J. DIFFERENTIAL GEOMETRY
15 (1980) 381–392

AUTOMORPHISM GROUPS OF SOME GEOMETRIC STRUCTURES

HSU-TUNG KU

1. Introduction

In this paper we shall investigate the gaps of the dimensions of compact classical Lie groups and the gaps of the dimensions of the automorphism groups of some geometric structures.

Let H be a closed subgroup of $O(n)$. In [12], Montgomery and Samelson have shown that dim H cannot fall into the following range if $n \neq 4$:

$$\langle n-1\rangle_{SO} + \langle 1\rangle_{SO} < \dim H < \langle n\rangle_{SO},$$

where $\langle s\rangle_{SO}$ denotes dim $SO(s)$.

We shall generalize this result by proving the following theorems.

Theorem A. *Let $H \subset G$ be a closed subgroup.*

(a) *If $G = O(n)$, then* dim H *cannot fall into any of the following ranges, i.e., there exist gaps*:

$$\langle n-k\rangle_{SO} + \langle k\rangle_{SO} < \dim H < \langle n-k+1\rangle_{SO},$$

where $1 \leqslant k \leqslant D_{SO}(n)$ *if* $n \geqslant 13$; *or* $1 \leqslant k \leqslant A_{SO(n)}$ *if* $n \geqslant 11$.

(b) *If $G = SU(n)$, then there exist gaps*:

$$\langle n-k\rangle_{SU} + \langle k\rangle_{U} < \dim H < \langle n-k+1\rangle_{SU},$$

where $1 \leqslant k \leqslant D_{SU(n)}$ *if* $n \geqslant 11$; *or* $1 \leqslant k \leqslant A_{SU(n)}$ *if* $n \geqslant 9$.

(c) *If $G = U(n)$, then there exist gaps*:

$$\langle n-k\rangle_{U} + \langle k\rangle_{U} < \dim H < \langle n-k+1\rangle_{SU},$$

where $1 \leqslant k \leqslant D_{U(n)}$ *if* $n \geqslant 11$; *or* $1 \leqslant k \leqslant A_{U(n)}$ *if* $n \geqslant 9$.

(d) *If $G = Sp(n)$, then there exist gaps*:

$$(1) \qquad \langle n-k\rangle_{Sp} + \langle k\rangle_{Sp} < \dim H < \langle n-k+1\rangle_{Sp},$$

where $\langle s\rangle_{SU} =$ dim $SU(s)$, $\langle s\rangle_{U} =$ dim $U(s)$, *and* $\langle s\rangle_{Sp} =$ dim $Sp(s)$, *and* $D_{SO(n)}$, $D_{SU(n)}$, $D_{U(n)}$ *and* $D_{SP(n)}$ *are the largest values of k for which the above inequalities in* (a), (b), (c) *and* (d) *are meaningful.* (*For notation A_X, $X = SO$, SU, U and Sp see Theorem* C).

Received November 16, 1978, and, in revised form, March 26, 1979, and October 27, 1979.

Theorem B. *Let $H \subset G$ be a closed subgroup, and $k_i (i = 0, 1, \cdots, s + 1)$ be any sequence of positive integers with $k_0 = n$.*

(a) *If $G = O(n)$, $k_{i+1} \leqslant D_{SO(k_i)}$ (resp. $k_{i+1} \leqslant A_{SO(k_i)}$), $0 \leqslant i \leqslant s$, and $k_s \geqslant 13$ (resp. $k_s \geqslant 11$), then there exists a gap*:

$$\sum_{i=0}^{s} \langle k_i - k_{i+1} \rangle_{SO} + \langle k_{s+1} \rangle_{SO} < \dim H < \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle_{SO} + \langle k_s - k_{s+1} + 1 \rangle_{SO}.$$

(b) *If $G = SU(n)$, $k_{i+1} \leqslant D_{SU(k_i)}$ (resp. $k_{i+1} \leqslant A_{SU(k_i)}$), $0 \leqslant i \leqslant s$, and $k_s \geqslant 11$ (resp. $k_s \geqslant 9$), then there exists a gap*:

$$\sum_{i=0}^{s} \langle k_i - k_{i+1} \rangle_{U} + \langle k_{s+1} \rangle_{SU} < \dim H < \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle_{SU} + \langle k_s - k_{s+1} + 1 \rangle_{SU}.$$

(c) *If $G = U(n)$, $k_{i+1} \leqslant D_{U(k_i)}$ (resp. $k_{i+1} \leqslant A_{U(k_i)}$), $0 \leqslant i \leqslant s$, and $k_s \geqslant 11$ (resp. $k_s \geqslant 9$), then there exists a gap*:

$$\sum_{i=0}^{s} \langle k_i - k_{i+1} \rangle_{U} + \langle k_{s+1} \rangle_{U} < \dim H < \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle_{SU} + \langle k_s - k_{s+1} + 1 \rangle_{SU}.$$

