J. DIFFERENTIAL GEOMETRY
16 (1981) 271–280

A C^∞ FLOW ON S^3 WITH A DENJOY MINIMAL SET

R. J. KNILL

The C^r Seifert conjecture is that every C^r flow on S^3 without singular points has a periodic orbit. Schweitzer gave a counterexample to the C^1 conjecture in 1974. The C^r conjecture remains open for $2 \leqslant r \leqslant \infty$. Recent work of Michael Handel suggests that a certain property of the Schweitzer counterexample is typical. Namely, if there is an isolated one-dimensional exceptional minimal set of a C^1 flow on S^3, then that minimal set should be a surface minimal set. The importance of this assertion is that a flow on a surface which has an exceptional minimal set can be no more than once differentiable.

We give here an example of a C^∞ flow on S^3 with an exceptional surface minimal set. Properties of this flow in a neighborhood of the minimal set are suggestive of properties that any C^∞ flow on S^3 with an exceptional minimal set should have. These properties include hyperbolicity of the Poincaré map on a cross section to the flow and consequent (in our context) existence of shift automorphisms and periodic trajectories arbitrarily close to the exceptional minimal set.

After describing our flow we will conclude with a discussion of some problems which it suggests–directions to take in pursuit of an answer to the C^r Siefert conjecture for high values of r.

The general idea of our construction is as follows. The Denjoy diffeomorphism f of S^1 is expansive on countably many subintervals of S^1 and contractive on complementary intervals. After adjusting Denjoy's example so that the expansive part of the diffeomorphism f expands by a factor of two (an "eigenvalue" of two), and the contractive part has an eigenvalue of one half, then the intervals on which f is expansive are embedded in the punctured $\mathbf{R}^2 \setminus 0$, tangent to one direction X, and the intervals on which f is contractive are embedded tangent to another direction Y. These directions are given in polar coordinates, i.e., $X = x_1\frac{\partial}{\partial r} + x_2\frac{\partial}{\partial \theta}$ and $Y = y_1\frac{\partial}{\partial r} + y_2\frac{\partial}{\partial \theta}$ so that one can achieve an embedding of the circle with a sort of Cantor radial

Communicated by R. Bott, May 23, 1980.

saw-tooth configuration. After making the embedding, f is conjugate to a map $\bar{f}$ on the Cantor saw-teeth which extends to a C^∞ map on a neighborhood. Near the Cantor saw-teeth, the differential of $\bar{f}$ has matrix M with eigenvalue two in the X-direction, and eigenvalue one-half in the Y-direction. Hence $\bar{f}$ is hyperbolic. By standard arguments every neighborhood of a point in the Cantor saw-teeth contains a shift automorphism. $\bar{f}$ is extended to an annulus in a C^∞ fashion, and then suspended to give a C^∞ flow in a flow box. This in turn is used to define the required C^∞ flow on S^3.

We would like to express our appreciation to Paul Schweitzer, Colin Rourke, and Jenny Harrison for openly sharing their insights into the problems associated with the Seifert conjecture.

1. Linearization of the Denjoy diffeomorphisms

Let us begin by summarizing some of the properties of the Denjoy diffeomorphisms of S^1. See [3], [5] and [7].

1.1. Theorem (*Denjoy*). *There is an orientation preserving C^1 diffeomorphism f: $S^1 \to S^1$ with the following properties*:

1.1.1. *There is a Cantor set K in S^1 invariant under f.*

1.1.2. *There are no periodic points of f.*

1.1.3. *The complement of K in S^1 is a union of countably many open intervals I_n, $n \in \mathbf{Z}$, such that $f(I_n) = I_{n+1}$ for each $n \in \mathbf{Z}$.*

1.1.4. *If μ is Haar measure on S^1, then $\mu(K) = 0$.*

Let us modify f and I_n, $n \in \mathbf{Z}$ to make our job easier. The modified function f will, however, only be a homeomorphism, and will no longer be C^1.

1.2. We assume f and I_n, $n \in \mathbf{Z}$, satisfy:

1.2.1. $\mu(I_n) = 2\pi/(3 \cdot 2^{|n|})$, $n \in \mathbf{Z}$.

1.2.2. Regarded as a function of arclength, suppose that $f|I_n$ is a linear map onto I_{n+1} for each $n \in \mathbf{Z}$.

