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MALCEV'S COMPLETION OF A GROUP AND DIFFERENTIAL FORMS

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1. Let G be a finitely generated group, and let $G_2 = (G, G)$ be the normal subgroup of G generated by the commutators $(a, b) = a^{-1}b^{-1}ab$; $a, b \in G$. Inductively we have the sequence of normal subgroups $G_{k+1} = (G, G_k)$, $k = 1, 2, \dots, G_1 = G$ of G and the corresponding tower of nilpotent groups $G/G_2 \leftarrow G/G_3 \leftarrow \cdots$. We assume that none of the groups G/G_k has an element of finite order. Then we talk about the group G without torsion.

A group $\mathcal{G}$ is said to be complete if for any positive integer n and any element $g \in \mathcal{G}$ the equation $x^n = g$ has at least one solution in $\mathcal{G}$. For any finitely generated nilpotent group N without torsion Malcev [4] constructed a complete nilpotent group $\overline{N}$ without torsion, called the completion of N, and an injection of N into $\overline{N}$. Furthermore he constructed a Lie algebra LN over the rationals and proved that there is a 1-1 correspondence between the complete nilpotent groups without torsion and rational Lie algebras. Thus for any finitely generated group G without torsion we have the tower of Malcev's completions

$$\overline{G/G_2} \leftarrow \overline{G/G_3} \leftarrow \cdots$$

and the tower of nilpotent rational Lie algebras

$$LG/G_2 \leftarrow LG/G_3 \leftarrow \cdots,$$

given by Malcev's theory. We talk about the Lie algebra LG of the group G. Each Lie algebra LG/G_k can be given a structure of a group by the Campbell-Hausdorff formula

$$x \circ y = x + y + \tfrac{1}{2}[x, y] + \cdots.$$

This group is isomorphic with $\overline{G/G_k}$.

On the other hand the rational homotopy type of the Eilenberg-McLane space $K(G, 1)$ is completely determined by a differential graded algebra which is free with a decomposable differential and is constructed inductively by the elementary extensions. Such algebras are said to be minimal by

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Sullivan [2]. The elements of degree one form a subalgebra $M = MK(G, 1) = MG$. A pair (M, ψ), where ψ is a morphism of the differential graded algebra M into the algebra of rational forms $A^*(K(G, 1))$ on $K(G, 1)$ which induces an isomorphism on the cohomology in dimension 1 and injectivity in dimension 2, is called the 1-minimal model for G. The duals of the indecomposables in M define a Lie algebra LM over the rationals with the bracket given by the differential.

The sequence of elementary extensions in the construction of a minimal algebra give an increasing filtration of the 1-minimal algebra $M = \bigcup M_k$, $M_2 \subset M_3 \subset \cdots$, by 1-minimal subalgebras; M_k is the 1-minimal algebra of the nilpotent group G/G_k. The corresponding Lie algebras give the tower of nilpotent Lie algebras

$$L/L_2 \leftarrow L/L_3 \leftarrow \cdots,$$

where $L = LM$, $L_{k+1} = [L, L_k]$, $k \geqslant 1$, $L/L_k = LM_k$. This tower will be called the Lie algebra $L = LMG$ of the 1-minimal model for G.

Although it has been accepted that these two rational Lie algebras LG and L constructed by Malcev and Sullivan are isomorphic, the proof of this fact seems to exist only for few examples. The purpose of this note is to give a complete proof of this fact.

Theorem. *The Lie algebra LG of a finitely generated group G without torsion and the Lie algebra $L = LM$ of the* 1*-minimal model of the group G are isomorphic as rational Lie algebras.*

By extending the coefficients of LG from the rationals to the reals we get the tower of real nilpotent Lie algebras. The exponential map exp: $LG/G_k \otimes R \to \mathcal{G}_k$ defines a simply connected Lie group $\mathcal{G}_k$ over R. The image of the rational Lie algebra LG/G_k in $\mathcal{G}_k$ under the map exp is a totally disconnected subgroup which is isomorphic with $\overline{G/G_k}$. Furthermore the inclusion of G/G_k into its completion $\overline{G/G_k}$ via this isomorphism defines a subgroup $G(k)$ of $\mathcal{G}_k$, which is isomorphic with G/G_k. The space of orbits $\mathcal{G}_k/G(k)$ is a real compact nilmanifold N_k.

The construction of the nilmanifolds could have been started with the Lie algebra L instead of LG. The exponential map exp: $LM_k \otimes R \to \mathcal{L}_k$ of the real extension of LM_k defines a simply connected Lie group $\mathcal{L}_k$, and the image of the rational Lie algebra LM_k in $\mathcal{L}_k$ gives a totally disconnected subgroup $S^k = SLM_k$ of $\mathcal{L}_k$. The isomorphism of the Lie algebras $L = LMG$ and LG is proved by verifying that the groups S^k and $\overline{G/G_k}$ are isomorphic. Then the simply connected Lie groups $\mathcal{L}_k$ and $\mathcal{G}_k$ are also isomorphic.

2. In this section we prove that for any finitely generated group G without torsion there is a 1-1 correspondance between the extension cocycles for the group and the extension cocycles for the Lie algebra of the 1-minimal model of G.