(d) *If $G = Sp(n)$, $k_{i+1} \leqslant D_{Sp(k_i)}$ or $A_{Sp(k_i)}$, $0 \leqslant i \leqslant s$, and $k_s \geqslant 11$, then there exists a gap*:

$$(2) \quad \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle_{P} + \langle k_s - k_{s+1} \rangle_{Sp} + \langle k_{s+1} \rangle_{Sp} < \dim H < \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle_{Sp} + \langle k_s - k_{s+1} + 1 \rangle_{Sp},$$

where $\langle s \rangle_P = \dim Sp(s) \times Sp(1)$.

Suppose now that G is a closed subgroup of the isometry group $I(M)$ of a connected Riemannian m-manifold M^m, and $H = G_x^0$ the identity component of a prinicipal isotropy subgroup G_x. It is well known that H is compact and $H \subset SO(m)$, [7], [13]. Moreover

$$\dim I(M) \leqslant \langle m + 1 \rangle_{SO}.$$

If G is a group of automorphism of a Hermitian manifold M of dimension $2n$, then $H \subset U(n)$, and if M is an almost quaternionic $4n$-manifold with a compatible Riemannian structure and G is a closed subgroup of $I(M)$, then $H \subset Sp(n)$; cf. [1].

There is a general pattern of gaps in the dimensions of the closed subgroups of the isometry group $I(M)$, that is, the dimension of G cannot fall into any of the following ranges if $m \geqslant 11$, [11] (cf. also [8]),

$$\langle m-k+1\rangle_{SO} + \langle k+1\rangle_{SO} < \dim G < \langle m-k+2\rangle_{SO},$$

where $1 \leqslant k \leqslant A_{SO(m)}$.

In [1], Cattani-Mann have proved the existence of gaps for the dimensions of the closed subgroups of the isometry groups $I(M)$ for both Hermitian manifolds and almost quaternionic manifolds with a compatible Riemannian structure. We shall establish the most general pattern of gaps which contain large families of gaps in the dimension of the automorphism groups for the Riemannian, Hermitian and almost quaternionic categories which extend far beyond all the existence results.

Throughout this paper, we shall always express a compact connected Lie group H in the following form:

$$(*) \qquad H = \overline{H}/N = (T^q \times S)/N = (T^q \times G_1 \times \cdots \times G_v)/N,$$

where T^q is a q-torus, $q \geqslant 0$, each G_j is a compact connected simply connected Lie group, S semi-simple and N is a finite normal subgroup of $\overline{H}$. We shall call the G_i's the normal factors of H. Let $F(H, M)$ denote the fixed point set of the action of H on M, $L \subset K$, and L be a subgroup of K.

The main results concerning the gaps of the dimension of the automorphism groups are the following:

Theorem C. *Suppose M is a connected Riemannian m-manifold, and G a closed subgroup of the isometry group $I(M)$ with G_x a principal isotropy subgroup. Let $H = G_x^0$, and w be any nonnegative integer satisfying $q \geqslant w$ and* $r = \dim G(x)$.

(a) *If $r \leqslant n + w$, $H \subset SO(n) \subset SO(m)$ and $n \geqslant 11$, then there exist gaps:*

$$w + \langle n-k+1\rangle_{SO} + \langle k+1\rangle_{SO} < \dim G < w + \langle n-k+2\rangle_{SO},$$

where $1 \leqslant k \leqslant A_{SO(n)}$.

(b) *If $r \leqslant 2n + w$, $H \subset SU(n) \subset SO(m)$ and $n \geqslant 9$, then there exist gaps:*

$$w + \langle n-k+1\rangle_{SU} + \langle k+1\rangle_{SU} - 1 < \dim G < w + \langle n-k+2\rangle_{SU} - 1.$$

where $1 \leqslant k \leqslant A_{SU(n)}$.

(c) *If $r \leqslant 2n + w$, $H \subset U(n) \subset SO(m)$ and $n \geqslant 9$, then there exist gaps:*

$$w + \langle n-k+1\rangle_{SU} + \langle k+1\rangle_{SU} < \dim G < w + \langle n-k+2\rangle_{SU} - 1,$$

where $1 \leqslant k \leqslant A_{U(n)}$.

