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RATIONAL PONTRYAGIN CLASSES AND KILLING FORMS

ALLEN BACK

It is well known that equivariant isomorphism classes of K-vector bundles over the homogeneous space K/H are in one-to-one correspondence with equivalence classes of representations of H on the fiber V over the basepoint. Also, if the isotropy representation α: $H \to \mathrm{Aut}(V)$ extends to a homomorphism $\bar{\alpha}$: $K \to \mathrm{Aut}(V)$, then it is easy to see that the associated equivariant bundle is trivial. The main purpose of this note is to use these observations together with the Chern-Weil theory of characteristic classes to prove the following.

Theorem 1. *If the Killing form of a compact Lie group K restricts to a multiple of the Killing form of H, then the first Pontryagin class of the tangent bundle of K/H is torsion.*

If H is a simple Lie group, then any two H-invariant forms are proportional. Consequently, the simplest case in which to apply Theorem 1 is:

Corollary 2. *If K is compact and H is simple, then the first rational Pontryagin class of K/H is trivial.*

We shall in general follow the conventions and notations of [1]. The notations $\underline{K}$ for the Lie algebra of K, Ad_K for the adjoint action of K on $\underline{K}$, and ad_K for its derivative will also be used.

Since K is compact, choice of an Ad_K invariant metric on $\underline{K}$ allows us to write $\underline{K} = \underline{H} + \underline{M}$ where $\underline{M}$ is the orthogonal complement of $\underline{H}$. The representation Ad_K restricted to H leaves $\underline{M}$ invariant, and gives rise to the tangent bundle β_1 of K/H. If we let β_2 and β_3 be the bundles associated to Ad_H and to Ad_K restricted to H respectively, then we clearly have $\beta_3 = \beta_1 + \beta_2$ topologically.

Obviously β_3 comes from a representation extending to K, so β_3 is trivial. Consequently the first Pontryagin class of β_2 is the negative of that of β_1.

Proof of Theorem 1. Equip β_2 and β_3 with the K-invariant canonical connections of the first kind (i.e., the vector $\bar{X}$ induced by the element X of $\underline{M}$

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is horizontal at the base point). The Lie algebra valued curvature forms of β_2 and β_3 are then given by $\Omega_2(\bar{X}, \bar{Y}) = -\mathrm{ad}_H([X, Y]_{\underline{H}})$ and $\Omega_3(\bar{X}, \bar{Y}) = -\mathrm{ad}_K([X, Y]_H)$ respectively where $[X, Y]_H$ is the $\underline{H}$ component of $[X, Y]$ in $\underline{K} = \underline{H} + \underline{M}$.

Now the first Pontryagin form p_1 of a connection with $\mathrm{End}(V)$ valued curvature form Ω is given by

$$p_1(\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4) = c\,\mathrm{Tr}_V[\Omega(X_1, X_2)\Omega(X_3, X_4) - \Omega(X_1, X_3)\Omega(X_2, X_4) + \Omega(X_1, X_4)\Omega(X_2, X_3)],$$

where c is a universal constant, and $\mathrm{Tr}_V[f]$ is the trace of the element $f \in \mathrm{End}(V)$.

For the bundle β_3, $\mathrm{Tr}_K[\Omega_3(\bar{X}, \bar{Y})\Omega_3(\bar{Z}, \bar{W})]$ is simply $\langle [X, Y]_{\underline{H}}, [Z, W]_{\underline{H}} \rangle_K$ where $\langle\ ,\ \rangle_K$ is the Killing form of K. So for β_3,

$$p_1(\bar{X}_1, \bar{X}, \bar{X}_3, \bar{X}_4) = c\big[\langle [X_1, X_2]_{\underline{H}}, [X_3, X_4]_{\underline{H}} \rangle_K - \langle [X_1, X_3]_{\underline{H}}, [X_2, X_4]_{\underline{H}} \rangle_K + \langle [X_1, X_4]_{\underline{H}}, [X_2, X_3]_{\underline{H}} \rangle_K\big].$$

Since β_3 is trivial, this form is exact.

But the first Pontryagin form of β_2 is given by the same expression except that all K-Killing forms $\langle\ ,\ \rangle_K$ are replaced by H-Killing forms $\langle\ ,\ \rangle_H$. So if the Killing form of K restricts to a nonzero multiple of that of H, it is immediate that $p_1(\beta_2)$ is rationally trivial. If the Killing form of K restricts to zero on H, then the semisimple part of H is trivial, and all Pontryagin classes of K/H will be zero.

It is interesting to note that the proof of Theorem 1 may also be carried out using the natural Riemannian connections on the β_i.

Other examples to which Theorem 1 applies may be readily constructed; e.g., $SU(P)/(SU(m))^n$. More generally, if H_1 and H_2 are semisimple, then any $\mathrm{Ad}_{H_1 \times H_2}$ invariant form on $\underline{H}_1 + \underline{H}_2$ will have $\underline{H}_1$ and $\underline{H}_2$ orthogonal to each other. Consequently we have

Corollary 3. *If H_1 and H_2 are simple and conjugate inside K, then $p_1(K/(H_1 \times H_2))$ will be torsion.*

One might also notice that the proof of Theorem 1 immediately generalizes to higher characteristic forms. Given $A_i \in \underline{K}$ $(1 \leqslant i \leqslant n)$, define the "higher Killing form" $B_n(A_1, A_2, \cdots, A_n)$ to be $\mathrm{Tr}_{\underline{K}}(\mathrm{ad}_{\underline{K}} A_1\, \mathrm{ad}_{\underline{K}} A_2 \cdots \mathrm{ad}_{\underline{K}} A_n)$ (or the symmetrized version). Then the argument of Theorem 1 will express the symmetric sum characteristic forms s_n (i.e. the image under the Chern-Weil homomorphism of the symmetric polynomial $X \to \mathrm{Tr}(X^n)$) of β_2 and β_3 in terms of the higher Killing forms of H and K respectively. Hence

Theorem 4. *If the higher Killing form B_n of K restricts to a nonzero multiple of that of H, then the s_n-characteristic class of the stable normal bundle of K/H is torsion.*

References

[1] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vols. I, II, Wiley, New York, 1963, 1969.

[2] A. Borel & F. Hirzebruch, *Characteristic classes and homogeneous spaces*. I, Amer. J. Math. **80** (1958) 458–538.

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