Suppose that $A^*(G_k/G_{k+1}) = A^*(K(\overline{G_k/G_{k+1}}, 1))$, $\overline{G_k/G_{k+1}} = (G_k/G_{k+1}) \otimes Q$, and $A^*(G/G_k) = A^*(K(\overline{G/G_k}, 1))$ are the complexes of rational forms on the respective Eilenberg-McLane spaces, and $H^*(G_k/G_{k+1}) = H^*(K(\overline{G_k/G_{k+1}}, 1); Q)$ and $H^*(G/G_k) = H^*(K(\overline{G/G_k}, 1); Q)$ are the singular cohomology rings of those spaces. Let $x_1^{(k)}, \cdots, x_n^{(k)}$ be the generators for $H^1(G/G_k)$, and let $y_1^{(k)}, \cdots, Y_{m_k}^{(k)}$ be the generators for $H^1(\underline{G_k/G_{k+1}})$. The transgression τ: $H^1(\underline{G_k/G_{k+1}}) \to H^2(G/G_k)$ for the fibration $K(\overline{G_k/G_{k+1}}, 1) \to K(\overline{G/G_{k+1}}, 1) \to K(\overline{G/G_k}, 1)$ sends the generators $y_j^{(k)}$ into the elements $z_j^{(k)} = \tau y_j^{(k)}, j = 1, 2, \cdots, m_k$. We choose the forms $a_j^{(k)} \in A^1(G/G_{k+1})$ which restricted to the fibre $K(\overline{G_k/G_{k+1}}, 1)$ represent the cohomology classes $y_j^{(k)}$ and the representatives $b_j^{(k)}$ for the transgressive elements $z_j^{(k)}, j = 1, 2, \cdots, m_k$. Then the transgression is given by the map τ: $a_j^{(k)} \to b_j^{(k)}, j = 1, 2, \cdots, m_k$. Let $c_1, \cdots, c_n$ be elements of $A^1(G/G_2)$ representing the cohomology classes $x_1^{(2)}, \cdots, x_n^{(2)}$ which generate $H^1(G/G_2)$. The successive pullbacks of the forms $c_1, \cdots, c_n$, $\underline{a_1^{(2)}, \cdots, a_{m_2}^{(2)}}, \cdots, \underline{a_1^{(k-1)}, \cdots, a_{m_{k-1}}^{(k-1)}}$ by the projections $K(\overline{G/G_k}, 1) \to K(\overline{G/G_{k+1}}, 1) \to \cdots K(\overline{G/G_2}, 1)$ are the elements of $A^1(G/G_k)$, and will be denoted by $c_1^{(k)}, \cdots, c_{n_k}^{(k)}$.

On the module $\bigoplus A^*(G/G_k) \otimes \overline{G_k/G_{k+1}}$ we define the differential $d \otimes 1$, where d is the differential on forms. $H^1(G_k/G_{k+1}) = \mathrm{Hom}(H_1(G_k/G_{k+1}), Q)$ is generated by the $y_j^{(k)}$'s and $H_1(G_k/G_{k+1}) = \overline{G_k/G_{k+1}}$. We denote the generators for the abelian group $\overline{G_k/G_{k+1}}$ also by $y_j^{(k)}, j = 1, 2, \cdots, m_k$.

Proposition 2.1. *The transgression*

$$\tau\colon H^1(G_k/G_{k+1}) \to H^2(G/G_k)$$

gives the extension cocycle for the group in the form

$$\sum_{j=1}^{m_k} b_j^{(k)} \otimes y_j^{(k)} \in A^2(G/G_k) \otimes \overline{G_k/G_{k+1}}.$$

Proof. The transgression on homology

$$\tau\colon H_2(G/G_k) \to H_1(G_k/G_{k+1}) = \overline{G_k/G_{k+1}}$$

determines the τ on the cohomology. It is clearly represented by the above cocycle.

This extension cocycle is completely determined by the 1-minimal model M for the group G. We define $M = \bigcup M_k$ inductively as in [2]. $M_2 = \bigwedge(\omega_1, \cdots, \omega_n)$ with the differential $d\omega_i = 0, i = 1, 2, \cdots, n$, and degree of ω_i,

$+1$, and there is the algebra morphism $\psi\colon M_2 \to A^*(\overline{G/G_k})$, $\psi(\omega_i) = c_i$, $i = 1, 2, \cdots, n$. In M_2^2 (elements of degree 2 in M_2) there are elements β_j such that $\psi(\beta_j) = b_j$, $j = 1, 2, \cdots, m$; and rational numbers β_j^{rs} such that

$$\beta_j = \sum \beta_j^{rs} \omega_r \wedge \omega_s.$$

Then define $M_3 = \bigwedge(\omega_1, \cdots, \omega_n; \gamma_1, \cdots, \gamma_m)$, where $\gamma_1, \gamma_2, \cdots, \gamma_m$ are new generators in degree $+1$ and $d\omega_i = 0$, $d\gamma_j = \beta_j$, $\psi(\gamma_j) = a_j$. In this way the morphism ψ is a morphism of differential graded algebras, and it induces an isomorphism on cohomology in all dimensions. Inductively we have $M_k = \bigwedge(\omega_1^{(k)}, \cdots, \omega_{n_k}^{(k)})$ with the differential d_k, and again in M_k^2 there are elements

$$\beta_j^{(k)} = \sum \beta_j^{(k)rs} \omega_r^{(k)} \wedge \omega_s^{(k)}$$

such that $\psi_k\colon M_k \to A^*(G/G_k)$ maps $\beta_j^{(k)}$ to $b_j^{(k)}$. The elementary extension $M_{k+1} = \bigwedge(\omega_1^{(k)}, \cdots, \omega_{n_k}^{(k)}; \gamma_1^{(k)}, \cdots, \gamma_{m_k}^{(k)})$, $\gamma_j^{(1)} = 0$, with the new elements $\gamma_j^{(k)}$ of degree $+1$ and the differential $d_{k+1} = d_k$ on the $\omega_i^{(k)}$'s and $d_{k+1}\gamma_j^{(k)} = \beta_j^{(k)}$, $j = 1, 2, \cdots, m_k$, and also $\psi_{k+1} = \psi_k$ on M_k and $\psi_{k+1}(\gamma_j^{(k)}) = a_j^{(k)}$. From this construction we have