The following theorem will be proven in §2.

2.1. Main theorem. *There is a continuous embedding*

$$g : S^1 \to \mathbf{E}^2$$

such that

2.1.1. *there exist positive numbers $a < b$ such that*

$$g(S^1) \subset \{(r, \theta) : a \leqslant r \leqslant b\} = \mathcal{Q}.$$

Here (r, θ) are the polar coordinates of a point.

2.1.2. *there is a C^∞ diffeomorphism F: $\mathcal{Q} \to \mathcal{Q}$ such that*

$$Fg|K = gf|K.$$

Actually we prefer to lift f to a map $\tilde{f}$ on $\mathbf{R}$, so that if p: $\mathbf{R} \to S^1$ is the map $p(t) = \exp(it)$, then $p\tilde{f} = \tilde{f}p$. For each $n \in \mathbf{Z}$, let $J_n = p^{-1}(I_n)$. Then J_n is a union of intervals whose centers differ by an integer multiple of 2π, all of which have the same length $\mu(I_n)$.

From Properties 1.2 of I_n and f we obtain

1.3. *Properties of J_n and $\tilde{f}$:*

1.3.1. *The length of each component of J_n is $2\pi/(3 \cdot 2^{|n|})$, $n \in \mathbf{Z}$.*

1.3.2. *The derivative of $\tilde{f}|J_n$ is*

$$(\tilde{f}|J_n)' = 2 \textit{ for } n < 0,$$

$$(\tilde{f}|J_n)' = \tfrac{1}{2} \textit{ for } n \geqslant 0.$$

1.3.3. *Let λ: $\mathbf{R} \to \mathbf{R}$ be the measurable function which has value 0 on $\tilde{K} = p^{-1}(K)$, value 2 on J_n, $n \geqslant 0$ and value $\frac{1}{2}$ on J_n for $n \geqslant 0$. Then*

$$\tilde{f}(x) = \tilde{f}(0) + \int_0^x \lambda(t)\, dt. \tag{1}$$

1.3.4. *$\tilde{f}(x + 2k\pi) = \tilde{f}(x) + 2k\pi$ for each integer k.*

Now choose vectors $X = (1, 0)$, and $Y = (-2, 3)$ in $\mathbf{R}^2$, so that X and Y are linearly independent and $\frac{2}{3}X + \frac{1}{3}Y = (0, 1)$.

What we shall now do is to define an embedding

$$\gamma: \mathbf{R} \to \mathbf{R}^2$$

such that the following two properties will hold.

1.4. *Desired properties of γ:*

1.4.1. *γ is $\mathbf{Z}$ equivariant in the sense that for any integer k,*

$$\gamma(x + 2k\pi) = \gamma(x) + (0, 2k\pi), \quad x \in \mathbf{R}^1.$$

1.4.2. *Let $\bar{f} = \gamma\tilde{f}\gamma^{-1}$. Then $\bar{f}|\gamma(\tilde{K})$: $\gamma(\tilde{K}) \to \gamma(\tilde{K})$ has differential M where M is the linear transformation of $\mathbf{R}^2$ which has eigenvalues 2 and $\frac{1}{2}$, with corresponding eigenvectors X and Y, respectively.*

We take our cue for defining γ from Property 1.3.3. Namely let $\gamma(x)$ be defined as

$$\gamma(x) = \gamma(0) + \int_0^x \phi(t)\, dt, \tag{2}$$

where ϕ is the measurable vector valued function

$$\phi: \mathbf{R} \to \mathbf{R}^2$$

defined by: *ϕ has value 0 on $\tilde{K} = p^{-1}(K)$, it has the value X on j_n if $n \leqslant 0$, and it has the value Y on J_n for $n > 0$.* ($\gamma(0)$ will be specified in §2.)

