9 is proved.

4.4. Expression of the variation of $I_H(u)$. First of all we study the variation of the functional

$$\Phi_H(u) = -\int_G u(x)\omega(w, de), \tag{4.37}$$

where $u(x) \in C_0^-(\overline{G})$ and $w(x)$ is the convex function spanning $u(x)$ from below on the convex closed domain $\overline{H} \subset G$. From Theorem 9 it follows that

$$\begin{aligned}\Phi_H(u) &= (n+1)V_H(w) = -\int_{H_u^*} w^*(p)\sigma(S_H(w), de') \\ &= -\int_{H_u^*} u^*(p)\sigma(S_H(w), de'),\end{aligned} \tag{4.38}$$

where H_u^* is a closed subset of $\overline{H}^* = \gamma_1(\overline{H})$, $w^*(p) = u^*(\dot{p})$ and $V_H(w)$ is the volume of the part of the convex cone $K(\partial G^*)$ situated under the dual convex hypersurface $S_H(w)$.

Now we want to complement $S_H(w)$ to the whole closed convex hypersurface. The boundary $\nu(w)$ of $S_H(w)$ lies on the conic convex hypersurface $L(\partial G^*) = \partial\{K(\partial G^*)\}$ and homeomorphic to the $(n-1)$-sphere. We can evidently find two numbers m_1 and m_2 depending on $\|w\|$, $\operatorname{dist}\{\overline{H}, \partial G\} = h_H > 0$ and $\operatorname{dist}\{\overline{G}^*, \partial S_-^n\}$ such that

$$0 < m_1 \leqslant \operatorname{dist}\{\tilde{0}, \nu(w)\} \leqslant m_2 < +\infty. \tag{4.39}$$

We denote by $S_+^n(r)$ the hemisphere

$$p_1^2 + p_2^2 + \cdots + p_{n+1}^2 = r^2, \qquad p_{n+1} \geqslant 0, \tag{4.40}$$

and by $U_+^n(r)$ the set

$$p_1^2 + p_2^2 + \cdots + p_{n+1}^2 \leqslant r^2, \qquad p_{n+1} \geqslant 0. \tag{4.41}$$

We will only consider the functions $u(x) \in C_0^-(\overline{G})$ such that for the convex functions $w(x) = F_H(u(x))$ spanned by $u(x)$ from below on $\overline{H}$ the inequalities

$$m_0 \leqslant \|w(x)\| \leqslant M_0 \tag{4.42}$$

hold (see Theorem 8). Then there exist the common numbers $0 < m_1 < m_2 < +\infty$ such that for all functions $u(x) \in C_0^-(\overline{G})$ the inequalities

$$0 < m_1 \leqslant \operatorname{dist}\{\tilde{0}, \nu(w)\} \leqslant m_2 < +\infty \tag{4.43}$$

hold, if (4.42) are fulfilled $w(x) = F_H(u(x))$. Thus we will be able to construct all the bounded convex bodies $\Pi_H(w)$.

Now consider the supporting function of $\Pi_H(x)$. We denote this function by $h_H(p)$ where p runs through the whole unit sphere S^n: $p_1^2 + p_2^2 + \cdots + p_n^2 + p_{n+1}^2 = 1$.

The closed convex hypersurfaces $\Lambda_H(w)$ has at least two ribs, $\nu(w)$ and $\nu(m_1 + 2m_2)$, which are the boundaries for three domains $S_H(w)$, $Z(m_1 + 2m_2)$ and $T(w, U_+^n(m_1 + 2m_2)) \subset L(\partial G^*)$.

Therefore

$$h_H^*(p) = \begin{cases} w^*(p) < 0 & \text{if } p \in G^*, \\ 0 & \text{if } p \in \partial G^*, \\ \text{takes positive} & \\ \text{values} & \text{if } p \in S^n \setminus \left(\overline{G}^* U \left\{ \dfrac{1}{m_1 + 2m_2} Z(m_1 + 2m_2) \right\} \right), \\ m_1 + 2m_2 & \text{if } p \in \dfrac{1}{m_1 + 2m_2} Z(m_1 + 2m_2) \subset S_+^n . \end{cases}$$

Note that

$$\sigma(\Lambda_H(w), S^n) \setminus \left(\overline{G}^* U \frac{1}{m + 2m_2} Z(m_1 + 2m_2) \right) = 0.$$