(d) *If* $r \leqslant 4n + w$, $H \subset Sp(n) \subset SO(m)$ *and* $n \geqslant 11$, *then there exist gaps*:

$$w + \langle n - k + 1\rangle_{Sp} + \langle k + 1\rangle_{Sp} - 6 < \dim G < w + \langle n - k + 2\rangle_{Sp} - 3,$$

$1 \leqslant k \leqslant A_{Sp(n)}$, *where* $A_{SO(n)}$, $A_{SU(n)}$, $A_{U(n)}$ *and* $A_{Sp(n)}$ *are the largest values of* k *for which the above inequalities in* (a), (b), (c) *and* (d) *respectively are meaningful.*

Theorem D. *Suppose* M *is a connected Riemannian* m*-manifold, and* G *a closed subgroup of the isometry group* $I(M)$ *and* G_x *a principal isotropy subgroup. Let* $H = G_x^0$, w *be any nonnegative integer satisfying* $q \geqslant w$, $r = \dim G(x)$, *and* $k_i (i = 0, \cdots, s + 1)$ *be any sequence of positive integers with* $k_0 = n$.

(a) *If* $r \leqslant n + w$, $H \subset SO(n) \subset SO(m)$, $k_{i+1} \leqslant A_{SO(k_i)}$, $0 \leqslant i \leqslant s$, *and* $k_s > 11$, *then there exists a gap*:

$$w + \sum_{i=0}^{s} \langle k_i - k_{i+1} + 1\rangle_{SO} + \langle k_{s+1} + 1\rangle_{SO}$$
$$< \dim G < W + \sum_{i=0}^{s-1} \langle k_i - k_{i+1} + 1\rangle_{SO} + \langle k_s - k_{s+1} + 2\rangle_{SO}.$$

(b) *If* $r \leqslant 2n + w$, $H \subset SU(n) \subset SO(m)$, $k_{i+1} \leqslant A_{SU(k_i)}$, $0 \leqslant i \leqslant s - 1$, $k_{s+1} \leqslant \bar{A}_{SU(k_s)}$, *and* $k_s \geqslant 9$, *then there exists a gap*:

$$w + \sum_{i=0}^{s} \langle k_i - k_{i+1} + 1\rangle_{SU} + \langle k_{s+1} + 1\rangle_{SU} - 1$$
$$< \dim G < w + \sum_{i=0}^{s-1} \langle k_i - k_{i+1} + 1\rangle_{SU} + \langle k_s - k_{s+1} + 2\rangle_{SU} - 1 - s.$$

(c) *If* $r \leqslant 2n + w$, $H \subset U(n) \subset SO(m)$, $k_{i+1} \leqslant A_{U(k_i)}$, $0 \leqslant i \leqslant s - 1$, $k_{s+1} \leqslant \bar{A}_{U(k_s)}$, *and* $k_s \geqslant 9$, *then there exists a gap*:

$$w + \sum_{i=0}^{s} \langle k_i - k_{i+1} + 1\rangle_{SU} + \langle k_{s+1} + 1\rangle_{SU}$$
$$< \dim G < w + \sum_{i=0}^{s-1} \langle k_i - k_{i+1} + 1\rangle_{SU} + \langle k_s - k_{s+1} + 2\rangle_{SU} - 1 - s.$$

(d) *If* $r \leqslant 4n + w$, $H \subset Sp(n) \subset SO(m)$, $k_{i+1} \leqslant A_{Sp(k_i)}$, $0 \leqslant i \leqslant s - 1$, $k_{s+1} \leqslant \bar{A}_{Sp(k_s)}$, *and* $k_s \geqslant 11$, *then there exists a gap*:

$$w + \sum_{i=0}^{s} \langle k_i - k_{i+1} + 1\rangle_{Sp} + \langle k_{s+1} + 1\rangle_{Sp} - 6$$
$$< \dim G < w + \sum_{i=0}^{s-1} \langle k_i - k_{i+1} + 1\rangle_{Sp} + \langle k_s - k_{s+1} + 2\rangle_{Sp} - 3(s + 1),$$

where $\bar{A}_{SU(k_s)}$, $\bar{A}_{U(k_s)}$ *and* $\bar{A}_{Sp(k_s)}$ *are the largest values of* k_{s+1} *for which the inequalities in* (b), (c) *and* (d) *respectively are meaningful. In other words,* $k_{s+1} \leqslant \bar{A}_{X(k_s)}$ *satisfying the following inequalities*:

$$\langle k_s - k_{s+1} + 1\rangle_{SU} + \langle k_{s+1} + 1\rangle_{SU} - 1 < \langle k_s - k_{s+1} + 2\rangle_{SU} - 1 - s, \quad \text{for } X = SU,$$

$$\langle k_s - k_{s+1} + 1\rangle_{SU} + \langle k_{s+1} + 1\rangle_{SU} < \langle k_s - k_{s+1} + 2\rangle_{SU} - 1 - s, \quad \text{for } X = U,$$

$$\langle k_s - k_{s+1} + 1\rangle_{Sp} + \langle k_{s+1} + 1\rangle_{Sp} - 6 < \langle k_s - k_{s+1} + 2\rangle_{Sp} - 3(s + 1), \quad \text{for } X = Sp.$$

Clearly, $\bar{A}_{X(k_s)} \leqslant A_{X(k_s)}$ for $X = SU$, U, or Sp. The assumption $q \geqslant w$ is necessary in both Theorem C and Theorem D. For the special case $w = 0$, Theorem C (b), (c) and (d) are results in [1], and Theorem D(a) was proved in [8].