Proof of 1.4.1. Since $\tilde{K} + 2k\pi = \tilde{K}$ and $J_n + 2k\pi = J_n$ for any integers k and n, ϕ is periodic of period 2π. Thus for any x

$$\gamma(x + 2\pi) - \gamma(x) = \int_x^{x+2\pi} \phi(t)\, dt = \int_0^{2\pi} \phi(t)\, dt. \tag{3}$$

At this point we make the assumption, whose motivation would not have been clear earlier, that 0 is the left endpoint of a component of J_0. Thus $0 \in \tilde{K}$. It follows that each J_n has exactly one component in $[0, 2\pi]$, called $\tilde{I}_n$, of length $2\pi/(3 \cdot 2^{|n|})$. Since $\phi(t)$ is zero on $\tilde{K}$, is X on $\tilde{I}_n$ for $n \leqslant 0$, and is Y on $\tilde{I}_n$ for $n > 0$, we have

$$\begin{aligned}\int_0^{2\pi} \phi(t)\, dt &= \sum_{-n=0}^{\infty} \mu(\tilde{I}_n)X + \sum_{n=1}^{\infty} \mu(\tilde{I}_n)Y \\ &= \frac{2\pi}{3}\left(\sum_{n=0}^{\infty} \frac{1}{2^n} X + \sum_{n=1}^{\infty} \frac{1}{2^n} Y\right) \\ &= 2\pi\left(\frac{2}{3}X + \frac{1}{3}Y\right) = (0, 2\pi),\end{aligned}$$

which together with (3) proves 1.4.1.

Proof of 1.4.2. We shall prove the stronger statement that $\bar{f}|\gamma(\mathbf{R}^1 - J_0)$ has derivative M. In fact we shall show local affineness of $\bar{f}|\gamma(\mathbf{R}^1 - J_0)$:

1.4.2′. *If x and y are in a common component of $\mathbf{R}^1 - J_0$, then*

$$\bar{f}(\gamma(y)) - \bar{f}(\gamma(x)) = M(\gamma(y) - \gamma(x)).$$

Proof. Let $h = y - x$. Then

$$\begin{aligned}\bar{f}\gamma(y) - \bar{f}\gamma(x) &= \gamma\tilde{f}\gamma^{-1}\gamma(x + h) - \gamma\tilde{f}\gamma^{-1}\gamma(x) \\ &= \gamma\tilde{f}(x + h) - \gamma\tilde{f}(x) \\ &= \int_{\tilde{f}(x)}^{\tilde{f}(x+h)} \phi(t)\, dt.\end{aligned}$$

Making the substitution

$$t = \tilde{f}(s),$$

we find that since

$$\tilde{f}'(s) = \lambda(s) \text{ (almost everywhere)},$$

then

$$dt = \lambda(s)\, ds.$$

Hence we obtain

$$\begin{aligned}\int_{\tilde{f}(x)}^{\tilde{f}(x+h)} \phi(t)\, dt &= \int_x^{x+h} \phi(\tilde{f}(s))\lambda(s)\, ds \\ &= \int_x^{x+h} \phi(s)\lambda(s)\, ds, \qquad (4)\end{aligned}$$

since $\phi(\tilde{f}(s)) = \phi(s)$, $s \notin J_0$. Furthermore for $s \notin J_0$, $\phi(s)\lambda(s) = M(\phi(s))$ since the eigenvalues of M corresponding to the vectors $\phi(s) = 0, X, Y$ are $\lambda(s) = 0, 2\ \frac{1}{2}$, respectively. Thus we have

$$\begin{aligned} \int_x^{x+h} \phi(s)\lambda(s)\, ds &= \int_x^{x+h} M(\phi(s))\, ds \\ &= M\int_x^{x+h} \phi(s)\, ds \qquad (5) \\ &= M(\gamma(x+h) - \gamma(x)), \end{aligned}$$

which together with (4) and $y = x + h$ proves 1.4.2′.

2. Imbedding of S^1 into $\mathbf{E}^2$ with polar coordinates

A point of $\mathbf{E}^2$ with polar coordinates (r, θ) has rectangular coordinates $\binom{u}{v}$ where

$$u = r\cos\theta, \quad v = r\sin\theta. \tag{6}$$

Let ρ: $\mathbf{R}^2 \to \mathbf{E}^2$ be the map $\rho(r, \theta) = \binom{u}{v}$, defined by (6). It maps $\gamma(\mathbf{R}^1)$ onto a topological circle $\rho\gamma(\mathbf{R}^1)$ in $\mathbf{E}^2$, provided $\gamma(0)$ is chosen so that $\gamma(x) = (r_x, \theta_x)$ always has a positive radial coordinate r_x. Evidently this may be accomplished if we let

$$X = (1, 0), \quad Y = (-2, 3), \quad \gamma(0) = \left(\tfrac{10}{3}\pi, 0\right). \tag{7}$$