Therefore the volume of $\Pi_H(w)$ can be found by the formula

$$\begin{aligned} V(\Pi_H(w)) &= \frac{1}{n+1} \int_{S^n} h_H^*(p) \sigma(\Lambda_H(w), de') \\ &= \frac{\sigma(K(\partial G^*))}{n+1} (m_1 + 2m_2)^{n+1} + \frac{1}{n+1} \int_{\overline{G}^*} w^*(p) \sigma(S_H(w), de) \\ &= \frac{\sigma(K(\partial G^*))}{n+1} - \frac{1}{n+1} \int_{H_u^*} u^*(p) \sigma(S_H(w), de), \end{aligned}$$

where $\sigma(K(\partial G^*))$ is the solid angle of the convex cone $K(\partial G^*)$. Thus

$$V(\Pi_H(w)) = \frac{\sigma(K(\partial G^*))}{n+1} + \frac{1}{n+1} \phi(u). \tag{4.45}$$

If we change the point of the reference of distances with the sign to supporting hyperplanes to any convex body, then the supporting function of this body changes its values. If such a point p_0 coincides with the inner point of $\Pi_H(w)$, then the supporting function takes only positive values and is some strictly positive function on S^n.

Minkowski, Alexandrov, Fenchel and Jessen investigated the variation of the volume in the class of bounded convex bodies and established the formulas for the weak differential (the first variation) by different conditions (see [1], [10], [11], [15], [16]). The main methods and techniques of these investigations were the theory of Minkowski mixed volumes and the Brunn-Minkowski inequality.

Alexandrov proved that if $h_0(p)$ is any strictly positive continuous function on the unit sphere S^n: $|p| = 1$ and H_0 is the closed convex body defined by intersection of all the halfspaces $(p, z) \leq h_0(p), p \in S^n$, then

$$\lim_{t \to 0} \frac{V(H_t) - V(H)}{t} = \int_{S^n} \eta(p)\sigma(H_0, de'), \tag{4.47}$$

where $\eta(p)$ is any continuous function on S^n, t is a real parameter converging to zero, $\sigma(H_0, e')$ is the surface function of H_0 and H_t is the closed bounded convex body defined by intersections of all the halfspaces $(p, z) \leq h_0(p) + t\eta(p)$.

Remark. Since $h_0(p)$ and $\eta(p)$ are continuous on S^n and $h_0(p)$ is strictly positive, then $h_0(t) + t\eta(p)$ is also positive for sufficiently small t and the bodies H_t will be constructable.

Since 1) all terms of (4.46) are independent on the point of reference to the supporting hyperplanes and 2) it is possible to take any function $\eta \neq 0$ only on any closed set $\overline{H}_1 \subset H$, then from (4.46), (4.45) and Theorem 9 it follows that

$$\lim_{t \to 0} \frac{I_H(u + t\eta) - I_H(u)}{t} = (n+1)\left[\int_{H_u} \eta[-\omega(w, de) + \psi_H(de)]\right], \tag{4.48}$$

where $w = F_H(u)$. From (4.48) and Theorem 8 it follows that the function

$$w_0(x) \in U(H, m_0, M_0) \subset W_H^+(\overline{G}),$$

realizing the absolute minimum of $I_H(u)$ in $C_0^-(G)$, is a generalized solution of the Dirichlet problem

$$\omega(w, e) = \psi_H(e), \qquad w|_{\partial G} = 0. \tag{4.49}$$

Since the Dirichlet problem (4.49) has only one generalized solution and this solution belongs to $W_H^+(\overline{G})$ (see Theorem 14, §6), then there exists only one function realizing the absolute minimum of the functional $I_H(u)$ in the set $\overline{C}_0(\overline{G})$ and this function belongs to $W_H^+(\overline{G})$. Thus the following main result is proved.

Theorem 10. *There is only one function realizing the absolute minimum of the functional $I_H(u)$ in the set $C_0^-(\overline{G})$. This function $w_H(x)$ belongs to $W_H^+(\overline{G})$ and is the generalized solution of the Dirichlet problem* (4.49).

5. Generalizations

The functionals $I_H(u)$ considered above were concentrated on some fixed compact subdomains $\overline{H}$ of a given bounded open domain G. In this section we want to become free from this restriction by means of suitable passage to the limit. It requires some additional assumptions with respect to ∂G and the behavior of the set function $\psi(e)$ near ∂G. These assumptions are the following:

A.1. There exists the n-ball U_{x_0} of the radius r_{x_0} such that $x_0 \in \partial U_{x_0}$, $\overline{G} \subset \overline{U}_{x_0}$ and

$$r_{x_0} \leqslant r_0 = \text{const} < +\infty, \tag{5.1}$$

where x_0 is any point of ∂G.

A.2. Let $\psi(e)$ be a nonnegative, completely additive set function of the Borel subsets of G satisfying the two conditions

$$\psi(G) < +\infty \tag{5.2}$$

and

$$\psi(e) \leqslant a \operatorname{mes} e \tag{5.3}$$

for all Borel subsets $e \subset V_\delta \cap G$, where V_δ is a sufficient small neighborhood of ∂G in E^n, $a = \text{const} > 0$ and mes e in the Lebesque measure of e (V_δ can be considered as the union of all open n-balls of the sufficient small radius δ with centres in the set ∂G).