2. Preliminaries

Let K be a compact Lie group acting on a space X. The *ineffective kernel* K_0 of K is the largest normal subgroup of K which acts trivially on X. There is a naturally induced effective action of the group K/K_0 on X. The following lemma will be used to estimate the dimension of the ineffective kernel K_0.

Lemma 2.1. (*Hsiang* [6]). *Let* $G \times K$ *be a compact Lie group acting almost effectively on a manifold* M, *and let* H *be a principal isotropy subgroup of the* G-*action on* M. *Assume that* K_0 *is the ineffective kernel of the action of* K *on the orbit space* M/G. *Then*

$$\dim K_0 \leqslant \dim N(H, G)/H$$

where $N(H, G)$ *denotes the normalizer of* H *in* G.

The following lemma is easy to verify.

Lemma 2.2. *Let* $X = SO, SU, U$ *or* Sp.

(i) $\langle s\rangle_X + \langle t\rangle_X \leqslant \langle s + t\rangle_X$.

(ii) *If* $t \geqslant n/2$, *then*

$$\langle t\rangle_X + \langle n - t\rangle_X \leqslant \langle t + 1\rangle_X + \langle n - t - 1\rangle_X.$$

Proposition 2.3. (*Hsiang* [5], [6], [10, *Proposition B*]). *Let* G *be a compact Lie group acting effectively on a homogeneous space* G/H *and*

$$\dim G > r \dim G/H, r \geqslant 3.$$

Then $\bar{G}$ (*for notation see* (*)) *acts almost effectively on* G/H *such that* $\bar{G}/H' = G/H$, *and there is at least one normal factor, say* G_1, *such that*

(a) $\dim G_1 + \dim N(H_1, G_1)/H_1 > r \dim G_1/H_1$,

(b) $\dim H_1 > ((r-2)/(r-1)) \dim G_1 \geqslant (1/2) \dim G_1$, *where* $H_1 = G_1 \cap H'$. *If* $r \geqslant 13/4$, G_1 *is a classical group which is isomorphic to one of the following*:

(i) $Spin(n)$, $n > 2r$,

(ii) $SU(n)$, $n > 2r - 1$,

(iii) $Sp(n)$, $n > 2r - 2$.

Although this proposition combines two results of W. Y. Hsiang in [10, Proposition B] and [6], it is observed in [9] that the normal factor G_1 satisfies (i), (ii) or (iii) can be chosen to satisfy (a) and (b).

3. Proof of Theorems A and B

Proof of Theorem A. Since the proofs are identical for all four cases, we shall only give the proof for the case $H \subset Sp(n)$. Suppose $\dim H$ falls into the range (1). The assumption $k \leqslant D_{Sp(n)}$ implies $k \leqslant [(-5 + \sqrt{32n + 41})/4]$, where $[s]$ denotes the largest integer less than or equal to s. If $n \geqslant 11$, it is easy to see that

$$\dim H > \langle n - k \rangle_{Sp} + \langle k \rangle_{Sp} \geqslant n^2 + 3n - 1,$$

so that $\dim Sp(n)/H < (n-1)^2$. Since $H \subset Sp(n)$, by [4, Theorem 1.20], H contains a subgroup $Sp(t)$, $t > n/2$ which is conjugate to a standard imbedding and

$$H = Sp(t) \times K \subset Sp(t) \times Sp(n-t), K \subset Sp(n-t). \tag{3}$$

Hence

$$\dim Sp(t) = \langle t \rangle_{Sp} \leqslant \dim H < \langle n - k + 1 \rangle_{Sp},$$

by (1), whence $t \leqslant n - k$. By (3) and Lemma 2.2, we have

$$\dim H \leqslant \langle t \rangle_{Sp} + \langle n - t \rangle_{Sp} \leqslant \langle n - k \rangle_{Sp} + \langle k \rangle_{Sp}.$$

which is a contradiction to (1). A similar proof will show the theorem for $k \leqslant A_{Sp(n)}$.

It is easy to show the following:

Lemma 3.1. *Let* $X = SO, SU, U$ *or* Sp.

(i) $A_{X(n)} \leqslant D_{X(n)}$.

(ii) *Suppose* $k_1 \leqslant D_{X(k_0)}$ *and* $k_2 \leqslant D_{X(k_1)}$. *Then*

$$\langle k_0 - k_1 - 1 \rangle_X + \langle k_1 + 1 \rangle_X \leqslant \langle k_0 - k_1 \rangle_X + \langle k_1 - k_2 \rangle_X,$$

if $k_1 \geqslant 11, 9, 9$ *or* 8 *according to* $X = SO, SU, U$ *or* Sp *respectively*.

