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A LOWER BOUND FOR THE FIRST EIGENVALUE OF A NEGATIVELY CURVED MANIFOLD

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There has been much work in recent years on the relation of the low eigenvalues of a compact Riemannian manifold to the geometry of the manifold. For Riemann surfaces with positive genus, it was observed by P. Buser [1] that one can find a compact hyperbolic surface of fixed genus (hence fixed area) with arbitrarily small first eigenvalue (see [10] for more information on this problem). For hyperbolic manifolds of dimension larger than two, Mostow's theorem implies that the topology uniquely determines the geometry, so the above phenomenon for λ_1 is likely to be a two-dimensional phenomenon. In this note we show that this is the case. Precisely, let M^n be a compact Riemannian manifold with sectional curvature bounded between two negative constants. We show here that if $n \geqslant 3$, then $\lambda_1(M)$ has a lower bound depending only on the volume of M. Actually, for $n > 3$, Gromov [7] has shown that an upper bound on volume implies an upper bound on diameter (for negatively curved M). Using this result, a bound such as ours would follow from a general result of S. T. Yau [11]. For $n = 3$, the diameter is not bounded in terms of volume (see [2, 3.13]) so our result seems to be of most interest in this case. Buser [2] has observed that our dependence on the inverse square of the volume is best possible.

The case $n = 3$ of our theorem was announced in the Hawaii Symposium in 1979. In this note we give a simplified version, valid for all $n > 2$, of our original proof. We wish to thank P. Buser for pointing out reference [9] which is used in the proof of Lemma 1.

The main results

We will assume throughout that M^n is a compact n dimensional manifold. We state our main result.

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Theorem. *If the sectional curvatures of M satisfy the inequality $-1 \leqslant K_M \leqslant -\kappa^2$ for some $\kappa \in (0, 1)$ and if $n \geqslant 3$, then the first eigenvalue $\lambda_1(M)$ satisfies*

$$\lambda_1(M) \geqslant \min\left\{\frac{(n-1)^2\kappa^2}{4}, \frac{\delta_n}{\mathrm{Vol}^2(M)}\right\} \geqslant \frac{\delta'_{n,\kappa}}{\mathrm{Vol}^2(M)},$$

where $\delta_n = 4^{-1}\omega_{n-1}^2[\varepsilon_n e^{-\varepsilon_n(1-\kappa)}]^{2n-2}$, $\varepsilon_n = 4^{-(n+3)}$, ω_n = volume of the unit ball in R^n, and

$$\delta'_{n,\kappa} = (n-1)^2 4^{-1}\kappa^2\omega_n^2\varepsilon_n^{2n}.$$

We now introduce some terminology. Given a hypersurface Σ in M and a local orthonormal frame $e_1, \cdots, e_{n-1}$ tangent to Σ, the mean curvature vector is given by

$$H = \frac{1}{n-1}\sum_{i=1}^{n-1}\left(D_{e_i}e_i\right)^{\mathrm{Nor}},$$

where D is the Levi-Civita connection on M, and $(\ )^{\mathrm{Nor}}$ means projection normal to Σ. We will need three preliminary lemmas. The first is an isoperimetric inequality.

Lemma 1. *Suppose Σ is a closed (possibly disconnected) hypersurface in M which bounds a region Ω in M. Suppose the mean curvature vector H points everywhere into Ω, and assume the inequalities*

$$|H| \geqslant 1, \qquad \mathrm{Ric}_M \geqslant -(n-1).$$

Then we have $\mathrm{Vol}(\Sigma) \geqslant (n-1)\mathrm{Vol}(\Omega)$.

Proof. This result follows from the paper of Heintze-Karcher [9]. The estimates of [9, p. 453] applied on one side of Σ give the inequality

$$\mathrm{Vol}(\Omega) \leqslant \left(\int_0^\infty \left(\cosh r - \left(\min_\Sigma |H|\right)\sinh r\right)_+^{n-1} dr\right)\mathrm{Vol}(\Sigma),$$

where $(\ )_+$ indicates the positive part of a function. Using the fact that $|H| \geqslant 1$, we get immediately the conclusion of Lemma 1.