Indeed the radial coordinate of $\gamma(x)$ is at least the radial coordinate of Y times the measure, $2\pi/3$, of the sums of the lengths of the intervals $\tilde{I}_n$ where ϕ has value Y, plus the radial coordinate of $\gamma(0)$. In short

$$r_x \geqslant -\tfrac{4}{3}\pi + \tfrac{10}{3}\pi = 2\pi. \tag{8}$$

Likewise r_x is at most the radial coordinate of X times the measure, $4\pi/3$, of the sums of the lengths of the intervals $\tilde{I}_n$ where ϕ has value X, plus $10\pi/3$, so the image $\rho\gamma(\mathbf{R}^1)$ lies in an annular region in the plane between the two circles, centered at the origin, or radii 2π and $14\pi/3$.

Let F: $\rho\gamma(\mathbf{R}^1 - J_0) \to \mathbf{E}^2$ be defined by

$$F(\rho\gamma(x)) = \rho\gamma(\tilde{f}(x)) = \rho\bar{f}\gamma(x). \tag{9}$$

Since $\bar{f}$ is locally affine on $\gamma(\mathbf{R}^1 - J_0)$, F is C^∞, i.e., has a natural C^∞ extension to a neighborhood of $\rho\gamma(\mathbf{R}^1 - J_0)$ in $\mathbf{E}^2$. The differential of F is readily computed from (9) and 1.4.2′ as

$$DF = D\rho \circ M \circ D\rho^{-1}. \tag{10}$$

We would like to extend $F|\rho\gamma(\mathbf{R}^1 - J_0)$ to a diffeomorphism of an open neighborhood of $\rho\gamma(\mathbf{R}^1)$ onto the same open neighborhood. We would like to choose the neighborhood to be a topological annulus. It is easier to work with $\bar{f}$ first, then apply definition (9) again. We have been careful to restrict $\bar{f}$ to $\gamma(\mathbf{R}^1 - J_0)$, because its differential there takes X to $2X$, while on $\gamma(J_0)$, $\bar{f}$ is also affine but its directional derivative along $\gamma(J_0)$ takes X to $\frac{1}{2}Y$. Thus there can be no continuous differentiability of $\bar{f}$ at the endpoints of $\gamma(J_0)$, although its "left" and "right" derivatives exist at the endpoints of $\gamma(J_0)$.

If we are to extend $\bar{f}$ (and hence F) to an ε neighborhood of $\gamma(\mathbf{R}^1 - J_0)$ in a C^∞ fashion it makes sense to do so with the same derivative M as indicated by 1.4.2′. This we do, being careful to choose ε to be less than $\frac{1}{3}$ the minimal distance between distinct components of $\gamma(\mathbf{R}^1 - J_0)$. We sketch $\gamma(\tilde{I}_0)$ together with its endpoints A and B in Illustration 1. From the definition of γ in terms of ϕ and the fact that $\gamma(2\pi) = \gamma(0) + (0, 2\pi)$ we see that between $B = \gamma(2\pi/3)$ and $\gamma(2\pi)$, the curve $\gamma[2\pi/3, 2\pi]$ lies in the region outlined with dotted lines (Illustration 1).