We denote by $\Xi = \{H_\alpha\}$ the set of all convex open subdomains H_α of G such that

$$\operatorname{dist}(\overline{H}_\alpha, \partial G) = h_H > 0. \tag{5.4}$$

Evidently Ξ can be considered as a semi-ordered set with respect to the inclusion of convex subdomains H_α. Let

$$\psi_H(e) = \psi(e \cap H) \tag{5.5}$$

be the set function which was used in the definition of the functional $I_H(u)$ (see §2.3). The Dirichlet problem

$$\omega(w, e) = \psi_H(e), \tag{5.6}$$

$$w\,|_{\partial G} = 0 \tag{5.7}$$

has only one generalized solution, $w_H(x)$, in the class of convex functions and this solution belongs to $W_H^+(\bar{G})$ (see §6.2). From Theorem 10 (see §4) it follows that

$$\inf_{C_0^-(\bar{G})} I_H(u) = I_H(w_H). \tag{5.8}$$

If $H_\alpha \subset H_\beta$ then there are two important inequalities

$$w_{H_\alpha}(x) \geqslant w_{H_\beta}(x) \tag{5.9}$$

for all $x \in G$ and

$$I_{H_\alpha}(w_{H_\alpha}) \geqslant I_{H_\beta}(w_{H_\beta}). \tag{5.10}$$

The first one follows directly from Theorem 15 (see §6.2) and the second one is the corollary of the equality (5.8). Since

$$\psi_{H_\alpha}(G) = \psi(G \cap H_\alpha) \leqslant \psi(G) < +\infty \tag{5.11}$$

then from (2.8) (see §2.1) it follows that

$$-\left\{\frac{\psi(G)}{\mu_n}\right\}^{1/n} \operatorname{diam} G \leqslant w_{H_\alpha}(x) \leqslant 0 \tag{5.12}$$

for all sets $H_\alpha \in \Xi$.

Theorem 11. *There exists*

$$\lim_{\Xi} w_{H_\alpha}(x) = w(x) \tag{5.13}$$

where $w(x)$ is the solution of the Dirichlet problem

$$\omega(w, e) = \psi(e), \qquad w|_{\partial G} = 0. \tag{5.14}$$

Moreover $w_{H_\alpha}(x)$ uniformly converge to $w(x)$ in $\bar{G}$.

Proof. Since the convex functions $w_{H_\alpha}(x)$ are uniformly bounded and satisfy the monotonicity property (5.9), then there exists the limit (5.13) where $w(x)$ is some convex function in G. Now we can apply Theorem 13 (see §6.1), because G and the set functions $\psi_{H_\alpha}(e)$ and $\psi(e)$ satisfy conditions A.1 and A.2. Hence $w(x)$ is the solution of the Dirichlet problem (5.14). Moreover $w_{H_\alpha}(x)$ uniformly converge to $w(x)$. The theorem is proved.

Theorem 12. *There exists*

$$\inf_{\Xi} I_{H_\alpha}(w_{H_\alpha}) = \lim_{\Xi} I_{H_\alpha}(w_{H_\alpha}) = n\int_G w(x)\psi(de). \tag{5.15}$$

Proof. Since $w_{h_\alpha}(x)$ is the solution of the Dirichlet problem (5.7)–(5.8), then

$$I_H(w_H) = n\int_G w_{H_\alpha}(x)\psi_{H_\alpha}(de). \tag{5.16}$$

Since $w_{H_\alpha}(x)$ uniformly converge to $w(x)$, then using the uniform estimate (5.12) and condition A.2 we obtain

$$\lim_{\Xi} I_{H_\alpha}(w_{H_\alpha}) = n\int_G w(x)\psi(de)$$

by means of standard considerations. The equality

$$\lim_{\Xi} I_{H_\alpha}(w_{H_\alpha}) = \inf_{\Xi} I_{H_\alpha}(w_{H_\alpha})$$

of the monotonicity property (5.10) and of the uniform estimates

$$0 \geqslant I_{H_\alpha}(w_{H_\alpha}) \geqslant -n\left[\frac{\psi(G)}{\mu_n}\right]\psi(G)\operatorname{diam} G.$$

The last estimate is the consequence of (5.12) and (5.16). The theorem is proved.

The contents of Theorems 11 and 12 is the third main result of this paper.

6. Some main concepts and facts of the theory of elliptic solutions of the Monge-Ampère equations

In this section we briefly present a few main facts for elliptic generalized solutions of Monge-Ampère equations

$$R(\operatorname{grad} u)\det\|u_{ij}\| = f(x) \tag{6.1}$$

constructed by general convex functions. The detailed presentation of these and significantly more general classes of Monge-Ampère equations can be found in [6, §§18, 20], [13], [7], [9].

