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EXPANSIONS ASSOCIATED TO CLEAN INTERSECTIONS

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1. Introduction

In the papers [7], [8], [9], Patodi and the author studied certain expansions associated to compact group actions on compact differentiable manifolds. More specifically, let $f\colon X \to X$ be an isometry of a compact differentiable manifold X of dimension m. The fixed point set Ω of f is the disjoint union of closed connected submanifolds N of dimension n. If Δ denotes the Laplace operator of X for functions with eigenvalues λ, then f induces linear maps $f_\lambda^{\#}$ on the eigenspaces of Δ, and there exists an asymptotic expansion as $t\downarrow 0$:

$$\sum_{\lambda} \operatorname{Tr}(f_\lambda^{\#})e^{-t\lambda} \sim \sum_{n\in\Omega} (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k \int_N b_k(f, a)\,dvol_N(a).$$

The $b_k(f, a)$ have a simple description using invariant theory.

These expansions are closely related to the Atiyah-Singer-Lefschetz formulas for compact group actions. In fact, one easily generalizes to obtain expansions for the Laplacians of all the classical elliptic complexes. By taking an alternating sum, a local integral formula for the Lefschetz number is found. Invariant theory [1], [2], [3] may be applied to identify this integrand with the Atiyah-Singer integrand. The main result of [8] is a new proof of the G-Signature theorem of Atiyah-Singer via this method.

In this paper we study similar expansions associated to clean intersections in the sense of [6]. Let $N_1, N_2 \subset M$ be two submanifolds of the compact Riemannian manifold M having clean intersection Ω, the disjoint union of compact connected submanifolds N. Denote by n, n_1, n_2, and d respectively the dimensions of N, N_1, N_2, and M, and by $K(t, u, v)$ the fundamental solution of the heat equation on M for the Laplacian of functions. Then by evaluating the solution of the heat equation with distribution N_2 as initial

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conditions on the distribution N_1, one obtains an asymptotic expansion:

$$K_t[N_1 \times N_2] = \int_{N_1 \times N_2} K(t, u, v) dvol_{N_1}(u) dvol_{N_2}(v)$$

$$\sim \sum_{N \in \Omega} (4\pi t)^{(-d+n_1+n_2-n)/2} \sum_{l=0}^{\infty} t^l \int_N R_l(u) dvol_N(u).$$

If N_1, N_2 are totally geodesic submanifolds of M, then their intersection Ω is necessarily clean. If N is any component of Ω, then there are certain tensors C, D associated with straightening the angle between N_1 and N_2 at N. We denote by $\mathcal{C}$ the collection of tensors consisting of C, D, the curvature tensor R of M, and the covariant derivatives of R. If $a \in N$, then we may decompose $T_aM = T_aN_1^\perp \oplus T_aN \oplus T_aN_2^\perp \oplus T_aN_{12}^\perp$. Here $T_aN_1^\perp$ is the space of vectors in T_aN_1 which are orthogonal to T_aN; $T_aN_{12}^\perp$ is the space of vectors in T_aM which are orthogonal to both T_aN_1 and T_aN_2; and $T_aN_2^\perp$ is the space of vectors in T_aM which is orthogonal to $T_aN_1 \oplus T_aN_{12}^\perp$. To simplify notation we assume now that $n_1 + n_2 = d$. Then there is a natural action of $O(n_1 - n) \times O(n) \times O(n_2 - n) \times O(n)$ on $\mathcal{C}$, corresponding to the decomposition $T_aM = T_aN_1^\perp \oplus T_aN \oplus T_aN_2^\perp \oplus T_aN_{12}^\perp$. In the totally geodesic case, the terms $R_l(u)$ in the expansion derived above are $O(n_1 - n) \times O(n) \times O(n_2 - n) \times O(n)$ invariant polynomial maps from to functions on N.

In the special case where $M = X \times X$ is a product manifold, N_1 is the graph of an isometry f, and N_2 is the diagonal in $X \times X$, recovers the expansion of [7].

It is more interesting to study the corresponding expansions for the Laplacian on forms; these expansions are related to intersection theory. In §4, we consider the special case where M is an even dimensional compact oriented manifold, and $N_1 = N_2 = N$ is an oriented totally geodesic submanifold with half the dimension of M. If $K^n(t, x, y)$ is the fundamental solution of the heat equation for n-forms on M, then denote $K(t, x)[*N]$ the solution to the heat equation with initial data given by the distribution $*[N]$, where $*$ is the Hodge star operator of M. Denote i: $N \to M$ the inclusion. Then for $x \in N$ one has the asymptotic expansion:

$$i^*K(t, x)[*N] \sim (4\pi t)^{-n/2} \sum_{j=0}^{\infty} t^j \gamma_j(x).$$

The $\gamma_j(x)$ are $O(n) \times SO(n)$ invariant polynomial maps (corresponding to the decomposition $TM = TN \oplus TN^\perp$) from metrics on M to n-forms on N.

It is easily verified that for all t,

$$\int_N i^* K(t, x)[{}^*N] = [N] \cap [N],$$

where $[N] \cap [N]$ denotes the self-intersection number of N in M. Consequently

$$\int_N \gamma_j(x) = 0, \quad j \neq \frac{n}{2},$$

$$\int_N \gamma_j(x) = [N] \cap [N], \quad j \neq \frac{n}{2}.$$

In Theorem 4.3, we give a corresponding local result

$$\gamma_j(x) = 0, \qquad j < n/2,$$
$$\gamma_j(x) = \chi^{\perp}(\Omega), \quad j = n/2,$$

where $\chi^{\perp}(\Omega)$ is the Euler form of the normal bundle to N in M. If one takes $M = X \times X$ to be a product manifold, and N to be the diagonal in M, then Theorem 4.3 reduces to Patodi's theorem [11].

More generally we obtain a similar local vanishing theorem for two totally geodesic oriented submanifolds $N_1, N_2 \subset M$ when $n_1 + n_2 = d$. If one takes $M = X \times X$ to be a product manifold, N_1 to be the graph of an isometry f, and N_2 to be the diagonal in $X \times X$, then the local Lefschetz theorem for the Euler complex [9] is recovered.

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2. Asymptotic expansion

Let $N_1, N_2 \subset M$ be two submanifolds of the compact Riemannian manifold M. One says that N_1 and N_2 have clean intersection [6] if the following conditions are satisfied: (i) The intersection $\Omega = N_1 \cap N_2$ is the disjoint union of compact connected submanifolds N, and (ii). For each $N \in \Omega$ and $a \in N$ one has $T_aN = T_aN_1 \cap T_aN_2$, where T_aN denotes the tangent space of N. The dimensions of N, N_1, N_2, and M will be denoted by n, n_1, n_2, and d respectively.

Theorem 2.1. *Let $N_1, N_2 \subset M$ be two submanifolds with clean intersection Ω, and let $K_t(x, y) = K(t, x, y)$ be the fundamental solution of the heat equaton for functions on M. Then there exists an asymptotic expansion*:

$$K_t[N_1 \times N_2] = \int_{N_1 \times N_2} K(t, x, y)\, dvol_{N_1}(x)\, dvol_{N_2}(y)$$
$$\sim \sum_{N \in \Omega} (4\pi t)^{(-d+n_1+n_2-n)/2} \sum_{l=0}^{\infty} t^l \int_N \mathfrak{R}_l(x)\, dvol_N(x), \quad t \downarrow 0,$$

where $dvol_{N_i}$, $i = 1, 2$, *denotes the measure induced by the Riemannian metric of* N_i *as a submanifold of* M, *and the* $\mathfrak{R}_l(x)$ *are local invariants depending only on the germs of the submanifolds* N_1, N_2 *and the Riemannian metric of* M *near* $x \in N$.

Proof. Let U_{N_i} be a tubular neighborhood of N in N_i, $1 = 1, 2$. U_{N_i} may be identified with a neighborhood of the zero section of the normal bundle to N in N_i, $i = 1, 2$. π_i: $U_{N_i} \to N$ is the natural projection corresponding to the exponential map along N in N_i.