Proposition 2.2. *The extension cocycle for the group $\overline{G/G_{k+1}}$ is the ψ-image of the element*

$$\sum_{j=1}^{m_k} \beta_j^{(k)} \otimes y_j^{(k)} \in M_k^2 \otimes \overline{G_k/G_{k+1}}\,.$$

The decomposable elements in $M = \bigcup M_k$ together with the differential give the Lie algebra $L = LM$ with the filtration by ideals $L = L_1 \supset L_2 \supset \cdots, L_{k+1} = [L, L_k]$, $k = 1, 2 \cdots$. Let $X_1, \cdots, X_n$ be the dual elements to the generators $\omega_1, \cdots, \omega_n$ of M_2; $Y_1, \cdots, Y_m$ the dual elements to the generators $\gamma_1, \cdots, \gamma_m$, and in general $Y_1^{(k)}, \cdots, Y_{m_k}^{(k)}$ the duals to $\gamma_1^{(k)}, \cdots, \gamma_{m_k}^{(k)}$; $\omega_i^{(1)} = \omega_i$, $\gamma_j^{(2)} = \gamma_j$, $m_2 = m$, $n_2 = n$. Denote by $X_1^{(k)}, \cdots, X_{n_k}^{(k)}$ the lifts of the elements $X_1, \cdots, X_n$, $Y_1, \cdots, Y_m, \cdots, Y_1^{(k-1)}, \cdots, Y_{m_{k-1}}^{(k-1)}$. The Lie algebra L as a module is spanned by $X_1^{(k)}, \cdots, X_{n_k}^{(k)}, \cdots$. The bracket on L modulo L_{k+1} = the $(k+1)$st bracket on the generators $X_1, \cdots, X_n$ is defined by $\langle d_{k+1}\gamma_j^{(k)}; X_p^{(k)}, X_q^{(k)}\rangle = \langle \gamma_j^{(k)}, [X_p^{(k)}, X_q^{(k)}]\rangle$ and $[X_p^{(k)}, Y_j^{(k)}] = 0$, $[Y_r^{(k)}, Y_s^{(k)}] = 0$. But $\langle d_{k+1}\gamma_j^{(k)}; X_p^{(k)}, X_q^{(k)}\rangle = \langle \sum_{r,s} \beta_j^{(k)rs}\omega_r^{(k)} \wedge \omega_s^{(k)}; X_p^{(k)}, X_q^{(k)}\rangle = \sum_{r,s}(\beta_j^{(k)rs}\delta_{rp}\delta_{sq} - \beta_j^{(k)sr}\delta_{sp}\delta_{rq})$. Thus we have

Proposition 2.3. *With the above choice of the generators the nilpotent Lie algebra L/L_{k+1} has the structure*

$$\left[X_p^{(k)}, X_q^{(k)}\right] = \sum_{r,s,j} \left(\beta_j^{(k)rs}\delta_{rp}\delta_{sq} \not\equiv \beta_j^{(k)sr}\delta_{rp}\delta_{sq}\right) Y_j^{(k)}$$

$$+ \textit{sum of combinations of } X_i^{(k)},\ s, i > \max(p, q)$$

with the coefficients determined inductively by k;

$$[X_p^{(k)}, Y_r^{(k)}] = 0,\ [Y_r^{(k)}, Y_s^{(k)}] = 0,$$
$$p, q = 1, 2, \cdots, n_k;\ r, s = 1, 2, \cdots, m_k;$$

the abelian groups $\overline{G_k/G_{k+1}}$ *and* L_k/L_{k+1} *are isomorphic by the very construction, and the isomorphism sends* $y_j^{(k)}$ *to* $Y_j^{(k)}$.

Proposition 2.4. *The extension cocycle for the nilpotent Lie algebra* L/L_{k+1}, *as an extension* $0 \to L_k/L_{k+1} \to L/L_{k+1} \overset{\pi}{\to} L/L_k \to 0$, *is the element*

$$\sum_{j=1}^{m_k} \beta_j^{(k)} \otimes Y_j^{(k)} \in M_k^2 \otimes L_k/L_{k+1}.$$

Proof. For any two elements $X_p^{(k)}, X_q^{(k)}$ in L, modulo L_k we have

$$\left\langle \sum_{j=1}^{m_k} \beta_j^{(k)} \otimes Y_j^{(k)}; X_p^{(k)}, X_q^{(k)} \right\rangle = \sum \beta_j^{(k)rs} \langle \omega_r^{(k)} \wedge \omega_s^{(k)}; X_p^{(k)}, X_q^{(k)} \rangle$$
$$= \sum \left(\beta_j^{(k)rs} \delta_{rp} \delta_{sq} - \beta_j^{(k)sr} \delta_{rp} \delta_{sq} \right) Y_j^{(k)}$$
$$= [X_p^{(k)}, X_q^{(k)}],$$

where the last bracket is in L modulo L_{k+1}.