For a point $P \in M$, let $i(P)$ denote the injectivity radius of M at P. Our next lemma gives an estimate of $i(P)$ for points along a hypersurface in terms of the volume of the hypersurface.

Lemma 2. *Suppose M satisfies $K_M \leqslant -\kappa^2$ for some $\kappa \geqslant 0$, and let Σ be a hypersurface in M with mean curvature H satisfying $|H| \leqslant \Lambda$. Suppose also that* $\mathrm{Vol}(\Sigma) < \infty$ *and $\mathcal{H}^{n-2}(\overline{\Sigma} \sim \Sigma) = 0$ where $\mathcal{H}^s$ denotes Hausdorff s dimensional measure. Then for every point $P \in \overline{\Sigma}$ we have*

$$i(P)e^{-i(P)(\Lambda-\kappa)_+} \leqslant \left[\omega_{n-1}^{-1}\mathrm{Vol}(\Sigma)\right]^{1/n-1},$$

where ω_n denotes the volume of the unit ball in R^n.

Proof. The proof is a modificatiion of a well known monotonicity inequality for the area of a submanifold of R^n. We do the proof assuming that Σ is closed since an easy cutoff argument can then be used to prove the general case. By standard comparison theorems, if r denotes the distance function to a point, $P \in M$, then we have the Hessian comparison

$$\tfrac{1}{2}D_{x,x}r^2 \geqslant (1 + \kappa r)\,|\,x\,|^2,$$

provided $r < i(P)$. Restricting this inequality to Σ and taking the trace we have

$$\tfrac{1}{2}\Delta_\Sigma r^2 \geqslant (n-1)(1 - r(\Lambda - \kappa)_+).$$

Integrating this inequality over $\Sigma_\tau = \Sigma \cap B_\tau(P)$ and applying Stokes theorem we get

$$\tau\int_{\partial\Sigma_\tau} |\,\nabla r\,| \geqslant (n-1)(1 - \tau(\Lambda - \kappa)_+)\mathrm{Vol}(\Sigma_\tau),$$

where ∇ is the connection on Σ. Since for any regular value τ of $r\,|_\Sigma$ we have

$$\frac{d}{d\tau}\mathrm{Vol}(\Sigma_\tau) = \int_{\partial\Sigma_\tau} |\,\nabla r\,|^{-1},$$

and since $|\,\nabla r\,| \leqslant 1$ on Σ, we get the differential inequality

$$\tau\frac{d}{d\tau}\mathrm{Vol}(\Sigma_\tau) \geqslant (n-1)(1 - \tau(\Lambda - \kappa)_+)\mathrm{Vol}(\Sigma_\tau).$$

Integrating from ε to $i(P)$ we have

$$\left[\varepsilon e^{-\varepsilon(\Lambda-\kappa)_+}\right]^{1-n}\mathrm{Vol}(\Sigma_\varepsilon) \leqslant \left[i(P)e^{-i(P)(\Lambda-\kappa)_+}\right]^{1-n}\mathrm{Vol}(\Sigma).$$

Letting $\varepsilon \downarrow 0$ then gives the conclusion of Lemma 2.

The third preliminary lemma we need is a version of the Margulis lemma.

Lemma 3. *Suppose* $-1 \leqslant K_M < 0$, *and define a set* $\mathcal{O}$ *by* $\mathcal{O} = \{P \in M\colon i(P) < \varepsilon_n := 4^{-(n+3)}\}$. *The set* $\mathcal{O}$ *is an open set having finitely many components* $\mathcal{O}_1, \cdots, \mathcal{O}_l$. *Each component* $\mathcal{O}_i$ *is a neighborhood of a simple closed geodesic* Γ_i *with length* $(\Gamma_i) < 2 \cdot \varepsilon_n$. *Moreover, each* $\mathcal{O}_i$ *is topologically equivalent to* $S^1 \times B^{n-1}$, *and is star-shaped with respect to* Γ_i *in the sense that every point of* $\mathcal{O}_i$ *is connected to* Γ_i *by a unique geodesic arc lying within* $\mathcal{O}_i$ *and meeting* Γ_i *orthogonally.*