One has

$$K_t[N_1 \times N_2] = \int_{N_1}\int_{N_2} K(t, u_1, u_2)dvol_{N_1}(u_1)dvol_{N_2}(u_2)$$

$$\sim \sum_{N \in \Omega} \int_{U_{N_1}}\int_{U_{N_2}} K(t, u_1, u_2)dvol_{N_1}(u_1)dvol_{N_2}(u_2).$$

We proceed with a detailed analysis of each term in the above sum. Fix any component N of Ω and denote

$$I = \int_{U_{N_1}}\int_{U_{N_2}} K(t, u_1, u_2)dvol_{N_1}(u_1)dvol_{N_2}(u_2).$$

If a, b are points of N, then by letting $\bar{x}, \bar{y}$ be normal coordinates on $\pi_1^{-1}(a)$, $\pi_2^{-1}(b)$ respectively we have

$$I = \int_{N \times N}\left(\int_{\pi_1^{-1}(a)\pi_2^{-1}(b)} K(t, \bar{x}, \bar{y})\psi_1(\bar{x})\psi_2(\bar{y})\, d\bar{x}\, d\bar{y}\right)dvol_N(a)dvol_N(b),$$

where ψ_i, $i = 1, 2$, are defined by

$$dvol_{N_1}(\bar{x}) = \psi_1(\bar{x})\, d\bar{x}\pi_1^*(dvol_N(a)),$$
$$dvol_{N_2}(\bar{y}) = \psi_2(\bar{y})\, d\bar{y}\pi_2^*(dvol_N(b)).$$

Denote by U_D a sufficiently small neighborhood of the diagonal in $N \times N$. Then

$$I \sim \int_{U_D}\left(\int_{\pi_1^{-1}(a)}\int_{\pi_2^{-1}(b)} K(t, \bar{x}, \bar{y})\psi_1(\bar{x})\psi_2(\bar{y})\, d\bar{x}\, d\bar{y}\right)dvol_N(a)dvol_N(b).$$

By means of the well-known asymptotic expansion of Minakshisundaram [5]:

$$K(t, \bar{x}, \bar{y}) \sim \frac{\exp(-r^2/4t)}{(4\pi t)^{d/2}}\left(\sum_{i=0}^{\infty} t^i u_i(\bar{x}, \bar{y})\right),$$

where r^2 is the square of the distance from $\bar{x}$ to $\bar{y}$, we thus have

$$I \sim (4\pi t)^{-d/2} \int_{U_D} \left(\int_{\pi_1^{-1}(a)} \int_{\pi_2^{-1}(b)} \exp\left(\frac{-r^2}{4t}\right) \sum_{i=0}^{\infty} t^i u_i(\bar{x}, \bar{y}) \cdot \psi_1(x)\psi_2(y)\, d\bar{x} d\bar{y} \right) dvol_N(a) dvol_N(b).$$

It is a consequence of the classical Morse lemma [10, p. 6] that there exists a change of coordinates $x = x(\bar{x}, \bar{y})$, $y = y(\bar{x}, \bar{y})$ so that

$$\exp\left(\frac{-r^s}{4t}\right) = \exp\left(\frac{-r^2(a, b)}{4t}\right) \exp\left(\frac{-\sum x_j^2}{4t}\right) \exp\left(\frac{-\sum y_k^2}{4t}\right).$$

Then

$$I \sim (4\pi t)^{-d/2} \int_{U_D} \exp\left(\frac{-r^2(a, b)}{4t}\right) \left(\int_{\pi_1^{-1}(a) \times \pi_2^{-1}(b)} \exp\left(\frac{-\sum x_j^2}{4t}\right) \exp\left(\frac{-\sum y_k^2}{4t}\right) \cdot \sum_{i=0}^{\infty} t^i \bar{\mathfrak{L}}_i(x, y)\, dx\, dy \right) dvol_N(a) dvol_N(b),$$

where $\bar{\mathfrak{L}}_i(x, y) = u_i(\bar{x}, \bar{y})\psi_1(\bar{x})\psi_2(\bar{y})J(\bar{x}, \bar{y}, x, y)$. One denotes by $J(\bar{x}, \bar{y}, x, y)$ the Jacobian determinant of the above change of variables.

By appeal to the Taylor series expansion of the $\bar{\mathfrak{L}}_i(x, y)$ one may deduce

$$I \sim (4\pi t)^{-d/2} \int_{U_D} \exp\left(\frac{-r^2(a, b)}{4t}\right) \sum_{i=0}^{\infty} t^i \sum_{\alpha,\beta} \frac{1}{(2\alpha)!\,(2\beta)!} \left(\frac{\partial^{2|\alpha|+2|\beta|}}{\partial x^{2\alpha} \partial y^{2\beta}} \bar{\mathfrak{L}}_i \right)(a, b) \cdot \left(\int_{\pi_1^{-1}(a)} \int_{\pi_2^{-1}(b)} x^{2\alpha} y^{2\beta} \exp\left(\frac{-\sum x_j^2}{4t}\right) \exp\left(\frac{-\sum y_k^2}{4t}\right) dx\, dy \right) dvol_N(a) dvol_N(b),$$

where the sum is over multi-indices α of length $n_1 - n$ and multi-indices β of length $n_2 - n$.

We now make the change of variables $w = x/\sqrt{t}$, $z = y/\sqrt{t}$. Then

$$I \sim (4\pi t)^{-d/2} \int_{U_D} \exp\left(\frac{-r^2(a, b)}{4t}\right) \sum_{i=0}^{\infty} t^i \sum_{\alpha,\beta} \frac{1}{(2\alpha)!\,(2\beta)!} \left(\frac{\partial^{2|\alpha|+2|\beta|}}{\partial x^{2\alpha} \partial y^{2\beta}} \bar{\mathfrak{L}}_i \right)(a, b) \cdot \left[t^{|\alpha|+|\beta|+((n_1+n_2)/2)-n} \int_{R^{n_1-n}} w^{2\alpha} \exp\left(\frac{-\sum w_j^2}{4}\right) dw \cdot \int_{R^{n_2-n}} z^{2\beta} \exp\left(\frac{-\sum z_k^2}{4}\right) dz \right] dvol_N(a) dvol_N(b).$$

Moreover,

$$I \sim (4\pi t)^{-d/2+(n_1+n_2)/2-n} \int_{U_D} \exp\left(\frac{-r^2(a,b)}{4t}\right)$$
$$\cdot \sum_{i=0}^{\infty} t^i \sum_{\alpha,\beta} \frac{1}{(2\alpha)!\,(2\beta)!}\left(\frac{\partial^{2|\alpha|+2|\beta|}}{\partial x^{2\alpha}\partial y^{2\beta}} \bar{\mathfrak{L}}_i\right)(a,b)$$
$$\cdot t^{|\alpha|+|\beta|}\left[(4\pi)^{n-(n_1+n_2)/2} \prod_{j=1}^{n_1-n} \int_R w_j^{2\alpha_j} \exp\left(\frac{-w_j^2}{4}\right) dw_j\right.$$
$$\left.\cdot \prod_{k=1}^{n_2-n} \int_R z_k^{2\beta_k} \exp\left(\frac{-z_k^2}{4}\right) dz_k\right] dvol_N(a)\,dvol_N(b).$$

Applying the classical formula [12, p. 426]

$$\int_R x^{2s} \exp\left(\frac{-x^2}{4}\right) dx = \frac{1\cdot 3\cdot 5\cdots(2s-1)}{2^{-s-1}}(\sqrt{\pi}) \tag{2.2}$$

thus yields

$$I \sim (4\pi t)^{-d/2+(n_1+n_2)/2-n} \sum_{i=0}^{\infty} t^i \int_{U_D} \exp\left(\frac{-r^2(a,b)}{4t}\right)$$
$$\cdot \sum_{\alpha,\beta} \frac{1}{\alpha!\,\beta!}\left(\frac{\partial^{2|\alpha|+2|\beta|}}{\partial x^{2\alpha}\partial y^{2\beta}} \bar{\mathfrak{L}}_i\right)(a,b)\, t^{|\alpha|+|\beta|} dvol_N(a)\,dvol_N(b).$$

Now let

$$\Box_x = \sum_l \frac{\partial^2}{\partial x_l^2}, \Box_y = \sum \frac{\partial^2}{\partial y_m^2},$$
$$\mathfrak{L}_k(a,b) = \sum_{j=0}^{k} \frac{1}{j!}(\Box_x + \Box_y)^j \bar{\mathfrak{L}}_{k-j}(a,b).$$

Then

$$I \sim (4\pi t)^{-d/2+(n_1+n_2)/2-n} \sum_{k=0}^{\infty} t^k \int_{U_D}\left[\exp\left(\frac{-r^2(a,b)}{4t}\right)\mathfrak{L}_k(a,b)\right]$$
$$\cdot dvol_N(a)\,dvol_N(b).$$

For each $a \in N$, let U_a be a sufficiently small normal coordinate neighborhood of a, and denote by v_i, $1 \leqslant i \leqslant n$, a normal coordinate system on U_a. Then

$$I \sim (4\pi t)^{-d/2+(n_1+n_2)/2-n} \sum_{k=0}^{\infty} t^k \int_N\left[\int_{U_a} \exp\left(\frac{-\|v\|^2}{4t}\right)\mathfrak{L}_k(v)\,dv\right] dvol_N(a).$$

We deduce from the Taylor expansion of $\mathfrak{L}_k(v)$ that

$$I \sim (4\pi t)^{-d/2+(n_1+n_2)/2-n} \sum_{k=0}^{\infty} t^k \int_N \sum_\alpha \frac{1}{(2\alpha)!} \frac{\partial^{2\alpha}\mathfrak{L}_k}{\partial v^{2\alpha}}(a)$$

$$\cdot \left[\int_{U_a} \exp\left(\frac{-\sum v_i^2}{4t} \right) v^{2\alpha}\, dv \right] dvol_N(a),$$

where α is a multi-index of length n.