We can think of the 2-cocycle

$$\sum_{j=1}^{m_k} \beta_j^{(k)} \otimes y_j^{(k)} \in M_k^2 \otimes \overline{\cdot\ {}_k/G_{k+1}}$$

as a "universal" extension cocycle which on one side gives the extension of the groups

$$0 \to \overline{G_k/G_{k+1}} \to \overline{G/G_{k+1}} \to \overline{G/G_k} \to 1$$

by the map ψ and on the other hand the extension of the Lie algebras

$$0 \to L_k/L_{k+1} \to L/L_{k+1} \to L/L_k \to 0.$$

The structure equations for the nilpotent Lie algebra confirm the known fact that a nilpotent Lie algebra over Q is isomorphic to a rational subalgebra of a Lie algebra of the Lie group of the upper triangular matrices.

3. The essential step in the proof of the theorem is to enlarge the ring of the rationals to the real numbers.

Suppose that $M' = M \otimes R$, where M is the 1-minimal model for a finitely generated group G without torsion. We denote by $L' = L \otimes R$ the real Lie algebra associated with the Lie algebra L of M over the rationals. $\overline{\overline{G_k/G_{k+1}}} = G_k/G_{k+1} \otimes R$, and the real Malcev's completion $\overline{\overline{G/G_{k+1}}}$ of the nilpotent group $\overline{G/G_{k+1}}$ is defined inductively as a central extension of $\overline{G_k/G_{k+1}}$ by $\overline{G/G_k}$; $\overline{G/G_2} = G/G_2 \otimes R$;

$$0 \to \overline{G_k/G_{k+1}} \to \overline{G/G}_{k+1} \to \overline{G/G_k} \to 1.$$

The exponential map

$$\exp\colon L'/L'_p \to \mathcal{G}_k$$

defines the simply connected Lie group $\mathcal{G}_k$. We denote the subgroup of $\mathcal{G}_k$ by $S^k = SL/L_k$. The group structure on S^k is isomorphic to that on the nilpotent Lie algebra L/L_k with the product given by the Campbell-Hausdorff formula.

The central extensions of the Lie algebras and groups together with the exponential map make the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_k/L_{k+1} & \longrightarrow & L/L_{k+1} & \longrightarrow & L/L_k & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} & & \\ 0 & \longrightarrow & SL_k/L_{k+1} & \longrightarrow & S^{k+1} & \longrightarrow & S^k & \longrightarrow & 0. \end{array}$$

If we succeed in proving that there is an isomorphism between the groups $\overline{G/G_r}$ and S^r, $r = 2, 3, \cdots$, then the theorem follows by the argument at the end of part 1.

The proof that $\overline{G/G_r}$ and S^r are isomorphic for all r is given in two steps:

1. There is constructed a morphism of differential graded algebras

$$\phi_r\colon M_r \to A^*(S^r)$$

for $r = 2, 3, \cdots$, where $A^*(S^r)$ is the complex of real forms on the classifying space for the group S^r.

2. From the assumption that (M_r, ϕ_r) is the 1-minimal model for S^r, $r = 2, 3, \cdots, k$, it is proved that (M_{k+1}, ϕ_{k+1}) is also the 1-minimal model for S^{k+1}.

Because the kernels of the projections $S^{r+1} \to S^r$ and $\overline{G/G_{r+1}} \to \overline{G/G_r}$ are isomorphic abelian groups, we can apply the converse of the Hirsch lemma even over the reals. From here it follows that S^r and $\overline{G/G_r}$ are isomorphic for all $r = 2, 3, \cdots$.

Construction of the map ϕ_r

There is a weak homotopy equivalence between the Eilenberg-McLane space $K(S^r, 1)$ and the fat realization $|S^r|$ of the simplicial group NS^r. The cohomologies of the differential graded algebras of forms $A^*(NS^r)$ and $A^*(S^r) = A^*(K(S^r, 1))$ are isomorphic. We write $A^*(S^r)$ instead of $A^*(NS^r)$. Recall that with the simplicial group $NS^r = \{S^r_p\}, p = 0, 1, 2, \cdots$, there is given a family of face operators ε_i: $S_p^r \to S_{p-1}^r$, $i = 0, 1, \cdots, p$, and inclusions ε^i: $\Delta^{p-1} \to \Delta^p$ of the i-th face. An n-form $\phi \in A^n(NS^r)$ is a sequence of n-forms $\phi^{(p)} \in A^n(\Delta^p \times S_p^r)$ on the disjoint union $\amalg \Delta^p \times S_p^r$, with S_p^r discrete, satisfying the compatibility conditions $(\varepsilon^i \times \mathrm{id})^*\phi^{(p)} = (\mathrm{id} \times \varepsilon_i)^*\phi^{(p-1)}$ on $\Delta^{p-1} \times S_p^r$ for all $i = 0, 1, \cdots, p$ and $p = 1, 2, \cdots$.

The group S^r acts on the contractible Lie group $\mathcal{C}_r$ from the left. Hence with the universal S^r-bundle $NE^r \to NS^r$ there is associated an $\mathcal{C}_r$-bundle $NB^r = NE^r \times_{S^r} \mathcal{C}_r \to NS^r$. On the simplicial manifold NB^r there is a double complex of differential forms $\{\mathcal{C}^{k,l}(NB^r)\}$, $\mathcal{C}^{k,l}(NB^r) = A^l(B_k^r)$, where $A^l(B_k^r)$ are differential forms along the fibre with the usual differentials. From the Theorem 2.3 of Dupont [3] it follows that there exists a morphism of differential graded algebras

$$\delta_r\colon \mathcal{C}^*(NB^r) \to A^*(NS^r)$$

which induces an isomorphism on the cohomology.