Proof of Theorem B. Again we shall only give the proof of (d). The proof will be by induction on s. The assertion is certainly true when $s = 0$ by

Theorem A. If, on the contrary, the dimension of H falls into the range (2), then

$$\dim H > \langle k_0 - k_1 \rangle_p + \langle k_1 - k_2 \rangle_{Sp} > (n/4 + 1)(4n - 1),$$

because $D_{Sp(k_0)} \geqslant k_1 \geqslant 11$, hence $k_0 \geqslant 74$. By Proposition 2.3, H contains a normal factor G_1 which is a classical group. Since $H \subset Sp(n)$, we conclude that $G_1 = Sp(t)$, $t > n/2$. Since $\dim Sp(n)/H < (n-1)^2$, by [4, Theorem 1,20] we can assume that (3) holds, i.e., $H = Sp(t) \times K \subset Sp(t) \times Sp(n-t)$, and $K \subset Sp(n-t)$.

We proceed to show that $t = k_0 - k_1$. Suppose $t \geqslant k_0 - k_1 + 1$. Then $\dim Sp(t) \geqslant \langle k_0 - k_1 + 1 \rangle_{Sp}$. It follows from (2) and Lemma 2.2 that

$$\begin{aligned}\dim H &< \langle k_0 - k_1 \rangle_{Sp} + \left\langle \sum_{i=1}^{s-1} (k_i - k_{i+1}) + (k_s - k_{s+1} + 1) \right\rangle_{Sp} \\ &\leqslant \langle k_0 - k_1 \rangle_{Sp} + \langle k_1 \rangle_{Sp} < \langle k_0 - k_1 + 1 \rangle_{Sp} \quad (\text{since } k_1 < D_{Sp}(k_0)) \\ &\leqslant \dim Sp(t) \leqslant \dim H,\end{aligned}$$

which is clearly impossible. Hence $t \leqslant k_0 - k_1$. If $t \leqslant k_0 - k_1 - 1$, by Lemma 2.2, Lemma 3.1 and (3) we have

$$\begin{aligned}\dim H &\leqslant \langle t \rangle_{Sp} + \langle n - t \rangle_{Sp} \leqslant \langle k_0 - k_1 - 1 \rangle_{Sp} + \langle k_1 + 1 \rangle_{Sp} \\ &\leqslant \langle k_0 - k_1 \rangle_{Sp} + \langle k_1 - k_2 \rangle_{Sp} < \dim H,\end{aligned}$$

which is an obvious contradiction. Thus we have shown that $t = k_0 - k_1$. Now let $\overline{H} = Sp(k_0 - k_1) \times K$, where $K \subset Sp(k_1)$. Then

$$\begin{aligned}&\sum_{i=1}^{s-1} \langle k_i - k_{i+1} \rangle_P + \langle k_s - k_{s+1} \rangle_{Sp} + \langle k_{s+1} \rangle_{Sp} \\ &\qquad < \dim K < \sum_{i=1}^{s-1} \langle k_i - k_{i+1} \rangle_{Sp} + \langle k_s - k_{s+1} + 1 \rangle_{Sp},\end{aligned}$$

contradicts the inductive hypothesis. This completes the proof of Theorem B.

4. Gaps in the dimensions of the automorphism groups

In this section, we shall proceed to prove Theorems C and D. We shall consider only the case $H \subset Sp(n)$. The proofs of other cases follow the same line of arguments. Note that in the proofs of Theorems C and D we only use the fact that a Lie group G acts on a locally compact Hausdorff space M such that G acts effectively on an orbit $G(x)$ of dimension r with G_x compact. Thus the results are true in more general category of G-spaces, for instance, the topological Cartan G-manifolds (for the definition see [14]).

Proof of Theorem C(d). Suppose that dim G fall into the following range:

$$(**)\qquad \begin{aligned} w + \langle n-k+1\rangle_{Sp} + \langle k+1\rangle_{Sp} - 6 &< \dim G \\ &< w + \langle n-k+2\rangle_{Sp} - 3. \end{aligned}$$

Then $k \leqslant D_{Sp(n)}$ by Lemma 3.1(i). Since $r = \dim G(x) \leqslant 4n + w$, we have $\dim H = \dim G - r$, and

$$\begin{aligned} \langle n-k\rangle_{Sp} + \langle k\rangle_{Sp} &= w + \langle n-k+1\rangle_{Sp} + \langle k+1\rangle_{Sp} - 6 - (4n+w) \\ &\leqslant w + \langle n-k+1\rangle_{Sp} + \langle k+1\rangle_{Sp} - 6 - r \\ &< \dim G - r = \dim H \quad (\text{by } (**)). \end{aligned}$$