One now makes the change of variables $u = v/\sqrt{t}$. Then

$$I \sim (4\pi t)^{-d/2+(n_1+n_2)/2-n/2} \sum_{k=0}^{\infty} t^k \int_N \sum_\alpha \frac{1}{(2\alpha)!} \frac{\partial^{2\alpha}\mathfrak{L}_k}{\partial v^{2\alpha}}(a) t^{|\alpha|}$$

$$\cdot \left(\prod_{i=1}^{n} \int_R u^{2\alpha_i} \exp\left(\frac{-u_i^2}{4} \right) du_i \right) dvol_N(a),$$

which simplifies, in consequence of the classical formula (2.2) above, to become:

$$I \sim (4\pi t)^{-d/2+(n_1+n_2)/2-n/2} \sum_{k=0}^{\infty} t^k \int_N \left(\sum_\alpha \frac{1}{\alpha!} \frac{\partial^{2\alpha}\mathfrak{L}_k}{\partial v^{2\alpha}}(a) t^{|\alpha|} \right) dvol_N(a).$$

Now let $\square_v = \sum \partial^2/\partial v_i^2$. Then

$$I \sim (4\pi t)^{-d/2+(n_1+n_2)/2-n/2} \sum_{n=0}^{\infty} t^k \int_N \sum_{j=0}^{k} \frac{1}{j!} (\square_v^j \mathfrak{L}_{k-j})(a) dvol_N(a).$$

Finally by denoting

$$\mathfrak{R}_k = \sum_{j=0}^{k} \frac{1}{j!} (\square_v^j \mathfrak{L}_{k-j})(a),$$

we obtain

$$I \sim (4\pi t)^{-d/2+(n_1+n_2)/2-n/2} \sum_{k=0}^{\infty} t^k \int_N \mathfrak{R}_k(a) dvol_N(a).$$

This completes the proof of Theorem 2.1.

Let X be a compact connected Riemannian manifold. If Δ is the Laplace operator of X acting on functions, then we may decompose $L^2(X) = \sum_\lambda L^2_\lambda(X)$ where $L^2_\lambda(X)$ are the eigenspaces of Δ corresponding to the eigenvalues λ. The various eigenspaces are orthogonal with respect to the measure induced by the Riemannian metric of X. We denote by P_λ: $L^2(X) \to L^2_\lambda(X)$ the corresponding orthogonal projections.

Now suppose f: $X \to X$ is a differentiable map. Then f induces maps $f_\lambda^\#$: $L^2_\lambda \to L^2_\lambda$ defined by $f_\lambda^\#(\phi) = P_\lambda(\phi \circ f)$ for $\phi \in L^2_\lambda$. In Theorem 2.1 we may take M to be the product manifold $X \times X$, N_1 to be the graph G_f of f, and N_2 to be the diagonal $\mathfrak{D}$ in $X \times X$. Then one recovers the asymptotic expansion of [7]:

Corollary 2.3. *Let f: $X \to X$ be a diffeomorphism which preserves the measure induced by the Riemannian metric of X. Suppose that the fixed point set Ω of f satisfies the following conditions: (i) Ω is the disjoint union of compact connected submanifolds N of dimension n, and (ii) for each $N \in \Omega$ and $a \in N$ one has* $\det(I - f') \neq 0$ *where f': $T_aN^{\perp} \to T_aN^{\perp}$ is the induced map on the normal bundle of N. Denote* $\operatorname{Tr}(f_\lambda^{\#})$ *the trace of f_λ: $L_\lambda^2 \to L_\lambda^2$ with respect to the inner product on L_λ^2 induced by the Riemannian metric of X. Then there exists an asymptotic expansion as $t\downarrow 0$:*

$$\sum_{\lambda} \operatorname{Tr}(f_\lambda^{\#})e^{-t\lambda} \sim \sum_{N\in\Omega} (4\pi t)^{-n/2} \sum_{l=0}^{\infty} t^l \int_N b_l(f, a)dvol_N(a), \tag{2.3}$$

where the $b_l(f, a)$ are local invariants depending only on the germ of f and the Riemannian metric of X near a.

Proof. The components of the intersection of N_1, the graph of f, and the diagonal N_2 may be identified with the components of the fixed point set of f. Moreover, the assumption that $\det(I - f') \neq 0$ implies that for each component N of the fixed point set one has $T(N) = T(N_1) \cap T(N_2)$. Thus the hypotheses of Theorem 2.1 are satisfied, and we have

$$K_t[N_1 \times N_2] \sim \sum_{N\in\Omega} (\sqrt{2}\,)^n(4\pi t)^{-n/2} \sum_{l=0}^{\infty} t^l \int_N \mathfrak{R}_l(x)dvol_N(x),$$

where $dvol_N(x)$ is the volume form of N considered as a submanifold of X, and K_t is the heat kernel of $X \times X$.

Now we may write

$$K_t(x_1, x_2, x_3, x_4) = \sum_{i,j} e^{-t\lambda_i - t\lambda_j}\phi_i(x_1)\phi_j(x_2)\phi_i(x_3)\phi_j(x_4),$$

where ϕ_i are the eigenfunctions of the Laplacian on X. Then

$$\begin{aligned} K_t[N_1 \times N_2] &= \sum_{i,j} e^{-t\lambda_i - t\lambda_j} \int_{N_1} \phi_i(x_1)\phi_j(f(x_1))dvol_{N_1}(x_1) \\ &\quad \cdot \int_{N_2} \phi_i(x_3)\phi_j(x_3)dvol_{N_2}(x_3) \\ &= 2^{d/2}\sum e^{-t\lambda_i - t\lambda_j} \int_X \phi_i(x_1)\phi_j(f(x_1))dvol_X(x_1) \\ &\quad \cdot \int_X \phi_i(x_3)\phi_j(x_3)dvol_X(x_3), \end{aligned}$$

since f is volume-preserving. Therefore

$$K_t[N_1 \times N_2] = 2^{d/2} \sum_i e^{-2t\lambda_i} \int_X \phi_i(x_1)\phi_i(f(x_1)) dvol_X(x_1)$$
$$= 2^{d/2} \sum_\lambda \operatorname{Tr}(f_\lambda^{\#}) e^{-2t\lambda},$$

or

$$\sum_\lambda \operatorname{Tr}(f_\lambda^{\#}) e^{-2t\lambda} \sim \sum_{N \in \Omega} 2^{(n-d)/2} (4\pi t)^{n/2} \sum_{l=0}^{\infty} t^l \int_N \mathfrak{R}_l(x) dvol_N(x),$$

which gives the corollary with

$$b_l = 2^{n-l-d/2} \mathfrak{R}_l.$$

Remark. One may derive the expansion (2.3), using the method of [7], for a differentiable map f: $X \to X$ which is not necessarily measure-preserving. It is only necessary to assume Conditions (i) and (ii) of Corollary 2.3. Unfortunately the more general case does not seem to be a corollary of Theorem 2.1.

3. The totally geodesic case

Let N_1, $N_2 \subset M$ be two totally geodesic submanifolds. Then each component N of the intersection Ω of N_1 and N_2 is a totally geodesic submanifold N. Furthermore the intersection is necessarily clean. Thus one has the asymptotic expansion of Theorem 2.1. In the totally geodesic case, the terms $\mathfrak{R}_l$ will be more readily computable than in general. We restrict our attention to a single component N of Ω and define basic tensors, denoted C and D, associated with straightening the angle between N_1 and N_2 at N.

Let $\bar{x}_1, \cdots, \bar{x}_d$ be normal coordinates on M, centered at $a \in N$, such that $\bar{x}_1, \cdots, \bar{x}_{n_1}$ are normal coordinates on N_1, and $\bar{x}_{n_1-n+1}, \cdots, \bar{x}_{n_1}$ are normal coordinates on N. Similarly, let $\bar{y}_1, \cdots, \bar{y}_d$ be normal coordinates on M, centered at $a \in N$, such that $\bar{y}_{n_1-n+1}, \cdots, \bar{y}_{n_1}$ are normal coordinates on N, and $\bar{y}_{n_1-n+1}, \cdots, \bar{y}_{n_1}, \bar{y}_{n_1+1}, \cdots, \bar{y}_{n_1+n_2-n}$ are normal coordinates on N_2. We may assume that the $\bar{y}$ coordinates coincide with the $\bar{x}$ coordinates for points on N.

We define C to be the transition matrix from the $\bar{y}$ to the $\bar{x}$ coordinate system:

$$\bar{x}_l = \sum_{m=1}^{d} C_{lm} \bar{y}_m, \ 1 \leqslant l < d;$$

C is an orthogonal matrix.

The canonical form of the matrix C is independent of the chosen point $a \in N$. To see this, first observe that C is completely determined by the differentials $d\bar{x}_l$, $d\bar{y}_m$ at a. Now if $a_1, a_2 \in N$, we may choose a curve γ in N and parallel translate $d\bar{x}_l(a_1)$, $d\bar{y}_m(a_1)$ along to covectors at a_2. Since N_1, N_2 are totally geodesic, the parallel translates of $d\bar{x}_l(a_1)$, $d\bar{y}_m(a_1)$ will be the differentials of suitably normalized coordinates centered at a_2, in the sense specified above.

For the remainder of this section, we adopt the Einstein summation convention of summing over repeated indices. Unless otherwise indicated, we will allow the following ranges for our indices:

$$1 \leqslant i, j, k \leqslant n_1 - n, \qquad n_1 + 1 \leqslant \bar{i}, \bar{j}, \bar{k} \leqslant n_1 + n_2 - n,$$

$$n_1 - n + 1 \leqslant \alpha, \beta, \gamma \leqslant n_1, \qquad d - n + 1 \leqslant \bar{\alpha}, \bar{\beta}, \bar{\gamma} \leqslant d.$$

To make the notation less cumbersome, we will assume that $d = n_1 + n_2$. The more general case offers no new difficulties.