Suppose that $C^*(L/L_r)$ is the complex of R-valued skew-symmetric multilinear forms on L/L_r. Because L is the Lie algebra associated with the 1-minimal model M, there is a homomorphism of complexes

$$\mu_r\colon M_r \to C^*(L/L_r)$$

defined by duality on the generators. More precisely the maps μ_r are defined inductively as follows: μ_2: $M_2 \to C^*(L/L_2)$ sends ω_i to the dual of X_i which is ω_i itself. Thus μ_2 is an identity on the generators and extends multiplicatively. $M_3 = M_2 \otimes \bigwedge(\gamma_1^{(2)}, \cdots, \gamma_{m_2}^{(2)})$ and L/L_3 has the basis $X_1^{(2)}, \cdots, X_{n_2}^{(2)}$, $Y_1^{(2)}, \cdots, Y_{m_2}^{(2)}$ where $X_i^{(2)}$ is mapped onto X_i by the projection π: $L/L_3 \to L/L_2$. Therefore we can think of $X_i^{(2)}$ as of a lift of X_i. If there is a splitting λ_2 of the exact sequence $0 \to L_2/L_3 \to L/L_3 \overset{\pi}{\to} L/L_2 \to 0$, then we can define $\mu_3 \to C^*(L/L_3)$ by $\mu_3(\omega_i^{(2)}) =$ the dual of $\lambda_2(X_i)$ and $\mu_3(\gamma_j^{(2)})(A) = \gamma_j^{(2)}(A - \lambda_2\pi A)$ for each $A \in L/L_3$. Further extend μ_3 to an algebra morphism. Suppose that μ_r has been defined for $r = 2, 3, \cdots, k-1$ by a sequence of splittings λ_r of the exact sequences $0 \to L_r/L_{r+1} \to L/L_{r+1} \overset{\pi}{\to} L/L_r \to 0$. Then choose a splitting λ_k, and define $\Lambda_r = \mathrm{id} - \lambda_r\pi$: $L/L_{r+1} \to L_r/L_{r+1}$ for $r = 2, 3, \cdots, k$. Thus we define $\mu_k(\gamma_j^{(k)}) = \gamma_j^{(k)} \circ \Lambda_k$ and $\mu_k(\omega_i^{(k)}) =$ the dual to $\lambda_k(X_i^{(k-1)})$. Such a map μ_r induces an isomorphism on cohomology with real coefficients.

Next we identify L/L_r with the tangent vector space to $\mathcal{C}_r$ at the identity e and L/L_r with a fixed subspace of L'/L'_r. Let ν_r: $C^*(L/L_r) \to A_l^*(\mathcal{C}_r)$ be the map of complexes which sends an element $\alpha_0 \in C^p(L/L_r)$ to the left invariant p-form α on the Lie group $\mathcal{C}_r$ such that $\alpha(e) = \alpha_0$.

The pullback of α with respect to the projection $NE^r \times \mathcal{C}_r \to \mathcal{C}_r$, being S^r invariant, defines a unique p-form $\eta_r(\alpha) = \tilde{\alpha} \in \mathcal{C}^p(NB^r)$.

Finally the composition of all these maps gives a morphism of differential graded algebras

$$\phi_r = \delta_r \circ \eta_r \circ \nu_r \circ \mu_r \colon M_r \to A^*(NS^r).$$

From this construction it does not follow that ϕ_r is an isomorphism on cohomology.

(M_{k+1}, ϕ_{k+1}) *as* 1-*minimal model for* S^{k+1}

Now we assume that $\phi_r\colon M_r \to A^*(S^r)$ is the 1-minimal model for S^r, $r = 2, 3, \cdots, k$. Because $\psi_r\colon M_r \to A^*(\overline{G/G_r})$ is the 1-minimal model for all $r = 2, 3, \cdots,$ we know that there is an isomorphism of groups $\gamma_r\colon \overline{G/G_r} \to S^r$ for $r = 2, 3, \cdots, k$. The morphism ϕ_k and the isomorphism of the abelian groups exp: $L_k/L_{k+1} \to SL_k/L_{k+1}$ give the map of complexes $\phi_k = \phi_k \otimes \exp\colon M_k \otimes L_k/L_{k+1} \to A^*(S^k) \otimes SL_k/L_{k+1}$. The image of the extension cocycle

$$\beta^{(k)} = \sum_{j=1}^{m_k} \beta_j^{(k)} \otimes Y_j^{(k)} \in M_k^2 \otimes L_k/L_{k+1}$$

under ϕ_k is the 2-cocycle

$$\phi_k(\beta^{(k)}) \in A^2(S^k) \otimes SL_k/L_{k+1}.$$

$\phi_k(\beta^{(k)})$ defines the group extension

$$0 \to SL_k/L_{k+1} \to \tilde{S}^{k+1} \to S^k \to 1.$$

Lemma 3.1. *The morphism of differential graded algebras* $\phi_k\colon M_k \otimes L_k/L_{k+1} \to A^*(S^k) \otimes SL_k/L_{k+1}$ *maps the extension cocycle* $\beta^{(k)} \in M_k^2 \otimes L_k/L_{k+1}$ *for the elementary extension* $M_{k+1} = M_k \otimes \Lambda(\gamma_1^{(k)}, \cdots, \gamma_{m_k}^{(k)})$ *to the cocycle* $\phi_k(\beta^{(k)}) \in A^2(S^k) \otimes SL_k/L_{k+1}$ *for the extension of the group*

$$0 \to SL_k/L_{k+1} \to S^{k+1} \to S^k \to 1,$$

i.e., $S^{k+1} = \exp(L/L_{k+1})$ *and* $\tilde{S}^{k+1}$ *are isomorphic groups.*

This lemma follows from the geometric construction and two propositions.