It follows from Theorem A(d) that

$$(4)\qquad \langle n-k\rangle_{Sp} + \langle k\rangle_{Sp} < \langle n-k+1\rangle_{Sp} \leqslant \dim H.$$

We can repeat the argument of the proof of Theorem A(d) to obtain a subgroup $Sp(t) \subset H$, $t > n/2$ and $H = Sp(t) \times K \subset Sp(t) \times Sp(n-t)$. Now $n - k + 1 \leqslant t$. If not, $t \leqslant n - k$. This implies, in consequence of Lemma 2.2, that

$$\dim H \leqslant \langle t\rangle_{Sp} + \langle n-t\rangle_{Sp} \leqslant \langle n-k\rangle_{Sp} + \langle k\rangle_{Sp},$$

which is a contradiction to (4). Since $w \leqslant q$ by hypothesis, we obtain

$$\begin{aligned} w + \langle n-k+1\rangle_{Sp} &\leqslant q + \dim Sp(t) \leqslant \dim H, \\ r + w + \langle n-k+1\rangle_{Sp} &\leqslant r + \dim H \\ &= \dim G < w + \langle n-k+2\rangle_{Sp} - 3 \\ &= w + \langle n-k+1\rangle_{Sp} + 4(n-k+1). \end{aligned}$$

It follows that $r < 4(n-k+1)$. On the other hand, $Sp(n-k+1) \subset H \subset SO(r)$. Since $F(H, G(x))$ is not empty, $4(n-k+1) \leqslant r$ from the local representation of $Sp(n-k+1)$ and H in the neighborhood of a fixed point. This is an obvious contradiction.

Proof of Theorem D(d). Suppose dim G fall into the following range:

$$(5)\qquad \begin{aligned} &w + \sum_{i=0}^{s} \langle k_i - k_{i+1} + 1\rangle_{Sp} + \langle k_{s+1} + 1\rangle_{Sp} - 6 \\ &\quad < \dim G \\ &\quad < w + \sum_{i=0}^{s-1} \langle k_i - k_{i+1} + 1\rangle_{Sp} + \langle k_s - k_{s+1} + 2\rangle_{Sp} - 3(s+1). \end{aligned}$$

We shall proceed to get a contradiction. From (5) we have

$$\sum_{i=0}^{s-1} \langle k_i - k_{i+1}\rangle_P + \langle k_s - k_{s+1}\rangle_{Sp} + \langle k_{s+1}\rangle_{Sp}$$
$$\leqslant w + \sum_{i=0}^{s} \langle k_i - k_{i+1} + 1\rangle_{Sp} + \langle k_{s+1} + 1\rangle_{Sp} - 6 - r$$
$$< \dim H.$$

According to Lemma 3.1, we have $k_{i+1} \leqslant D_{Sp(k_i)}$, $0 \leqslant i \leqslant s$. Hence it follows from Theorem B(d) that

$$\sum_{i=0}^{s-1} \langle k_i - k_{i+1}\rangle_{Sp} + \langle k_s - k_{s+1} + 1\rangle_{Sp} \leqslant \dim H. \tag{6}$$

We shall prove by induction on s that under the hypothesis (6) the following holds:

$$Sp(k_0 - k_1) \times \cdots \times Sp(k_{s-1} - k_s) \times Sp(k_s - k_{s+1} + 1) \subset \overline{H}. \tag{7}$$

This is true for $s = 0$ from the proof of Theorem C(d). We may assume by induction that the assertion is true for closed subgroups of $Sp(k_1)$ and $k_i (i = 1, 2, \cdots, s + 1)$.

Let L be a principal isotropy subgroup of the action of $Sp(t)$ on the manifold $G(x)$. We proceed to show that L is conjugate to a standard imbedded $Sp(t - 1)$, hence the principal orbits of the $Sp(t)$ action will be of type S^{4t-1}. From Proposition 2.3 we see easily that

$$\dim Sp(t)/L < (1/(n/4)) \dim Sp(t) \leqslant (1/3) \dim Sp(t) \leqslant (t - 1)^2.$$

Again by [4, Theorem 1.20], $L = Sp(t - s) \times V \subset Sp(t)$ for some $s < t/2$, and $V \subset Sp(s)$. Notice that

$$4n - 1 > \dim Sp(t)/L \geqslant \dim Sp(t)/Sp(t - s) \times Sp(s) = 4ts - 4s^2,$$

hence $n > ts - s^2$. It follows from (a) in Proposition 2.3 that

$$\dim Sp(t) + \dim Sp(s) + 3 \geqslant \dim Sp(t) + \dim N(L, Sp(t))/L$$
$$\geqslant (n/4 + 1) \dim Sp(t)/Sp(t - s) \times Sp(s)$$
$$> ((ts - s^2)/4 + 1)(4ts - 4s^2).$$