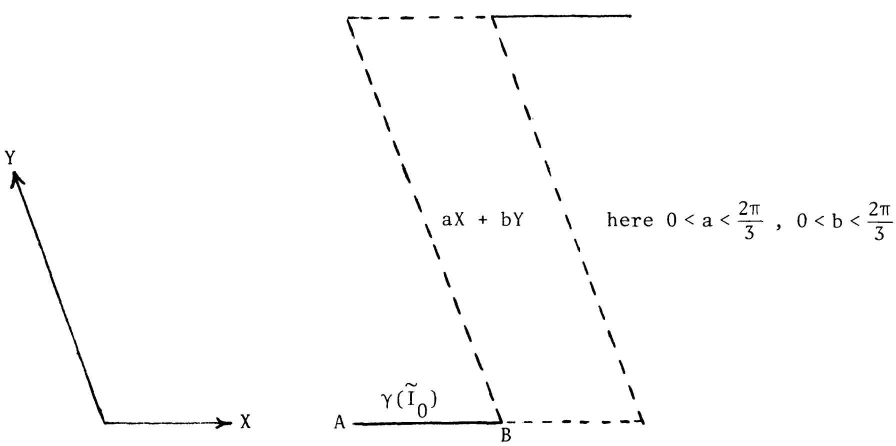

ILLUSTRATION 1

The reason the curve $\gamma(\mathbf{R}^1 - \bar{J}_0)$ never leaves the regions enclosed by the dotted lines is that for $2\pi/3 < x < 2\pi$, $\gamma(x) - \gamma(2\pi/3) = aX + bY$ where $0 < a < 2\pi/3 = \Sigma_{-n=1}^{\infty}\, \mu(\tilde{I}_n)$, $0 < b < 2\pi/3 = \Sigma_{n=1}^{\infty}\, \mu(\tilde{I}_n)$.

Now connect A and B with a C^∞ curve S which leaves A in the direction of Y, crosses the center of $\gamma(\tilde{I}_0)$ in the direction of $-Y$ and reaches B coming in the direction of Y. This is sketched in Illustration 2 together with a dotted neighborhood formed from the union of the dotted region in Illustration 1 together with ε neighborhoods of $\gamma(J_0)$ and $\bigcup_{k\in\mathbf{Z}}[S + (0, 2k\pi)]$.