Let $\mathcal{G}_2 \leftarrow \mathcal{G}_3 \leftarrow \mathcal{G}_4 \leftarrow \cdots$ be the tower of nilpotent contractible Lie groups defined by the tower of nilpotent Lie algebras $L'/L'_2 \leftarrow L'/L'_3 \leftarrow L'/L'_4 \leftarrow \cdots$. We choose a splitting ∇_k of the short exact sequence

$$(*) \qquad 0 \to L'_k/L'_{k+1} \to L'/L'_{k+1} \xrightarrow{\pi} L'/L'_k \to 0.$$

Such ∇_k determines the splitting $\omega^{(k)}$ by $\omega^{(k)}(X) = X - \nabla_k \pi X$, $X \in L'/L'_{k+1}$, and bilinear skew-symmetric map $\Omega^{(k)}\colon L'/L'_k \times L'/L'_k \to L'_k/L'_{k+1}$ by the formula

$$\Omega^{(k)}(A, B) = \tfrac{1}{2}\{[\nabla_k A, \nabla_k B] - \nabla_k[A, B]\}, \quad A, B \in L'/L'_k.$$

The extension cocycle for $(*)$ is; by definition, an element $\mathcal{F}_k \in C^2(L'/L'_k, L'_k/L'_{k+1}) = C^2(L'/L'_k) \otimes L'_k/L'_{k+1}$, $\mathcal{F}_k(A, B) = 2\Omega^{(k)}(A, B)$. The group $\mathcal{Q}_{k+1}$ is an extension $0 \to \mathcal{K}_{k+1} \to \mathcal{Q}_{k+1} \xrightarrow{\pi} \mathcal{Q}_k \to 1$, where $\mathcal{K}_{k+1}$ is an abelian group isomorphic with $\overline{G_k/G_{k+1}}$ and with L'_k/L'_{k+1}. $\mathcal{Q}_{k+1}$ acts on itself and via the projection π also on $\mathcal{Q}_k$, from the left.

Let $T(\mathcal{Q}_{k+1})$ be the tangent bundle, and $T(\mathcal{Q}_{k+1})/\mathcal{Q}_{k+1}$ the quotient with respect to the right action. Let $F(\mathcal{Q}_{k+1}) = \{X \in T(\mathcal{Q}_{k+1}) | \pi(X) = 0\}$. Then $\mathcal{Q}_{k+1}$ acts from the left on each term of the exact sequence of vector bundles over $\mathcal{Q}_k$

$$(**) \qquad 0 \to F(\mathcal{Q}_{k+1})/\mathcal{Q}_{k+1} \to T(\mathcal{Q}_{k+1})/\mathcal{Q}_{k+1} \to T(\mathcal{Q}_k) \to 0.$$

This sequence of vector bundles, restricted to the identity $e \in \mathcal{Q}_k$, coincides with the exact sequence $(*)$ if we disregard the Lie algebra structure. Therefore the splitting ∇_k defines a left invariant splitting $\overline{\nabla}_k$ of the exact sequence $(**)$, and on the principal bundle $\mathcal{Q}_{k+1} \xrightarrow{\pi} \mathcal{Q}_k$ it defines an L'_k/L'_{k+1}-valued connection 1-form $\overline{\omega}^{(k)}$. The curvature of this connection is the 2-form $\overline{\Omega}^{(k)} = d\overline{\omega}^{(k)} + \frac{1}{2}[\overline{\omega}^{(k)}, \overline{\omega}^{(k)}]$ on $\mathcal{Q}_{k+1}$ with values in the abelian Lie algebra L'_k/L'_{k+1}. Therefore $[\overline{\omega}^{(k)}, \overline{\omega}^{(k)}] = 0$, and $\overline{\Omega}^{(k)}$ is a pullback of a 2-form from $\mathcal{Q}_k$. Since it is the pullback of the left invariant 2-form $\Omega^{(k)}$ on $\mathcal{Q}_k$ defined above, we have $\overline{\Omega}^{(k)} = \pi^*\Omega^{(k)} = d\overline{\omega}^{(k)}$. All this can be formulated as follows.

Proposition 3.1. *A splitting ∇_k of the exact sequence $(*)$ determines a left invariant connection on the bundle $\mathcal{Q}_{k+1} \xrightarrow{\pi} \mathcal{Q}_k$. The curvature of this connection is an L'_k/L'_{k+1}-valued left invariant 2-form $\Omega^{(k)}$ on $\mathcal{Q}_k$ which restricted to the identity $e \in \mathcal{Q}_k$ gives an element $\Omega_e^{(k)}$ which determines the extension cocycle $\mathcal{F}_k = 2\Omega_e^{(k)}$ for the Lie algebra.*

Now we construct a map ι: $\mathcal{Q}_k \times \mathcal{Q}_k \to L'_k/L'_{k+1}$. For any $(a_1, a_2) \in \mathcal{Q}_k \times \mathcal{Q}_k$ construct an oriented 2-simplex with vertices a_1a_2, a_1, e as an immersion Δ: $\Delta^2 \to \mathcal{Q}_k$, $\Delta^2 = \langle v_0, v_1, v_2 \rangle$. Suppose that we are given the euclidean metric on L'/L'_k and the left invariant induced metric on $\mathcal{Q}_k$. Let α_1 be the geodesic from a_1 to e, and α_2 the left a_1-image of the geodesic from a_2 to e. We denote the geodesic from e to a_1a_2 by $\alpha_1\alpha_2$. Then the map Δ is defined on the faces by $\Delta(\langle v_0, v_1 \rangle) = \alpha_2$, $\Delta(\langle v_1, v_2 \rangle) = \alpha_1$, $\Delta(\langle v_2, v_0 \rangle) = \alpha_1\alpha_2$; $\Delta(\langle v_0 \rangle) = a_1a_2$, $\Delta(\langle v_1 \rangle) = a_1$, $\Delta(\langle v_2 \rangle) = e$. Denote the image $\Delta(\Delta^2)$ by $\Delta(a_1, a_2)$. The map ι is defined by the formula