Hence

$$s^4 - 2ts^3 + s^2(t^2 - 6) + s(4t - 1) - 2t^2 - t - 3 < 0.$$

This is possible only if $s = 1$. Since $F(Sp(t), G(x))$ is not empty, $Sp(t)$ acts orthogonal on a neighborhood of a fixed point, and so L conjugate to a standard imbedded $Sp(t - 1)$. Moreover

$$H = Sp(t) \times K \subset Sp(t) \times Sp(n - t), \quad K \subset Sp(n - t). \tag{8}$$

By assumption $T^q \subset \overline{H}$, and T^q acts on $G(x)$ with identity element as the connected principal isotropy subgroup, hence the dimension of the principal orbits of H is at least $4t - 1 + q$. Since $q \geqslant w$, it follows that

$$r \geqslant 4t - 1 + w. \tag{9}$$

Since $k_1 \leqslant A_{Sp(k_0)}$, we have

$$w + \langle k_0 - k_1 + 1\rangle_{Sp} + \langle k_1 + 1\rangle_{Sp} - 6 < w + \langle k_0 - k_1 + 2\rangle_{Sp} - 3,$$

or

$$\langle k_0 - k_1\rangle_{Sp} + \langle k_1\rangle_{Sp} \leqslant \langle k_0 - k_1 + 1\rangle_{Sp} - 4k_1 + 3. \tag{10}$$

We will proceed to show that $t = k_0 - k_1$. If $t \leqslant k_0 - k_1 - 1$, then

$$\begin{aligned}\dim H &\leqslant \langle k_0 - k_1 - 1\rangle_{Sp} + \langle k_1 + 1\rangle_{Sp} \leqslant \langle k_0 - k_1\rangle_{Sp} + \langle k_1 - k_2\rangle_{Sp} \\ &< \dim H,\end{aligned}$$

by (8), Lemma 2.2 and Lemma 3.1 which contradicts (6). Hence $k_0 - k_1 \leqslant t$. Suppose now that $t \geqslant k_0 - k_1 + 1$. Then by (9) we have

$$r \geqslant 4(k_0 - k_1) + 3 + w. \tag{11}$$

It is easy to see from (5) that

$$\begin{aligned}\dim H < \sum_{i=0}^{s-1} \langle k_i - k_{i+1}\rangle_{Sp} + \langle k_s - k_{s+1} + 1\rangle_{Sp} \\ + 4(k_0 - k_{s+1} + 1) + w - r.\end{aligned} \tag{12}$$

It follows from (11), (12) and Lemma 2.2 that

$$\begin{aligned}\dim H &< \langle k_0 - k_1\rangle_{Sp} + \left\langle \sum_{i=1}^{s-1} (k_i - k_{i+1}) + (k_s - k_{s+1} + 1)\right\rangle_{Sp} \\ &\quad + 4(k_0 - k_{s+1} + 1) + w - 4(k_0 - k_1) - 3 - w \\ &\leqslant \langle k_0 - k_1\rangle_{Sp} + \langle k_1 - k_{s+1} + 1\rangle_{Sp} + 4(k_1 - k_{s+1}) + 1 \\ &\leqslant \langle k_0 - k_1\rangle_{Sp} + \langle k_1\rangle_{Sp} + 4(k_1 - k_{s+1}) + 1 \\ &\leqslant \langle k_0 - k_1 + 1\rangle_{Sp} - 4k_1 + 3 + 4(k_1 - k_{s+1}) + 1 \qquad \text{(by (10))} \\ &\leqslant \langle k_0 - k_1 + 1\rangle_{Sp} + 4(1 - k_{s+1}) \\ &\leqslant \langle k_0 - k_1 + 1\rangle_{Sp} \leqslant \dim Sp(t) \leqslant \dim H,\end{aligned}$$

which is an obvious contradiction. Hence $t = k_0 - k_1$.

Now the group K acts on $G(x)/Sp(t)$, and hence on $F(Sp(t), G(x))$ with

$$\dim F(Sp(t), G(x)) \leqslant 4(k_0 - t) + w = 4k_1 + w,$$

and ineffective kernel K_0 satisfying

$$\dim K_0 \leqslant \dim N(Sp(t-1), Sp(t))/Sp(t-1) = 3.$$

Furthermore, $K/K_0 \subset Sp(n-t) = Sp(k_1)$. It follows from Theorem B(d) that there exists a gap:

$$\sum_{i=1}^{s-1} \langle k_i - k_{i+1} \rangle_P + \langle k_s - k_{s+1} \rangle_{Sp} + \langle k_{s+1} \rangle_{Sp} < \dim K/K_0 < \sum_{i=1}^{s-1} \langle k_i - k_{i+1} \rangle_{Sp} + \langle k_s - k_{s+1} + 1 \rangle_{Sp}. \tag{13}$$