If $\bar{x} \in N_1$ and $\bar{y} \in N_2$, we denote by $d(\bar{x}, \bar{y})$ the Riemannian distance from $\bar{x}$ to $\bar{y}$. $\pi_i: N_i \to N$, $i = 1, 2$, will be the projections, defined on neighborhoods of N in N_i, associated to the exponential maps along N. The first term in the Taylor expansion of $d^2(\bar{x}, \bar{y}) - d^2(\pi_1(\bar{x}), \pi_2(\bar{y}))$ defines a quadratic form, and we would like to find a linear change of variables which diagonalizes this quadratic form. Now

$$d^2(\bar{x}, \bar{y}) - d^2(\pi_1(\bar{x}), \pi_2(\bar{y})) = \sum (\bar{x}_i - C_{i\bar{j}}\bar{y}_{\bar{j}})^2 + \sum (C_{\bar{j}\bar{l}}\bar{y}_{\bar{l}})^2 + 0((\bar{x}, \bar{y})^4).$$

If we define

$$w_i = \bar{x}_i - C_{i\bar{j}}\bar{y}_{\bar{j}}, \; z_{\bar{j}} = C_{\bar{j}\bar{l}}\bar{y}_{\bar{l}}, \tag{3.1}$$

then

$$d^2(\bar{x}, \bar{y}) - d^2(\pi_1(\bar{x}), \pi_2(\bar{y})) = \sum w_i^2 + \sum z_{\bar{j}}^2 + O((w, z)^4).$$

The matrix $C_{\bar{j}\bar{l}}$ is necessarily invertible, and we denote its inverse matrix by $D_{\bar{l}\bar{m}}$, $C_{\bar{j}\bar{l}}D_{\bar{l}\bar{m}} = \delta_{\bar{j}\bar{m}}$. One defines $D_{i\bar{l}} = C_{i\bar{j}}D_{\bar{j}\bar{l}}$. Then

$$\bar{x}_i = w_i + D_{i\bar{l}}z_{\bar{l}}, \; \bar{y}_{\bar{j}} = D_{\bar{j}\bar{l}}z_{\bar{l}}. \tag{3.2}$$

If $a \in N$, then we decompose $T_aM = T_aN_1^\perp \oplus T_aN \oplus T_aN_2^\perp \oplus T_aN_{12}^\perp$. Here $T_aN_1^\perp$ is the space of vectors in T_aN_1 which are orthogonal to T_aN, $T_aN_{12}^\perp$ is the space of vectors in T_aM which are orthogonal to both T_aN_1 and T_aN_2, and $T_aN_1^\perp$ is the space of vectors in T_aM which is orthogonal to $T_aN_1 \oplus T_aN_{12}^\perp$.

Let $\mathcal{C}$ be the collection of tensors consisting of C, D, and the curvature tensor R of M and its covariant derivatives. There is a natural action of the product of orthogonal groups $O(n_1 - n) \times O(n) \times O(n_2 - n) \times O(n)$ on $\mathcal{C}$, corresponding to the decomposition $TM = TN_1^{\perp} \oplus TN \oplus TN_2^{\perp} \oplus TN_{12}^{\perp}$.

Define ρ to be the Ricci tensor of M, and ρ^1, ρ^2 to be the partial Ricci tensors obtained by taking the trace over an orthonormal basis for the normal bundles to N in N_1, N_2, respectively. One has

$$\rho(X, Y) = \sum_{j=1}^{d} R\big(X, d\bar{x}_j, Y, d\bar{x}_j\big),$$

$$\rho^{(1)}(X, Y) = \sum_{j=1}^{n_1 - n} R\big(X, d\bar{x}_j, Y, d\bar{x}_j\big),$$

$$\rho^{(2)}(X, Y) = \sum_{\bar{l}=n_1+1}^{n_1+n_2-n} R\big(X, d\bar{y}_{\bar{l}}, Y, d\bar{y}_{\bar{l}}\big),$$

$$\tau = \sum_{i=1}^{d} \rho\big(d\bar{x}_i, d\bar{x}_i\big),$$

where τ is the scalar curvature of M. Clearly the components of $\rho^{(2)}$ with respect to the $\bar{x}$ coordinate system are given by contractions in the components of C, R.

Theorem 3.3. *Let N_1, $N_2 \subset M$ be two totally geodesic submanifolds with intersection Ω, the disjoint union of compact connected totally geodesic submanifolds N. Then one has the asymptotic expansion of Theorem* 2.1:

$$K_t[N_1 \times N_2] \sim \sum_{N \in \Omega} (4\pi t)^{(-d+n_1+n_2-n)/2} \sum_{l=0}^{\infty} t^l \int_N \mathfrak{R}_l(x)\, dvol_N(x)$$

as $t \downarrow 0$. Moreover, if $d = n_1 + n_2$, we may write

$$\mathfrak{R}_l = |\det(D_{\bar{l}\bar{m}})|\,\mathfrak{R}_l',$$

where $\mathfrak{R}_l'$ is an $O(n_1 - n) \times O(n) \times O(n_2 - n) \times O(n)$ invariant polynomial map from the collection $\mathcal{C}$ of tensors to functions on N. In particular, we have

$$\mathfrak{R}_0 = |\det(D_{\bar{l}\bar{m}})|,$$

$$\begin{aligned}\mathfrak{R}_1 = {}& |\det(D_{\bar{l}\bar{m}})|(\tau/3 - 1/6\rho_{\bar{\alpha}\bar{\alpha}}) + \tfrac{2}{3}\rho^{(1)}_{r\bar{l}} D_{r\bar{l}} + \tfrac{1}{3}\rho^{(1)}_{\bar{l}\bar{l}} - \tfrac{2}{3}\rho^{(1)}_{\alpha\alpha} - \tfrac{1}{3}\rho^{(1)}_{ii} \\ & -\tfrac{1}{3}\rho^{(2)}_{lm} D_{l\bar{i}} D_{m\bar{i}} - \tfrac{1}{3}\rho^{(2)}_{\bar{l}\bar{l}} - \tfrac{2}{3}\rho^{(2)}_{l\bar{m}} D_{l\bar{m}} - R_{\bar{l}\bar{\alpha}\bar{l}\bar{\alpha}} - \tfrac{5}{3} R_{i\alpha j\alpha} D_{i\bar{l}} D_{j\bar{l}} \\ & -2R_{l\alpha\bar{m}\alpha} D_{l\bar{m}} + \tfrac{1}{3} R_{\bar{n}k\bar{l}h} D_{k\bar{l}} D_{h\bar{n}} + \tfrac{1}{3} R_{\bar{r}k\bar{l}h} D_{k\bar{r}} D_{h\bar{l}} + \tfrac{1}{3} R_{\bar{l}k\bar{l}h} D_{k\bar{m}} D_{h\bar{m}}.\end{aligned}$$

The components of the curvature tensor R refer to an $\bar{x}$ coordinate system.

Proof. It is clear that the coefficients $\mathfrak{R}_l$ are invariantly defined. The main point is to check that the $\mathfrak{R}_l'$ depend polynomially on the collection of tensors $\mathcal{C}$. This is illustrated by the explicit computation of $\mathfrak{R}_0$ and $\mathfrak{R}_1$.

We will use the notation of the proof of Theorem 2.1. The subscripts for the curvature tensor R and its covariant derivatives will refer to the $\bar{x}$ coordinate system. Some necessary geometric information is contained in the series expansions of [7, Section (2)].

One has

$$\begin{aligned}
\mathfrak{R}_0(a) &= \mathfrak{L}_0(a) = \bar{\mathfrak{L}}_0(a), \\
\bar{\mathfrak{L}}_0(a) &= u_0(a)\psi_1(a)\psi_2(a)J(\bar{x}, \bar{y}, x, y)(a) \\
&= J(x, y, w, z)J(w, z, x, y) \\
&= |\det(D_{\bar{l}\bar{m}})|,
\end{aligned}$$

where w, z are the variables defined earlier in this section. This implies that $\mathfrak{R}_0 = |\det(D_{\bar{l}\bar{m}})|$, the desired formula.