$$\iota(a_1, a_2) = \int_{\Delta(a_1,a_2)} \Omega^{(k)}.$$

The restriction gives the map ι: $S^k \times S^k \to L'_k/L'_{k+1}$. Composition of ι with the exponential defines the 2-cochain

$$\exp \circ \iota: S^k \times S^k \to SL'_k/L'_{k+1}.$$

In fact this is a cocycle representing a class in the cohomology of the group S^k with the coefficients in the abelian group SL'_k/L'_{k+1}. We denote by S^{k+1} the group which is an extension of SL'_k/L'_{k+1} by S^k determined by this cocycle; $0 \to SL'_k/L'_{k+1} \to S^{k+1} \to S^k \to 1$.

On the other hand we have the extension class for the group extension

$$(**) \qquad 0 \to SL_k/L_{k+1} \to S^{k+1} \to S^k \to 1.$$

This class is represented by a cocycle as follows: Let $\alpha_1, \alpha_2, \alpha_1\alpha_2$ be the boundary components of $\Delta(a_1, a_2)$. There exist elements $A_1, A_2, B \in L/L_k$ such that $\exp A_1 = a_1$, $\exp A_2 = a_2$, $\exp B = a_1a_2$. Let $\tilde{A}_i = \nabla_k A_i$, $i = 1, 2$ and let $\tilde{B} = \nabla_k B$. The element $\tilde{a}_1 = \exp \tilde{A}_1$ is the endpoint of the horizontal lift $\tilde{\alpha}_1$ of α_1 ending at $e \in \mathcal{Q}_{k+1}$. The horizontal lift $\tilde{\alpha}_2$ of α_2 ending at $\tilde{a}_1$ starts at $\tilde{a}_1\tilde{a}_2$, and the horizontal lift $\widetilde{\alpha_1\alpha_2}$ of $\alpha_1\alpha_2$ starts at $e \in \mathcal{Q}_{k+1}$ and ends at $\widetilde{a_1a_2} = \exp \tilde{B}$. By the definition the extension cocycle for (***) is the map $g_k\colon S^k \times S^k \to SL_k/L_{k+1}$ given by the formula $(\widetilde{a_1a_2})^{-1}\tilde{a}_1\tilde{a}_2 = g_k(a_1, a_2)$. This extension cocycle and the curvature defined above are related as follows.

Proposition 3.2.

$$\exp \int_{\Delta(a_1,a_2)} \Omega^{(k)} = g_k(a_1, a_2).$$

Proof. There exists a unique element $Z \in L_k/L_{k+1}$ such that the exponential map $\exp\colon L_k/L_{k+1} \to SL_k/L_{k+1}$ maps Z to $g_k(a_1, a_2)$. Define a curve $\zeta\colon [0, 1] \to \mathcal{Q}_{k+1}$ by $\zeta(t) = (\widetilde{a_1a_2}) \cdot \exp(tZ)$. This curve is in the fibre of $\mathcal{Q}_{k+1} \to \mathcal{Q}_k$. It is the left translate of $e \cdot \exp(tZ)$ by $\widetilde{a_1a_2}$ in $\mathcal{Q}_{k+1}$. Choose a 2-cell $\widetilde{\Delta(a_1, a_2)}$ in $\mathcal{Q}_{k+1}$ with the boundary $\widetilde{\alpha_1\alpha_2}, \zeta, \tilde{\alpha}_2, \tilde{\alpha}_1$ whose interior is diffeomorphic onto the interior of $\Delta(a_1, a_2)$ by the projection π. Then

$$\int_{\Delta(a_1, a_2)} \Omega^{(k)} = \int_{\widetilde{\Delta(a_1, a_2)}} \pi^*\Omega^{(k)} = \int_{\widetilde{\Delta(a_1, a_2)}} \bar{\Omega}^{(k)} = \int_{\widetilde{\Delta(a_1, a_2)}} d\bar{\omega}^{(k)} = \int_{\partial\widetilde{\Delta(a_1, a_2)}} \bar{\omega}^{(k)}.$$

But the boundary $\partial\widetilde{\Delta(a_1, a_2)}$ is the union of oriented arcs $\widetilde{\alpha_1\alpha_2}, \gamma, \tilde{\alpha}_2, \tilde{\alpha}_1$. Because the arcs $\widetilde{\alpha_1\alpha_2}, \tilde{\alpha}_2, \tilde{\alpha}_1$ are horizontal with respect to the connection $\bar{\omega}^{(k)}$, the connection 1-form $\bar{\omega}^{(k)}$ restricted to these arcs is zero. Therefore

$$\int_{\Delta(a_1, a_2)} \Omega^{(k)} = \int_{\gamma} \bar{\omega}^{(k)} = \int_{\exp(tZ)} \bar{\omega}^{(k)} = Z.$$

Proof of the Lemma. When the splitting λ_k is the restriction of ∇_k then μ_{k+1} and also ϕ_{k+1} are well defined maps. From the above propositions it follows that $\iota(S^k \times S^k) \subset L_k/L_{k+1}$, and that $\exp \cdot \iota = g_k$. Hence the groups S^{k+1} and $\tilde{S}^{k+1}$ are isomorphic.