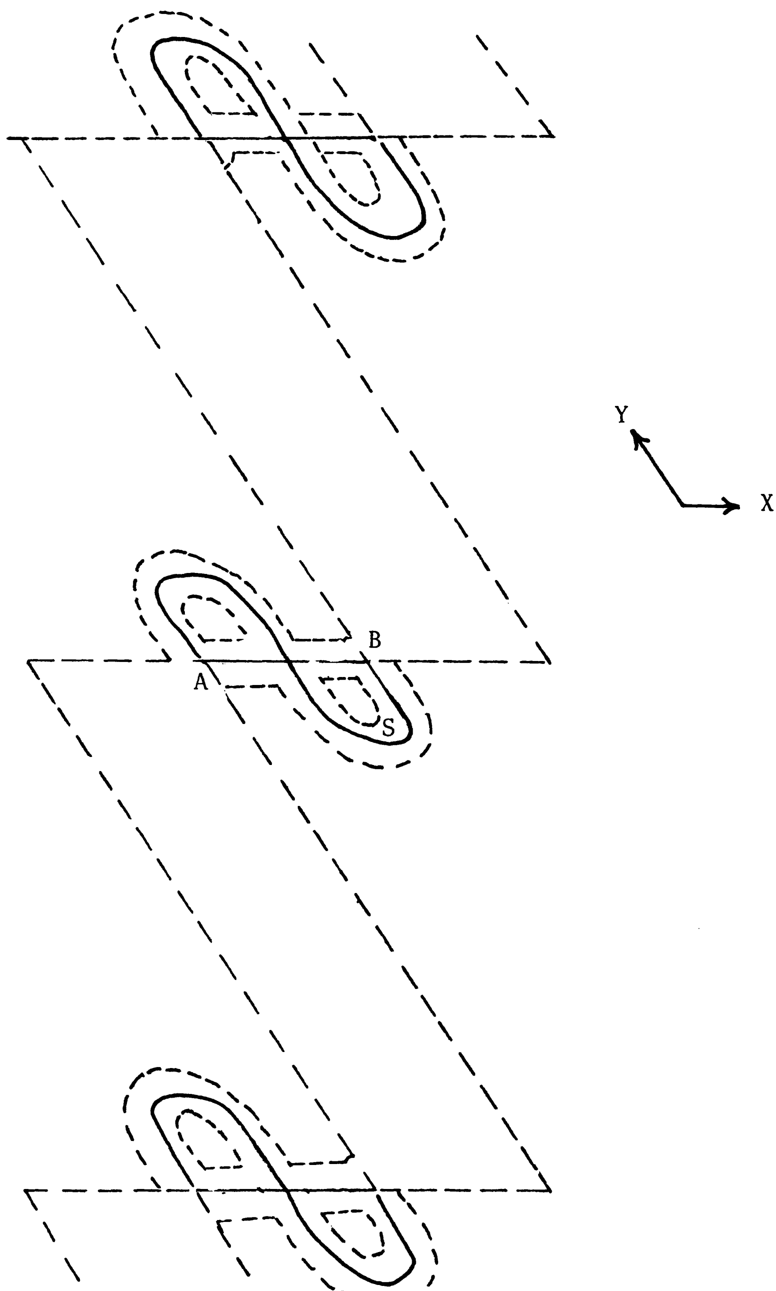

ILLUSTRATION 2

Now extend $\bar{f}$ so that it takes S onto $\gamma(\tilde{I}_1)$, and $\gamma(\tilde{I}_0)$ onto a C^∞ curve S' which leaves C in the direction of X, crosses $\gamma(\tilde{I}_1)$ at the midpoint in the direction of $-X$, and ends at 0, directed in the same direction as X. As indicated in Illustration 3, C and D are the lower and upper endpoints of $\gamma(\tilde{I}_1)$. We have sketched S' along with a neighborhood of $\gamma(\mathbf{R}^1) \cup S'$ below (Illustration 3).

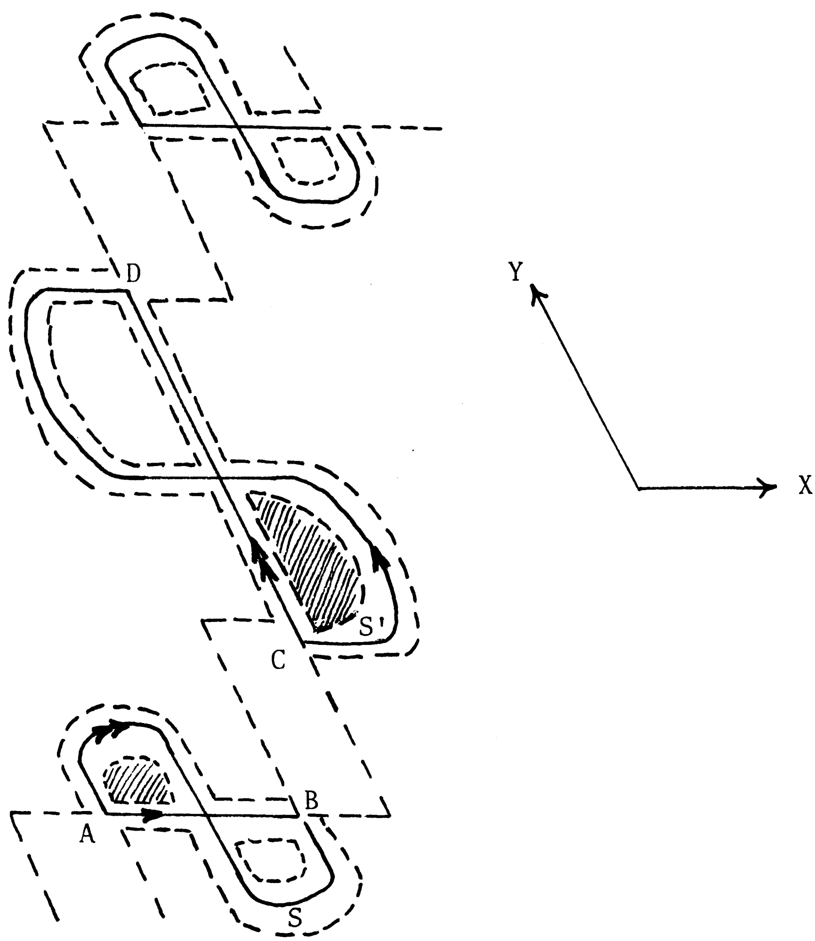

ILLUSTRATION 3

Since the two shaded regions (Illustration 3) are simply connected domains in $\mathbf{R}^2$, with C^∞ diffeomorphic boundaries, then $\bar{f}|S \cup \gamma(\tilde{\mathbf{R}}_0^1)$ extends to a C^∞ function of the sketched neighborhood, together with the shaded region, into

$\mathbf{R}^2$. Likewise $\bar{f}$ extends across the other bounded component of the complement of $S \cup \gamma(\tilde{I}_0)$, so that now $\bar{f}$ is defined on a simply connected neighborhood of $\gamma(\mathbf{R}^1)$. Further $\bar{f}$ as extended can be required to be $\mathbf{Z}$ equivariant in the sense that $\bar{f}(z + (0, 2k\pi)) = \bar{f}(z) + (0, 2k\pi)$ for every integer k and every z in the domain of $\bar{f}$.