More work is required to compute $\mathfrak{R}_1$.

$$\mathfrak{R}_1 = \square_v \mathfrak{L}_0 + \mathfrak{L}_1. \tag{3.4}$$

First we will compute the term $\mathfrak{L}_1 = (\square_x + \square_y)\bar{\mathfrak{L}}_0 + \bar{\mathfrak{L}}_1$:

$$\bar{\mathfrak{L}}_1(a) = u_1\psi_1\psi_2 J(\bar{x}, \bar{y}, x, y) = |\det(D_{\bar{l}\bar{m}})|(\tau/6), \tag{3.5}$$

where τ is the scalar curvature of M. This computation uses the well-known fact [4, p. 922] that $u_1(a) = \tau/6$. We turn our attention to

$$\begin{aligned}
(\square_x + \square_y)\bar{\mathfrak{L}}_0 &= (\square_x + \square_y)(u_0(x, y)\psi_1(x, y)\psi_2(x, y)J(\bar{x}, \bar{y}, x, y)) \\
&= |\det(D_{\bar{l}\bar{m}})|((\square_x + \square_y)(u_0(x, y)\psi_1(x, y)\psi_2(x, y)J(w, z, x, y)).
\end{aligned}$$

The Jacobian $J(w, z, x, y)$ arises from the change of variables required to represent $d^2(\bar{x}, \bar{y}) - d^2(\pi_1(\bar{x}), \pi_2(\bar{y}))$ as a sum of squares in x, y. It is a consequence of formula (2.2) of [7] that

$$\begin{aligned}
&d^2(\bar{x}, \bar{y}) - d^2(\pi_1(\bar{x}), \pi_2(\bar{y})) \\
&\quad = \sum w_i^2 + \sum z_{\bar{l}}^2 - \tfrac{1}{3} R_{rklh}\bar{x}_k\bar{x}_h C_{r\bar{j}}\bar{y}_{\bar{j}} C_{l\bar{s}}\bar{y}_{\bar{s}} - \tfrac{2}{3} R_{rk\bar{l}h}\bar{x}_k\bar{x}_h C_{r\bar{j}}\bar{y}_{\bar{j}} C_{\bar{l}\bar{s}}\bar{y}_{\bar{s}} \\
&\qquad - \tfrac{1}{3} R_{\bar{r}k\bar{l}h}\bar{x}_h\bar{x}_h C_{\bar{r}\bar{j}} y_{\bar{j}} C_{\bar{l}\bar{s}} y_{\bar{s}} + \cdots,
\end{aligned}$$

where the three dots $\cdots$ indicate terms which will make zero contribution to our computation of $(\square_x + \square_y)\bar{\mathfrak{L}}_0$.

Now, according to the formulas (3.1), (3.2), we may write

$$\begin{aligned}
&d^2(\bar{x}, \bar{y}) - d^2(\pi_1(\bar{x}), \pi_2(\bar{y})) \\
&\quad = \sum w_i^2 + \sum z_{\bar{l}}^2 - \tfrac{1}{3} R_{rklh} w_k w_h D_{r\bar{j}} z_{\bar{j}} D_{l\bar{s}} z_{\bar{s}} - \tfrac{2}{3} R_{rk\bar{l}h} w_k w_h D_{r\bar{j}} z_{\bar{j}} z_{\bar{l}} \\
&\qquad - \tfrac{2}{3} R_{rk\bar{l}h} w_k D_{h\bar{m}} z_{\bar{m}} D_{r\bar{j}} z_{\bar{j}} z_{\bar{l}} - \tfrac{2}{3} R_{\bar{r}k\bar{l}h} w_k D_{h\bar{m}} z_{\bar{m}} z_{\bar{r}} z_{\bar{l}} \\
&\qquad - \tfrac{1}{3} R_{\bar{r}k\bar{l}h} D_{k\bar{m}} D_{h\bar{n}} z_{\bar{m}} z_{\bar{n}} z_{\bar{r}} z_{\bar{l}} - \tfrac{1}{3} R_{\bar{r}k\bar{l}h} w_k w_h z_{\bar{r}} z_{\bar{l}} + \cdots.
\end{aligned}$$

We may choose

$$\begin{aligned} x_i &= w_i + \cdots, \\ y_{\bar l} &= z_{\bar l} - \tfrac{1}{6} R_{rksh} w_k w_h D_{r\bar j} z_{\bar j} D_{s\bar l} - \tfrac{1}{3} R_{rk\bar l h} w_k w_h D_{r\bar j} z_{\bar j} - \tfrac{1}{3} R_{rk\bar l h} w_k D_{h\bar m} z_{\bar m} D_{r\bar j} z_{\bar j} \\ &\quad - \tfrac{1}{3} R_{\bar r k\bar l h} w_k D_{h\bar m} z_{\bar m} z_{\bar r} - \tfrac{1}{6} R_{\bar r k\bar l h} D_{k\bar m} D_{h\bar n} z_{\bar m} z_{\bar n} z_{\bar r} - \tfrac{1}{6} R_{\bar r k\bar l h} w_k w_h z_{\bar r} + \cdots. \end{aligned}$$

Then

$$\begin{aligned} J(w, z, x, y) &= 1 + \tfrac{1}{6} R_{rksh} x_k x_h D_{r\bar l} D_{s\bar l} + \tfrac{1}{3} R_{rk\bar l h} x_k x_h D_{r\bar l} + \tfrac{1}{6} R_{\bar l k\bar l h} x_k x_h \\ &\quad + \tfrac{1}{6} R_{\bar l k\bar l h} D_{k\bar m} D_{h\bar n} y_{\bar m} y_{\bar n} + \tfrac{1}{6} R_{\bar r k\bar l h} D_{k\bar l} D_{h\bar n} y_{\bar n} y_{\bar r} \\ &\quad + \tfrac{1}{6} R_{\bar r k\bar l h} D_{k\bar m} y_{\bar m} D_{h\bar l} y_{\bar r} + \cdots. \end{aligned}$$

Formula (2.6) of [7] gives the expansion

$$\psi_1(\bar x) = 1 - \tfrac{1}{2} R_{i\alpha j\alpha} \bar x_i \bar x_j - \tfrac{1}{6} \rho^{(1)}_{ij} \bar x_i \bar x_j + \cdots.$$

Therefore

$$\begin{aligned} \psi_1(x, y) &= 1 - \tfrac{1}{2} R_{i\alpha j\alpha} x_i x_j - \tfrac{1}{2} R_{i\alpha j\alpha} D_{i\bar l} D_{j\bar m} y_{\bar l} y_{\bar m} - R_{i\alpha j\alpha} x_i D_{j\bar m} y_{\bar m} \\ &\quad - \tfrac{1}{6} \rho^{(1)}_{ij} x_i x_j - \tfrac{1}{6} \rho^{(1)}_{ij} D_{i\bar l} D_{j\bar m} y_{\bar l} y_{\bar m} - \tfrac{1}{3} \rho^{(1)}_{ij} x_i D_{j\bar m} y_{\bar m} + \cdots. \end{aligned}$$

Formula (2.6) of [7] also implies that

$$\begin{aligned} \psi_2(x, y) &= 1 - \tfrac{1}{2} R_{l\alpha m\alpha} D_{l\bar i} y_{\bar i} D_{m\bar j} y_{\bar j} - \tfrac{1}{2} R_{\bar l\alpha\bar m\alpha} y_{\bar l} y_{\bar m} - R_{l\alpha\bar m\alpha} D_{l\bar i} y_{\bar i} y_{\bar m} \\ &\quad - \tfrac{1}{6} \rho^{(2)}_{lm} D_{l\bar i} y_{\bar i} D_{m\bar j} y_{\bar j} - \tfrac{1}{6} \rho^{(2)}_{\bar l\bar m} y_{\bar l} y_{\bar m} - \tfrac{1}{3} \rho^{(2)}_{l\bar m} D_{l\bar i} y_{\bar i} y_{\bar m} + \cdots. \end{aligned}$$

Finally from [4, p. 923] we recall the formula:

$$u_0(x, y) = 1 + \frac{1}{12} \rho_{kh} x_k x_h + \frac{1}{12} \rho_{\bar k\bar h} y_{\bar k} y_{\bar h} + \cdots.$$

Thus

$$\Box_x(u_0\psi_1\psi_2 J) = \tfrac{1}{3} \rho^{(1)}_{rs} D_{r\bar l} D_{s\bar l} + \tfrac{2}{3} \rho^{(1)}_{r\bar l} D_{r\bar l} + \tfrac{1}{3} \rho^{(1)}_{\bar l\bar l} - \tfrac{1}{3} \rho^{(1)}_{ii} + \tfrac{1}{6} \rho_{hh}.$$

Moreover,

$$\begin{aligned} \Box_y(u_0\psi_1\psi_2 J) &= \tfrac{1}{3} R_{\bar l k\bar l h} D_{k\bar m} D_{h\bar m} + \tfrac{1}{3} R_{\bar n k\bar l h} D_{k\bar l} D_{h\bar n} + \tfrac{1}{3} R_{\bar r k\bar l h} D_{k\bar r} D_{h\bar l} \\ &\quad - R_{i\alpha j\alpha} D_{i\bar l} D_{j\bar l} - \tfrac{1}{3} \rho^{(1)}_{ij} D_{i\bar l} D_{j\bar l} - R_{l\alpha m\alpha} D_{l\bar i} D_{m\bar i} - R_{\bar l\alpha\bar l\alpha} \\ &\quad - 2 R_{l\alpha\bar m\alpha} D_{l\bar m} - \tfrac{1}{3} \rho^{(2)}_{lm} D_{l\bar i} D_{m\bar i} - \tfrac{1}{3} \rho^{(2)}_{\bar l\bar l} - \tfrac{2}{3} \rho^{(2)}_{l\bar m} D_{l\bar m} + \tfrac{1}{6} \rho_{\bar h\bar h}. \end{aligned}$$