The groups $\tilde{S}^{k+1}$ and $\tilde{\tilde{S}}^{k+1}$ are related by the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & SL_k/L_{k+1} & \longrightarrow & \tilde{S}^{k+1} & \longrightarrow & S^k & \longrightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & SL'_k/L'_{k+1} & \longrightarrow & \tilde{\tilde{S}}^{k+1} & \longrightarrow & S^k & \longrightarrow 1. \end{array}$$

Lemma 3.2. *Suppose that* (M_r, ϕ_r), $r = 1, 2, \cdots, k$, *is the* 1-*minimal model for the group* S^r. *Then* (M_{k+1}, ϕ_{k+1}) *is the* 1-*minimal model for the group* S^{r+1}.

Proof. This is proved by the comparison of the spectral sequences E for the elementary extension $M_{k+1} = M_k \otimes \Lambda(\gamma_1^{(k)}, \cdots, \gamma_{m_k}^{(k)})$ and the spectral sequence $\mathcal{E}$ for the de Rham cohomology of the fibration $K(S^{k+1}, 1) \to K(S^k, 1)$.

The map ϕ_{k+1} is a composition of maps

$$M_{k+1} \overset{\mu_{k+1}}{\to} C^*(L'_k/L'_{k+1}) \overset{\nu_{k+1}}{\to} A_l^*(\mathcal{C}_{k+1}) \overset{\eta_{k+1}}{\to} \mathcal{C}^*(NB^{k+1}) \overset{\delta_{k+1}}{\to} \mathcal{C}^*(S^{k+1}).$$

The projections $L'/L'_{k+1} \to L'/L'_k$, $\mathcal{C}_{k+1} \to \mathcal{C}_k$, $S^{k+1} \to S^k$ and the inclusion $M_k \to M_{k+1}$ induce decreasing bounded filtrations on each one of the complexes. The maps μ_{k+1} and ν_{k+1} preserve the filtrations by the very construction, and it follows from [2] that the maps η_{k+1} and δ_{k+1} are filtration preserving.

Let $\{F^pM_{k+1}\}$ be the filtration of the differential graded algebra M_{k+1}, where F^pM_{k+1} are elements of degree $\geq p$ in terms of the subalgebra M_k, and let $\{\mathcal{F}^pA^*(S^{k+1})\}$ be the filtration of $A^*(S^{k+1})$, $\mathcal{F}^pA^*(S^{k+1})$ are forms of degree $\geq p$ in the elements from $\pi^*A^*(S^k)$. Then the 1-st terms of the spectral sequences are

$$E_1^{p,q} \cong M_k^p \otimes H^q\big(\Lambda\big(\gamma_1^{(k)}, \cdots, \gamma_{m_k}^{(k)}\big)\big),$$
$$\mathcal{E}_1^{p,q} \cong A^p(S^k) \otimes H^q(SL_k/L_{k+1}),$$

with the differentials induced from the differentials on M_k and $A^*(S^k)$ respectively. Again from the construction of the map ϕ_{k+1} we get the isomorphism $E_2^{p,q} \cong \mathcal{E}_2^{p,q}$ for all $p \geq 0$, $q \geq 0$. Thus from the comparison of spectral sequences we can conclude that ϕ_{k+1} induces an isomorphism on cohomology.

Remark. It is instructive to check directly how the map ϕ_{k+1} induces an isomorphism on the 2-nd terms of the spectral sequences. That there is an isomorphism of algebras ϕ_{k+1}^*: $E_2^{p,q} \to \mathcal{E}_2^{p,q}$ is straightforward, and that ϕ_{k+1}^* commutes with the differentials d_2^E and $d_2^{\mathcal{E}}$ can be verified directly in the following way. Because the maps δ_r, η_r and ν_r are maps of the differential graded algebras for $r = 2, 3, \cdots$, we demonstrate that μ_{k+1} also commutes with the differentials.

The differential d_2^E is determined by the differential d_{k+1} on the generators of M_{k+1}. In particular $d_{k+1}\gamma_j^{(k)} = \beta_j^{(k)}$, $j = 1, 2, \cdots, m_k$, where $\gamma_j^{(k)}$ is a representative for the element $[\gamma_j^{(k)}] \in E_2^{0,1}$, and $\beta_j^{(k)}$ is the representative for $d_2^E([\gamma_j^{(k)}]) = [\beta_j^{(k)}] \in E_2^{2,0}$, d_2^e being the transgression.

On the other hand $\nu_{k+1} \circ \mu_{k+1}(\gamma_j^{(k)}) = \alpha_j^{(k)}$, where $\omega^{(k)} = \sum_{j=1}^{n_k} \alpha_j^{(k)} \otimes y_j^{(k)}$ is the left invariant connection 1-form on $\mathcal{C}_{k+1} \to \mathcal{C}_k$ and $[\alpha_j^{(k)}] \in \mathcal{E}_2^{0,1}$. $\nu_{k+1} \circ \mu_{k+1}(\beta_j^{(k)}) = \nu_{k+1} \circ \mu_{k+1}(\sum \beta_j^{(k)r,s} \omega_r^{(k)} \wedge \omega_s^{(k)}) = \sum \beta_j^{(k)r,s} \alpha_r^{(k)} \wedge \alpha_s^{(k)} = d\alpha_j^{(k)}$, where $[d\alpha_j^{(k)}] \in \mathcal{E}_2^{2,0}$. Hence ϕ_{k+1} induces the isomorphism of the 2-nd terms of the spectral sequences.

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