Proof. By a version of the Margulis lemma given by Buser-Karcher [3, 2.5.4] we have, under our hypotheses, that if α, β are loops at a point $q \in M$ which have lengths $|\,\alpha\,|$, $|\,\beta\,| \leqslant 2\varepsilon_n$ then α, β generate a cyclic subgroup of $\pi_1(M, q)$. Lemma 3 can be derived from this result as follows. Let $P \in \mathcal{O}$ be a given point, and let $\tilde{P}$ be a point in the universal cover $\tilde{M}$ of M lying above P. Since $i(P) < \varepsilon_n$, there is a deck transformation γ which translates $\tilde{P}$ a distance less than $2\varepsilon_n$. Because M has negative curvature, there is a unique geodesic σ which is preserved by γ. Let $\langle g\rangle$ denote the cyclic group of deck transformations which preserve σ. For any $h \in \langle g\rangle$, the function $\delta_h(x) = d(x, hx)$ is a convex function on M which achieves its minimum value on σ. The set $\tilde{\mathcal{O}}_i$ defined by

$$\tilde{\mathcal{O}}_i = \{x \in \tilde{M}: \delta_h(x) < \varepsilon_n \text{ for some } h \in \langle g\rangle\}$$

is therefore a finite union of convex neighborhoods of σ. Hence $\tilde{\mathcal{O}}_i$ is star-shaped with respect to σ. Now if k is a deck transformation such that both x and $k(x)$ lie in $\mathcal{O}_i$ for some x, then for some integers r, s, g^r (resp. g^s) translates x (resp. $k(x)$) a distance less than $2\varepsilon_n$. But then both g^r an $k^{-1}g^s k$ translate x less than $2\varepsilon_n$ and hence we have $k^{-1}g^s k \in \langle g\rangle$. From this it follows that g, k generate a solvable subgroup of π, which is cyclic by Preissman's theorem and hence $k \in \langle g\rangle$. Therefore, the set $\tilde{\mathcal{O}}_i/\langle g\rangle = \mathcal{O}_i$ is a domain in M containing the original point P and is a component of $\mathcal{O}$. This gives the conclusions of Lemma 3.

Proof of Theorem. To prove the theorem we will use the isoperimetric quantity $h(M)$ of Cheeger [4] defined by

$$h(M) = \inf\left\{\frac{\operatorname{Vol}(\Sigma^{n-1})}{\min\{V_1, V_2\}}\right\},$$

where the infimum is taken over all smooth embedded hypersurfaces Σ^{n-1} (not necessarily connected) which divide M into two components with volumes V_1, V_2. In [4], Cheeger proved the inequality

$$\lambda_1(M) \geqslant \tfrac{1}{4}h^2(M). \tag{1}$$

Thus we concentrate our efforts on giving a lower bound on $h(M)$. We will use the following existence theorem from minimal surface theory (see [5, Chapter 5], [6])

Existence Theorem. *For any v with $0 < v \leqslant \frac{1}{2}\operatorname{Vol}(M)$, there exist an open set $\Omega_v \subset M$ with $\operatorname{Vol}(\Omega_v) = v$, and a smooth embedded hypersurface Σ_v with the property that $\overline{\Sigma}_v = \partial\Omega_v$, $\mathcal{H}^s(\overline{\Sigma}_v \sim \Sigma_v) = 0$ for $s > n - 8$, and Ω_v has the extremal property*

$$\operatorname{Vol}(\Sigma_v) = \inf\{\operatorname{Vol}(\partial\Omega): \Omega \subseteq M \text{ with } \operatorname{Vol}(\Omega) = v\}.$$

Moreover, the mean curvature vector H of Σ_v satisfies $|H| \equiv H_v$ for a constant $H_v \geqslant 0$ as well as the property that H points everywhere into or everywhere out of Ω_v.