Combining the above two equations gives

$$
\begin{aligned}
(\square_x + \square_y)\bar{\mathcal{L}}_0 &= |\det(D_{i\bar{j}})|(\square_x + \square_y)(u_0\psi_1\psi_2 J) \\
&= |\det(D_{i\bar{j}})|\big(\tfrac{2}{3}\rho^{(1)}_{rl} D_{r\bar{l}} + \tfrac{1}{3}\rho^{(1)}_{\bar{l}\bar{l}} - \rho^{(1)}_{\alpha\alpha} - \tfrac{1}{3}\rho^{(1)}_{ii} + \tfrac{1}{6}\rho_{hh} + \tfrac{1}{3} R_{\bar{l}k\bar{l}h} D_{k\bar{m}} D_{h\bar{m}} \\
&\quad + \tfrac{1}{3} R_{\bar{n}k\bar{l}h} D_{k\bar{l}} D_{h\bar{n}} + \tfrac{1}{3} R_{\bar{r}k\bar{l}h} D_{k\bar{r}} D_{h\bar{l}} - 2R_{i\alpha j\alpha} D_{i\bar{l}} D_{j\bar{l}} - R_{\bar{l}\alpha\bar{l}\alpha} \\
&\quad - 2R_{l\alpha\bar{m}\alpha} D_{l\bar{m}} - \tfrac{1}{3}\rho^{(2)}_{\bar{l}\bar{m}} D_{l\bar{i}} D_{m\bar{i}} - \tfrac{1}{3}\rho^{(2)}_{\bar{l}\bar{l}} - \tfrac{2}{3}\rho^{(2)}_{l\bar{m}} D_{l\bar{m}} + \tfrac{1}{6}\rho_{\bar{h}\bar{h}}\big).
\end{aligned}
\tag{3.6}
$$

Finally one must compute the term

$$
\begin{aligned}
(\square_v \mathcal{L}_0)(a) = \big(\square_v \bar{\mathcal{L}}_0\big)(a) &= |\det(D_{i\bar{j}})|\square_v(u_0\psi_1\psi_2 J) \\
&= |\det(D_{i\bar{j}})|\square_v(u_0 J(w, z, x, y)).
\end{aligned}
$$

Recall that the v_i are the normal coordinates on N of $\pi_2(y)$ for a coordinate system centered at $\pi_1(x)$. To compute $\square_v(u_0 J(w, z, x, y))$ we may use formula (2.2) of [7] to write

$$
d^2(\bar{x}, \bar{y}) - d^2(\pi_1(\bar{x}), \pi_2(\bar{y})) = \sum w_i^2 + \sum z_{\bar{l}}^2 - \tfrac{1}{3} R_{\alpha k\beta h}\bar{x}_k\bar{x}_h v_\alpha v_\beta + \cdots,
$$

where the three dots $\cdots$ indicate terms which will make zero contribution to our computation. Then

$$
\begin{aligned}
& d^2(\bar{x}, \bar{y}) - d^2(\pi_1(\bar{x}), \pi_2(\bar{y})) \\
&\qquad = \sum w_i^2 + \sum z_{\bar{l}}^2 - \tfrac{1}{3} R_{\alpha k\beta h} w_k w_h v_\alpha v_\beta - \tfrac{2}{3} R_{\alpha k\beta h} w_k D_{h\bar{l}} z_{\bar{l}} v_\alpha v_\beta \\
&\qquad\quad - \tfrac{1}{3} R_{\alpha k\beta h} D_{k\bar{m}} D_{h\bar{l}} z_{\bar{l}} z_{\bar{m}} v_\alpha v_\beta + \cdots.
\end{aligned}
$$

One has

$$
J(w, z, x, y) = 1 + \tfrac{1}{6}\rho^{(1)}_{\alpha\beta} v_\alpha v_\beta + \tfrac{1}{6} R_{\alpha k\beta h} D_{k\bar{m}} D_{h\bar{m}} v_\alpha v_\beta.
$$

Moreover, from the well-known formula [4, p. 923] for u_0 we have

$$
u_0(\bar{x}, \bar{y}) = 1 + \frac{1}{12}\rho_{\alpha\beta} v_\alpha v_\beta + \cdots.
$$

Finally

$$
(\square_v \mathcal{L}_0)(a) = |\det(D_{i\bar{j}})|\big(\tfrac{1}{6}\rho_{\alpha\alpha} + \tfrac{1}{3}\rho^{(1)}_{\alpha\alpha} + \tfrac{1}{3} R_{\alpha k\alpha h} D_{k\bar{m}} D_{h\bar{m}}\big).
\tag{3.7}
$$

The required formula for

$$
\mathcal{R}_1 = \square_v \mathcal{L}_0 + \mathcal{L}_1
$$

follows by adding (3.5), (3.6) and (3.7).

To obtain the invariant theory characterization for the terms $\mathcal{R}_l$ one needs only to observe that

$$
\mathcal{R}_l = \sum_{j=0}^{l} \frac{1}{j!}\big(\square_v^j \mathcal{L}_{l-j}\big),
$$

where

$$\mathfrak{L}_k = \sum_{j=0}^{k} \frac{1}{j!}(\Box_x + \Box_y)^j \bar{\mathfrak{L}}_{k-j},$$

$$\bar{\mathfrak{L}}_i = |\det(D_{\bar{l}m})| u_i(\bar{x}, \bar{y}) \psi_1(\bar{x}) \psi_2(\bar{y}) J(w, z, x, y).$$

It follows from the expansion in [7, Section (2)] that each of the terms u_1, ψ_1, ψ_2, $J(w, z, x, y)$ has an expansion in x, y, v whose coefficients depend polynomially on the tensors in $\mathcal{C}$. This gives the invariant theory characterization of the terms $\mathfrak{R}_l$.

4. A generalization of Patodi's theorem

In this section we will assume that M is a compact oriented even dimensional Riemannian manifold of dimension $d = 2n$. Let N be a totally geodesic oriented submanifold with dimension n, half the dimension of M. We orient the normal bundle $TN^{\perp}$ of N in M to be compatible with the decomposition $TM = TN \oplus TN^{\perp}$.

The heat equation for n-forms on M has a fundamental solution $K^n(t, x, y)$ which is a smooth double form on $M \times M$. Since N is oriented we may integrate $K^n(t, x, y)$ over N in the y variable to obtain an asymptotic expansion similar to that of Theorem 2.1. It will be more interesting to consider instead the expansion associated to $*yK^n(t, x, y)$, where $*y$ is the Hodge star operator of M applied to the second variable y. We denote

$$K_t^n(x)[*N] = \int_N *yK_t^n(x, y),$$

where $K_t^n(x, y) = K^n(t, x, y)$, the fundamental solution of the heat equation for n-forms.

Theorem 4.1. *Suppose M is a compact oriented Riemannian manifold of even dimension $d = 2n$ and that N is a totally geodesic oriented submanifold of dimension n, half the dimension of M. Then for $x \notin N$ one has*

$$K_t^n(x)[*N] = O(e^{-c(x)/t}),$$

where $c(x) > 0$ is a constant depending on x. If $x \in N$, then there exists an asymptotic expansion

$$K_t^n(x)[*N] \sim (4\pi t)^{-n/2} \sum_{l=0}^{\infty} t^l \Gamma_l(x).$$

The n-forms $\Gamma_l(x)$ are local invariants of the Riemannian metric of M near x.

Denote i: $N \to M$ the inclusion map and $\gamma_l = i^*\Gamma_l$, the pull-backs of the Γ_l. Then the γ_l are $O(n) \times SO(n)$ invariant polynomial maps (corresponding to the *decomposition $TM = TN \oplus TN^{\perp}$) from the curvature tensor R of M and its covariant derivatives to n-forms on N.*

Proof. The proof is similar to that of Theorems 2.1 and 3.3. The $\gamma_l(x)$ are clearly $SO(n) \times SO(n)$ invariant. The only interesting point is to check that they are actually $O(n) \times SO(n)$ invariant. To see this, one needs only to observe that if one reverses the orientation of N while fixing the orientation of the normal bundle to N, then $K_t^n[{}^*N]$ is invariant.

One has

Lemma 4.2.

$$\int_N \gamma_l(x) = 0, \qquad l \neq \frac{n}{2},$$

$$\int_N \gamma_l(x) = [N] \cap [N], \quad l = \frac{n}{2},$$

where $[N] \cap [N]$ is the self-intersection number of N.

Proof. Let E_λ^n be an eigenspace of the Laplacian on n-forms corresponding to an eigenvalue $\lambda \neq 0$. It is a consequence of Hodge theory that there exists an orthogonal decomposition

$$E_\lambda^n = dE_\lambda^{n-1} \oplus d^* E_\lambda^{n+1},$$

where d is the exterior derivative, and d^* is the adjoint of d.

Now let $\{\phi_i(x)\}$ be an orthonormal basis for the eigenforms of the Laplacian on n-forms, where $\phi_i(x)$ corresponds to the eigenvalue λ_i. We may suppose, as observed in the above paragraph, that if $\lambda_i \neq 0$, then $\phi_i(x)$ is either exact or coexact. It is well-known [5] that

$$K^n(t, x, y) = \sum_i e^{-t\lambda_i} \phi_i(x) \otimes \phi_i(y),$$

so that

$$\int_{N \times N} {}^*yK^n(t, x, y) = \sum_i e^{-t\lambda_i} \left[\int_N \phi_i(x) \right] \left[\int_N {}^*\phi_i(y) \right]$$
$$= \sum_{\lambda_i = 0} \left[\int_N \phi_i(x) \right] \left[\int_N {}^*\phi_i(y) \right].$$

Since if $\lambda_i \neq 0$ then $\phi_i(x)$ is either exact or coexact. However, for $\lambda_i = 0$, the harmonic forms $\phi_i(x)$ represent a basis for $H^n(M, R)$, while ${}^*\phi_i(y)$ represent a dual basis. Thus

$$\int_{N \times N} {}^*yK^n(t, x, y) = [N] \cap [N].$$

However, by Theorem 4.1 we may write

$$\int_{N \times N} {*y} K^n(t, x, y) = \int_N K_t^n(x)[{*N}] \sim (4\pi t)^{-n/2} \sum_{l=0}^{\infty} t^l \int_N \gamma_l(x).$$

The left-hand side was just computed above, and is equal to $[N] \cap [N]$, independent of t. Thus the lemma follows.