This shows that F may be defined on a region in $\mathbf{E}^2$ containing $\rho\gamma(\mathbf{R}^1)$ and C^∞ diffeomorphic to an annulus. Finally we extend F to a C^∞ diffeomorphism of an annular region of the form $\{(r, \theta): a \leqslant r \leqslant b\}$ onto itself.

In summary we have

2.1. Theorem. *There is a continuous embedding*

$$g = \rho\gamma p^{-1} : S^1 \to \mathbf{E}^2$$

such that

2.1.1. *there exist positive numbers $a < b$ such that*

$$g(S^1) \subset \{(r, \theta) : a \leqslant r \leqslant b\} = \mathcal{A},$$

2.1.2. *there is a C^∞ diffeomorphism F: $\mathcal{A} \to \mathcal{A}$ such that*

$$Fg|K = gf|K.$$

3. Concluding discussion

By suspending F we obtain a flow G: $T^2 \times [a, b] \times \mathbf{R}^1 \to T^2 \times [a, b]$ which has a minimal set Σ_G topologically conjugate to the minimal set Σ of the suspension of the Denjoy diffeomorphism f. Using the techniques of [7], one may use G to define a C^∞ flow on S^3 which contains Σ_G as a minimal set. Unfortunately this does not provide us with a counterexample to the Seifert conjecture, nor would any C^∞ nearby modification of G do it, for since M has eigenvalues 2 and $\frac{1}{2}$, F is hyperbolic near $\rho\gamma(\tilde{K})$. Since $F|\rho\gamma(\tilde{K})$ has chain recurrent points, this implies by standard techniques [1] that near each point x of $\rho\gamma(\tilde{K})$, there is an arbitrarily small neighborhood N of x, and an integer k such that $F^k|N$ contains a shift automorphism analogous to that found in the Smale horseshoe [9]. Thus $F|N$ contains infinitely many periodic points. This analysis suggests two questions.

Question 1. In order to smooth Denjoy diffeomorphisms, is it necessary to introduce hyperbolicity of the diffeomorphism on a neighborhood of a minimal set?

"To smooth" is taken to mean to define a C^0 homeomorphism h of the circle into $\mathbf{R}^2$ and a C^∞ diffeomorphism H of $\mathbf{R}^2$ to $\mathbf{R}^2$ such that $Hh(x) = hf(x)$ for $x \in K$, where f and K are as in Theorem 1.1.

Question 2. In view of the work of Michael Handel [4], is it true that a nonsingular flow on S^3 having an isolated minimal set which is one-dimensional and not a closed orbit, cannot be C^∞?

Of course one could pose these questions for C^r ($r \geqslant 2$) diffeomorphisms rather than C^∞ diffeomorphisms.

References

[1] R. Bowen, *Markov partitions and minimal sets for axiom A diffeomorphisms*, Amer. J. Math. **92** (1970) 907–918.

[2] C. Conley, *Hyperbolic sets and shift automorphisms*, MRC Technical Summary Report §1502, Madison, WI, February, 1975.

[3] A. Denjoy, *Sur les courbes definées par les équations différentiables àla surface du tore*, J. Math. Pures Appl. **11** (1932) 333–375.

[4] M. Handel, *Isolated minimal sets and the Seifert conjecture*, Ann. of Math. **111** (1980) 35–66.

[5] Y. Katznelson, *Sigma-finite invariant measures for smooth mappings of the circle*, J. Analyse Math. **31** (1977) 1–18.

[6] Z. Nitecki, *Differentiable dynamics*, M.I.T. Press, Cambridge, MA, 1971.

[7] P. Schweitzer, *Counterexamples to the Seifert conjecture and opening closed leaves of foliations*, Ann. of Math. (2) **100** (1974) 386–400.

[8] H. Seifert, *Closed integral curves in 3-space and isotopic two-dimensional deformations*, Proc. Amer. Math. Soc. **1** (1950) 287–302.

[9] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967) 747–817.

TULANE UNIVERSITY