As $t = k_0 - k_1$, by (6) we have

$$\dim K/K_0 = \dim H - \langle k_0 - k_1 \rangle_{Sp} - \dim K_0 \geqslant \sum_{i=1}^{s-1} \langle k_i - k_{i+1} \rangle_{Sp} + \langle k_s - k_{s+1} + 1 \rangle_{Sp} - \dim K_0. \tag{14}$$

From the hypothesis $k_{s+1} \leqslant \overline{A}_{Sp(k_s)}$ we can easily obtain the following inequality:

$$\langle k_s - k_{s+1} \rangle_{Sp} + \langle k_{s+1} \rangle_{Sp} \leqslant \langle k_s - k_{s+1} + 1 \rangle_{Sp} - 4k_{s+1} + 3(1-s). \tag{15}$$

Hence it follows from (13), (14), and (15) that

$$\sum_{i=1}^{s-1} \langle k_i - k_{i+1} \rangle_{Sp} + \langle k_s - k_{s+1} + 1 \rangle_{Sp} \leqslant \dim K/K_0.$$

Therefore we can apply the inductive hypothesis of (6) to the action of K/K_0 on $F(Sp(t), G(x))$ to obtain

$$Sp(k_1 - k_2) \times \cdots \times Sp(k_{s-1} - k_s) \times Sp(k_s - k_{s+1} + 1) \subset K/K_0.$$

This proves (7). Since $H \subset SO(r)$, it follows from (7) that

$$4(k_0 - k_1) + \cdots + 4(k_{s-1} - k_s) + 4(k_s - k_{s+1} + 1) \leqslant r,$$

or

$$4(k_0 - k_{s+1} + 1) \leqslant r. \tag{16}$$

On the other hand, $w \leqslant q$, so by (7) and (12) we have

$$w + \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle_{Sp} + \langle k_s - k_{s+1} + 1 \rangle_{Sp} \leqslant \dim H < \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle_{Sp} + \langle k_s - k_{s+1} + 1 \rangle_{Sp} + 4(k_0 - k_{s+1} + 1) + w - r.$$

This leads to $r < 4(k_0 - k_{s+1} + 1)$ which contradicts (16).

In Theorems C and D, the assumption $q \geqslant w$ is necessary. For example, we can consider the action of $G = Sp(n)$ on $M = T^w \times S^{4n}$ by letting G acting trivially on T^w, and orthogonally on S^{4n}. If $3 < w < 4n - 16$, then

$$w + \langle n - 1\rangle_{Sp} + \langle 3\rangle_{Sp} - 6 < \dim G < w + \langle n\rangle_{Sp} - 3.$$

References

[1] E. H. Cattani & L. N. Mann, *Gaps in the dimensions of the automorphism groups of some geometric structures*, preprint.

[2] R. Hermann, *Differential geometry and the calculus of variations*, Academic Press, New York, 1968.

[3] W. C. Hsiang & W. Y. Hsiang, *Classification of differentiable actions of S^n, R^n and D^n with S^k as the principal orbit types*, Ann. of Math. **82** (1965) 421–433.

[4] ______, *Differentiable actions of compact connected classical groups*. I, Amer. J. Math. **89** (1967) 705–786.

[5] W. Y. Hsiang, *The natural metric on $SO(n)/SO(n-2)$ is the most symmetric metric*, Bull. Amer. Math. Soc. **72** (1967) 55–58.

[6] ______, *On the degree of symmetry and the structure of highly symmetric manifolds*, Tangkang J. Math. **2** (1971) 1–22.

[7] S. Kobayashi, *Transformation groups in differential geometry*, Springer, Berlin, 1972.

[8] H. T. Ku & M. C. Ku, *Gaps in the relative degree of symmetry*, London Math. Soc., Lecture Notes Series **26** (1977) 121–138.

[9] ______, *Degree of symmetry of manifolds*, Seminar Notes, University of Massachusetts, Amherst, 1976.

[10] H. T. Ku, L. N. Mann, J. L. Sicks & J. C. Su, *Degree of symmetry of a product manifold*, Trans. Amer. Math. Soc. **146** (1969) 133–149.

[11] L. N. Mann, *Gaps in the dimensions of isometry groups of Riemannian manifolds*, J. Differential Geometry **11** (1976) 293–298.

[12] D. Montgomery & H. Samelson, *Transformation groups of spheres*, Ann. of Math. **44** (1943) 454–470.

[13] S. B. Myers & N. E. Steenrod, *The group of isometries of a Riemannian manifold*, Ann. of Math. **40** (1939) 400–416.

[14] R. Palias, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. **73** (1961) 295–323.

University of Massachusetts