It is well-known [6, p. 196] that

$$[N] \cap [N] = \int_N \chi^{\perp}(\Omega),$$

where $\chi^{\perp}(\Omega)$ is the Euler form of the normal bundle to the totally geodesic submanifold N. Thus one is led to conjecture that the singular terms in the expansion

$$i^*(K_t^n(x)[{*N}]) \sim (4\pi t)^{-n/2} \sum_{l=0}^{\infty} t^l \gamma_l(x)$$

vanish, and that the constant term is equal to $\chi^{\perp}(\Omega)$. In fact, one has

Theorem 4.3. *Suppose M is a compact oriented manifold of even dimension $d = 2n$, and that N is a totally geodesic oriented submanifold of dimension n, half the dimension of M. Then in the expansion*

$$i^*(K_t^n(x)[{*N}]) \sim (4\pi t)^{-n/2} \sum_{l=0}^{\infty} t^l \gamma_l(x)$$

one has

$$\gamma_l = 0, \qquad l < n/2,$$
$$\gamma_l = \chi^{\perp}(\Omega), \quad l = n/2.$$

Proof. An invariant polynomial map P, from the curvature tensor of M and its covariant derivatives to n-forms on N, is said to have weight k if under a scaling $g \to c^2 g$ of the metric g on M one has $P \to c^k P$. It is well known [8, Theorem 5.1] that an invariant polynomial map P of weight k vanishes identically if $k > 0$. If $k = 0$, then P is a polynomial in (1) the Pontriagin forms of N, (2) the Euler form of the normal bundle to N, and (3) the Pontriagin forms of the normal bundle to N.

It is easy to see how the n-forms γ_l transform under a scaling $g \to c^2 g$ of the metric on M. In fact, if $g \to c^2 g$ then

$$K^n(t, x, y) \to K^n(c^{-2}t, x, y).$$

Moreover, since $d = 2n$, the Hodge star operator is invariant under scaling of the metric on M. Thus when $g \to c^2 g$, one has $\gamma_l \to c^{n-2l}\gamma_l$, and therefore $\gamma_l = 0$ for $l < n/2$ by Theorem 5.1 of [8].

Now suppose n is even and that $l = n/2$. By Theorem 5.1 of [8] one knows that γ_l is a polynomial in (1) the Pontriagin forms of N, (2) the Pontriagin forms of the normal bundle to N, and (3) the Euler form of the normal bundle to N. However, if one reverses the orientation of M, or equivalently of the normal bundle to N, while fixing the orientation of N, then $\gamma_l \to -\gamma_l$. Consequently γ_l must be a multiple of $\chi^{\perp}(\Omega)$. However

$$\int_N \gamma_l = \int_N \chi^{\perp}(\Omega) = [N] \cap [N].$$

So $\gamma_l = \chi^{\perp}(\Omega)$.

Let X be a compact oriented Riemannian manifold of dimension n. The diagonal $\mathfrak{D}$ is a totally geodesic submanifold of the product $X \times X$. This follows since $\mathfrak{D}$ is the fixed point set of the isometry $(x_1, x_2) \to (x_2, x_1)$. By applying Theorem 4.3 in the special case $N = \mathfrak{D}$, $M = X \times X$, one recovers a well-known result of Patodi [11].

Corollary 4.4 (*Patodi*). *Let X be a compact oriented Riemannian manifold. Denote $e^p(t, x, y)$ the fundamental solution of the heat equation for p-forms on X. Then*

$$\sum (-1)^p \operatorname{Tr}(e^p(t, x, x))^* 1 = \chi(\Omega) + O(t),$$

where $*$ is the Hodge star operator of X, and $\chi(\Omega)$ is the Euler form of X.

Proof. Suppose $\{\phi_{i,p}\}$ is an orthonormal basis, for the space of square integrable p-forms on X, consisting of eigenforms of the Laplacian with eigenvalues $\{\lambda_i\}$. Then $\{*\phi_{i,p}\}$ is an orthonormal basis for the space of square integrable $(n-p)$-forms on X, since $*$ is an isometry which commutes with the Laplacian. It is well-known [5] that we may write

$$e^p(t, x, y) = \sum_i e^{-t\lambda_{i,p}} \phi_{i,p}(x) \otimes \phi_{i,p}(y),$$

$$e^q(t, x, y) = \sum_i e^{-t\lambda_{i,p}} *\phi_{i,p}(x) \otimes *\phi_{i,p}(y)$$

for $p + q = n$.

The eigenforms of the Laplace operator of $M = X \times X$ acting on n-forms are spanned by wedge products $\psi(x_1, x_2) = \phi_{i,p}(x_1) \wedge \phi_{j,q}(x_2)$, $p + q = n$. Therefore the heat kernel for n-forms on $X \times X$ is

$$K^n(t, (x_1, x_2), (y_1, y_2)) = \sum_{p+q=n} e^p(t, x_1, y_1) \wedge e^q(t, x_2, y_2).$$

Moreover

$$\begin{aligned}
&{}^{*}yK^{n}(t, (x_1, x_2), (y_1, y_2)) \\
&= \sum_{p+q=n} (-1)^{q} {}^{*}y_1 e^{p}(t, x_1, y_1) \wedge {}^{*}y_2 e^{q}(t, x_2, y_2) \\
&= \sum_{p+q=n} (-1)^{q+pq} \Big(\sum_i e^{-t\lambda_{i,p}} \phi_{i,p}(x_1) \otimes {}^{*}\phi_{i,p}(y_1) \Big) \\
&\quad \wedge \Big(\sum_j e^{-t\lambda_{j,p}} {}^{*}\phi_{j,p}(x_2) \otimes \phi_{j,p}(y_2) \Big) \\
&= \sum_{p+q=n} (-1)^{q} \sum_{i,j} e^{-t\lambda_{i,p} - t\lambda_{j,q}} (\phi_{i,p}(x_1) \wedge {}^{*}\phi_{j,p}(x_2)) \\
&\quad \otimes (\phi_{j,p}(y_2) \wedge {}^{*}\phi_{i,p}(y_1)).
\end{aligned}$$

By integrating over $N = \mathfrak{D}$ one obtains

$$K_t^n(x_1, x_2)[{}^{*}N] = \sum_{p+q=n} (-1)^q \sum_i e^{-2t\lambda_{i,p}} \phi_{i,p}(x_1) \wedge {}^{*}\phi_{i,p}(x_2),$$

and thus

$$K_t^n(x, x)[{}^{*}N] = \sum_{p+q=n} (-1)^q \operatorname{Tr}(e_p(t, x, x))^{*}1.$$

Consequently, by Theorem 4.3,

$$\sum_p (-1)^p \operatorname{Tr}(e_p(t, x, x))^{*}1 = \chi^{\perp}(\Omega) + O(t),$$

where $\chi^{\perp}(\Omega)$ is the Euler form of the normal bundle to the diagonal $N = \mathfrak{D}$ in $M = X \times X$. Furthermore, the map $(v, v) \to (v, -v)$ is a connection preserving isomorphism of the tangent bundle and normal bundle to the diagonal. Therefore $\chi^{\perp}(\Omega) = \chi(\Omega)$, the Euler form of X, and Patodi's theorem follows.

Remark. If X is not necessarily orientable, one still has Patodi's theorem:

$$\sum (-1)^p \operatorname{Tr}(e^p(t, x, x)) = {}^{*}\chi(\Omega) + O(t).$$

This follows from Corollary 4.4 by considering the oriented double cover of X.

5. Invariant theory

It requires some preparation to develop further generalizations of Patodi's theorem. This section is devoted to some preliminary work which is mainly algebraic in nature. The methods used here are rather standard [1], [2], [3], [8].

Let M be a compact oriented Riemannian manifold of dimension d, and $N_1, N_2 \subset M$ be two totally geodesic oriented submanifolds of dimensions n_1,

n_2 respectively, and let $n_1 + n_2 = d$. We suppose that the intersection Ω of N_1 and N_2 is the disjoint union of oriented submanifolds, and focus our attention on a particular component $N \subset \Omega$, N of dimension n. We may use the notation of §3.