From the extremal property of Σ_v it is clear that

$$h(M) \geqslant \inf\{v^{-1}\operatorname{Vol}(\Sigma_v): 0 < v \leqslant \tfrac{1}{2}\operatorname{Vol}(M)\}. \tag{2}$$

We divide our proof into two cases. First, if $H_v \geqslant 1$ then Lemma 1 can be applied to give

$$(3) \qquad v^{-1}\operatorname{Vol}(\Sigma_v) \geqslant n-1.$$

Note that one has to take some care in applying Lemma 1 because for large n, Σ_v may have singularities. By the observation of Gromov [8], a nearest point to $\overline{\Sigma}_v$ from any given point of $M \sim \overline{\Sigma}_v$ is always a regular point and hence the methods of [9] are applicable.

The remaining case is $H_v < 1$. Now if it were true that

$$(4) \qquad \operatorname{Vol}(\Sigma_v) \geqslant \omega_{n-1}\left[\varepsilon_n e^{-\varepsilon_n(1-\kappa)}\right]^{n-1},$$

where $\varepsilon_n = 4^{-(n+3)}$, then we would be finished in light of (1)–(4). Therefore we assume that (4) does not hold. Then from Lemma 2 we would have $i(P) < \varepsilon_n$ for every $p \in \overline{\Sigma}_v$; that is, we have $\overline{\Sigma}_v \subseteq \mathcal{O}$ in the terminology of Lemma 3. Since $n > 2$, the set $M \sim \mathcal{O}$ is connected. Let U_v be the component of $M \sim \overline{\Sigma}_v$ which contains $M \sim \mathcal{O}$, and let $\Omega'_v = M \sim \overline{U}_v$ and $\Sigma'_v = \Sigma_v \cap \partial\Omega'_v$. By construction we have

$$(5) \qquad v^{-1}\operatorname{Vol}(\Sigma_v) \geqslant \operatorname{Vol}(\Omega'_v)^{-1}\operatorname{Vol}(\Sigma'_v)$$

(recall that $v \leqslant \frac{1}{2}\operatorname{Vol}(M)$). Let Ω be a component of Ω'_v and let $\Sigma = \Sigma_v \cap \partial\Omega$. Then $\overline{\Omega} \subset \mathcal{O}_i$ for some component $\mathcal{O}_i$ of $\mathcal{O}$. Since $\mathcal{O}_i$ is the quotient by a cyclic group of a star-shaped neighborhood of a geodesic σ in $\tilde{M}$ (see the proof of Lemma 3), the distance function to σ is a well defined function in $\mathcal{O}_i$ which we denote by ρ. By standard comparison methods we have $\Delta_M\rho \geqslant (n-1)\kappa$ in $\mathcal{O}_i$. Thus Stokes theorem applied in Ω gives

$$(n-1)\kappa\operatorname{Vol}(\Omega) \leqslant \operatorname{Vol}(\Sigma).$$

Since any two components of Ω'_v have disjoint closures, we can sum these inequalities over all components of Ω'_v to conclude

$$(6) \qquad \operatorname{Vol}(\Omega'_v)^{-1}\operatorname{Vol}(\Sigma'_v) \geqslant (n-1)\kappa.$$

Combining (1)–(6) we have

$$\lambda_1(M) \geqslant \min\left\{\frac{(n-1)^2\kappa^2}{4}, \frac{\delta_n}{\operatorname{Vol}^2(M)}\right\},$$

where $\delta_n = 4^{-1}\omega_{n-1}^2[\varepsilon_n e^{-\varepsilon_n(1-\kappa)}]^{2n-2}$, $\varepsilon_n = 4^{-(n+3)}$. The final inequality of the theorem follows from this because by Lemma 3 there is a point $P \in M$ with $i(P) \geqslant \varepsilon_n$ hence the volume satisfies

$$\operatorname{Vol}(M) \geqslant \omega_n\varepsilon_n^n.$$

Thus we have

$$\lambda_1(M) \geqslant \frac{\delta'_{n,\kappa}}{\mathrm{Vol}^2(M)},$$

where $\delta'_{n,\kappa} = 4^{-1}(n-1)^2\kappa^2\omega_n^2\varepsilon_n^{2n}$. (Note that for $n \geqslant 3$ we have $\delta'_{n,\kappa} \leqslant \delta_n$.) This completes the proof of the main theorem.

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