The equation (6.1) can be extended to the class of all convex functions by the set function equation

$$\omega(R, z, e) = \psi(e), \tag{6.2}$$

where $\omega(R, z, e)$ is the R-curvature of convex solutions $z(x)$ (see §2) and $\psi(e)$ is a given completely additive nonnegative set function.

We add a few new assumptions to A.1 and A.2 (see §5) with respect to coefficients of the equation (6.1).

A.3. The function $R(p)$ is locally summable in $R^n = \{p = (p_1, \cdots, p_n)\}$ and the inequality

$$R(p) \geqslant C_0(1 + |p|^2)^{-k}, \qquad C_0 = \text{const} > 0, k = \text{const} \geqslant 0, \tag{6.3}$$

holds for all $p \in R^n$.

A.4. There exists the neighborhood $S_{x_0} \subset E^n$ for any point $x_0 \in \partial G$ such that

$$\psi(e) \leqslant a\left\{\sup_e \{\operatorname{dist}(x, \partial G)\}\right\}^\lambda \operatorname{mes} e$$

holds for any Borel subset $e \subset S_{x_0} \cap G$, where $\lambda \geqslant 0$ and $a > 0$ are the common constants for all $x_0 \in \partial G$.

The assumption A.4 is the natural generalization of A.2. Really A.2 follows from A.4 if $\lambda = 0$.

A.5. Let $z_n(x) \in W^+(G) \cap C(\overline{G})$ be a sequence of convex functions pointwise convergent to some convex function $z(x) \in W^+(G) \cap C(\overline{G})$ only in the open domain G. We suppose also that all functions $z_n(x)$ and $z(x)$ vanish on ∂G.

6.1. Uniform convergence of convex functions. (For the detailed presentation see [6, §§17, 20], [7], [9].)

Theorem 13. *Suppose assumptions* A.1, A.3 *and* A.5 *are fulfilled. If there exists the neighborhood S_{x_0} for any point $x_0 \in \partial G$ such that the inequality*

$$\varliminf_{n\to\infty} \omega(R, z_n, e) \leqslant a\left\{\sup_e \operatorname{dist}(x, \partial G)\right\}^\lambda \operatorname{mes} e \tag{6.4}$$

holds for all Borel subsets $e \subset S_{x_0} \cap G$ and if

$$k \leqslant \frac{n+1+\lambda}{2} \tag{6.5}$$

then the sequence of convex functions $z_n(x)$ uniformly converges in $\overline{G}$. (*Here $\lambda \geqslant 0$ and $a > 0$ are the common constants for all $x_0 \in \partial G$.*)

Remark. If $R(p) = 1$, then $k = 0$ and (6.5) takes the form $n = 1 + \lambda \geqslant 0$. Hence we can take $\lambda = 0$, so that (6.4) takes the quite simple form

$$\varliminf_{n\to\infty} \omega(z_n, e) \leqslant a \operatorname{mes} e.$$

6.2. Existence, uniqueness and comparison theorems for the Monge-Ampère equations. (For the detailed presentation see [6, §§18, 20], [7], [9].)

Theorem 14. *The Dirichlet problem*

$$\omega(R, u, e) = \psi(e), \qquad u|_{\partial G} = 0 \tag{6.6}$$

has only one generalized convex solution, $u(x) \in W^+(G)$, if the assumptions A.1, A.3, A.4 *are fulfilled and if the inequalities*

$$k \leqslant \frac{n+1+\lambda}{2} \tag{6.7}$$

and

(6.8) $$\psi(G) < A(R) = \int_{R^n} R(p)\,dp$$

hold.

Remarks. (A) The condition (6.7) is sharp; the condition (6.8) is sufficient and interlocks with the necessary condition $\psi(G) = \omega(R, z, G) \leqslant A(R)$ (see §2.1).

(B) If $R(p) = 1$, then it is possible to take $\lambda = 0$ and not consider the inequality (6.7). The assumption A.4 is reduced to A.2, i.e. $\psi(G) < +\infty$ and $\psi(e) \leqslant a \operatorname{mes} e$ for $e \subset V_\delta \cap G$, where $a = \text{const} > 0$ and V_δ is a sufficient small neighborhood of ∂G in E^n (see §5).

Theorem 15 (*Comparison theorem*). *Let G be a bounded convex domain in E^n and let $u_1(x)$, $u_2(x)$ belong to $W^+(G)$ and be generalized convex solutions of the Dirichlet problems*

$$\omega(R, u_i, e) = \psi_i(e), \qquad u_i|_{\partial G} = 0 \qquad (i = 1, 2)$$

and $\psi_1(e) \leqslant \psi_2(e)$ for all Borel subsets e of the domain G. Then $u_1(x) \geqslant u_2(x)$ for all $x \in G$.

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