Consider the collection $\mathcal{C}$ of tensors, defined in §3, consisting of C, D and the curvature tensor R of M and its covariant derivatives. There is a natural action of $O(n_1 - n) \times O(n) \times O(n_2 - n) \times SO(n)$ on $\mathcal{C}$, corresponding to the decomposition $TM = TN_1^\perp \oplus TN \oplus TN_2^\perp \oplus TN_{12}^\perp$ and a chosen orientation of $TN_{12}^\perp$. To specify this action, we define a coordinate system $x_1, \cdots, x_d$ centered at $a \in N$ to be normalized if (1) $dx_1, \cdots, dx_{n_1-n}$ lie in $TN_1^\perp$; $dx_{n_1-n+1}, \cdots, dx_{n_1}$ lie in TN; $dx_{n_1+1}, \cdots, dx_{d-n}$ lie in $TN_2^\perp$; and $dx_{d-n+1}, \cdots, dx_d$ lie in $TN_{12}^\perp$, and (2) $dx_1, \cdots, dx_d$ are orthonormal at a. A function γ of the tensors $\mathcal{C}$ will be called an invariant polynomial map to n-forms on N if, with respect to any normalized coordinate system centered at $a \in N$, γ is an $O(n_1 - n) \times O(n) \times O(n_2 - n) \times SO(n)$ invariant polynomial map from $\mathcal{C}$ to n-forms on N.

One may denote by det the determinant map from tensors to functions on N:

$$\det\colon \otimes TN_{12}^\perp \to \Lambda^0(N)$$

corresponding to the orientation of $TN_{12}^\perp$.

Well-known results [3, p. 287], [13] from classical invariant theory imply that an invariant polynomial map γ to n-forms on N is a linear sum of elementary monomial invariants:

$$\operatorname{mon}(R, C, D) = \sum R_{F_1} \cdots R_{F_p} C_{G_1} \cdots C_{G_q} D_{H_1} \cdots D_{H_s},$$

where $F_1, \cdots, F_p$ are multi-indices containing indices which may refer to any of $TN_1^\perp$, TN, $TN_2^\perp$, or $TN_{12}^\perp$; $G_1, \cdots, G_q$, $H_1, \cdots, H_s$ are pairs of indices corresponding to $TN_1^\perp$ and $TN_2^\perp$ only. It is understood that n of the indices, necessarily among the F_i's, corresponding to TN must be alternated. Moreover, det may be applied to n-tuples of indices corresponding to $TN_{12}^\perp$. The remaining indices must be contracted in pairs.

An invariant polynomial map γ is said to be of weight k if under a scaling $g \to c^2 g$ of the metric on M one has $\gamma \to c^k \gamma$. Our first goal will be to characterize the elementary monomial invariants of nonnegative weight.

Lemma 5.1. *The weight of an elementary monomial invariant* $\operatorname{mon}(R, C, D)$ *is* $2p + n - \sum f_i$ *where* f_i *is the total number of indices in* F_i.

Proof Each R_{F_i}, C_{G_j}, and D_{H_k} has weight two. If (con) is the total number of contractions and (det) is the total number of times det is applied, then $\operatorname{mon}(R, C, D)$ has weight $2p + 2q + 2s - 2(\text{con}) - n(\det)$. On the other hand, $n + 2(\text{con}) + n(\det) = \sum f_i + 2q + 2s$. Thus $\text{mon}(R, C, D)$ has weight $2p + n - \sum f_i$.

Denote by ε_R the total number of covariant derivatives in all the $R_{F_i}'s$. Then $\sum f_i = 4p + \varepsilon_R$. Thus one obtains the formula $n = \text{weight}\ (\text{mon}(R, C, D)) + 2p + \varepsilon_R$.

Now we recall the classical identities satisfied by the curvature tensor R.

Lemma 5.2. *The Riemann curvature tensor R satisfies the identities*:

(5.2.1) $R_{\smile{ijkl}} = 0, \qquad R_{ij\smile{kl,r}} = 0,$

(5.2.2) $R_{ijkl} = -R_{jikl}, R_{ijkl} = -R_{ijlk}$

and consequently

(5.2.3) $R_{ijkl} = R_{klij}$,

(5.2.4) $R_{i[jk]l} = \frac{1}{2} R_{il[jk]}$,

where the bow $\smile$ *denotes alternation over three indices, and the bracket* [ ] *denotes alternation over two indices.*

The following lemma is a consequence of the curvature identities:

Lemma 5.3. *Let* $\text{mon}(R, C, D)$ *be an elementary monomial invariant of weight k.*

(i) *If $k > 0$, then* $\text{mon}(R, C, D) = 0$.

(ii) *If $k = 0$ and* $\text{mon}(R, C, D) \neq 0$, *then n is necessarily even. Moreover, in the above notation, $n = 2p$ and $\varepsilon_R = 0$. We may assume that precisely the last two indices are alternated in each R_{F_i}.*

Proof. The identities (5.2.1), (5.2.2), and (5.2.3) imply that for the terms R_{F_i} we may alternate over at most two of the first five indices, else $\text{mon}(R, C, D) = 0$. Thus as n is the total number of alternations in the R's, one has $n \leqslant 2p + \varepsilon_R$ with strict equality if $\varepsilon_R > 0$. However, $2p + \varepsilon_R = n-$ weight $(\text{mon}(R, C, D))$ as shown above. Thus, if weight $(\text{mon}(R, C, D)) > 0$ and $(\text{mon}(R, C, D)) \neq 0$, we must have $\varepsilon_R = \text{weight}\ (\text{mon}(R, C, D)) = 0$. Consequently, $n = 2p$. In particular, n is even.

Since $n = 2p$, we must alternate exactly two indices in each R. By (5.2.3), (5.2.4) we may suppose that the last two indices in each R are alternated.

The Euler form of $TN_{12}^{\perp}$, $\chi^{\perp}(\Omega)$ has the property that $\chi^{\perp}(\Omega) \to -\chi^{\perp}(\Omega)$ when the orientation on TN_{12} is reversed. In fact, $\chi^{\perp}(\Omega)$ is essentially the only such invariant of weight zero.

Lemma 5.4. *Let γ be an invariant polynomial map, to n-forms on N, of weight zero. Suppose that $\gamma \to -\gamma$ when the orientation on $TN_{12}^{\perp}$ is reversed. Then γ is a multiple of $\chi^{\perp}(\Omega)$.*

Proof. Since γ is always a sum of elementary monomial invariants, it suffices to treat the special case where $\gamma = \text{mon}(R, C, D)$ is an elementary monomial invariant.

By Lemma 5.3 one has $\varepsilon_R = 0$, $n = 2p$. Thus $\Sigma f_i = 4p$, and half of these indices are alternated. We may assume that the last two indices are alternated in each R_{F_i}.

Moreover, $\gamma \to -\gamma$ under reversal of the orientation on $TN_{12}^{\perp}$. Therefore the basic invariant map det must be applied to some indices in mon(R, C, D). However, C, D do not contain indices corresponding to $TN_{12}^{\perp}$. Thus det must be applied; to the $n = 2p$ indices consisting of the first two indices in each R_{F_i}, and we may write

$$\operatorname{mon}(R, C, D) = \chi^{\perp}(\Omega)(\operatorname{mon}(C, D)),$$

where mon(C, D) is an elementary invariant monomial map from the tensors C, D to functions on N. However, as observed in §3, the normal form of C, D is independent of the point of reference $a \in N$. Thus mon$(C, D) = b$. a constant, and mon$(R, C, D) = b\chi^{\perp}(\Omega)$.

6. A generalization of the local Lefschetz formula

Let M be a compact oriented Riemannian manifold of dimension d, and N_1, $N_2 \subset M$ be totally geodesic oriented submanifolds of dimension n_1, n_2 respectively, and let $n_1 + n_2 = d$. Suppose that the intersection Ω of N_1 and N_2 is the disjoint union of totally geodesic oriented submanifolds N of dimension n. Moreover, assume that the chosen orientations are compatible with the vector bundle isomorphisms:

$$TN_1 \cong TN \oplus TN_1^{\perp},$$
$$TN_2 \cong TN \oplus TN_2^{\perp},$$
$$TM \cong TN_1^{\perp} \oplus TN \oplus TN_2^{\perp} \oplus TN_{12}^{\perp}.$$

Denoting by $K^{n_1}(t, x, y)$ the fundamental solution of the heat equation for n_1-forms on M, one defines

$$K_t^{n_1}(x)[*N_2] = \int_{N_2} *yK^{n_1}(t, x, y).$$

The right-hand side is an n_1-form whose asymptotic behavior as $t\downarrow 0$ is local in nature. In particular,

$$K_t^{n_1}[*N_2] = O(e^{-c(x)/t}),\ x \notin N_1 \cap N_2,$$

where $c(x) > 0$ is a constant depending on x.

Now suppose that a lies in some component N of $N_1 \cap N_2$. Denote U_{N_1} a tubular neighborhood of N in N_1; U_{N_1} may be identified with a neighborhood of N in its normal bundle, via the exponential map along N, considered as a submanifold of N_1. It is interesting to integrate $K_t^{n_1}(x)[*N_2]$ over the fiber F_a of U_{N_1}. In fact, one has

Theorem 6.1. *Suppose that M is a compact oriented Riemannian manifold of dimension d, and that N_1, N_2 are two totally geodesic oriented submanifolds of dimensions n_1, n_2 respectively, and $n_1 + n_2 = d$. Let N, of dimension n, be any oriented component of the intersection $N_1 \cap N_2$. If $a \in N$, then there exists an asymptotic expansion*:

$$\int_{F_a} K_t^{n_1}(x)[*N_2] \sim (4\pi t)^{-n/2} \sum_{l=0}^{\infty} t^l \Gamma_l(a),$$

where the n-forms $\Gamma_l(a)$ are local invariants of the Riemannian metric of M near a.

Denote $i\colon N \to M$ the inclusion and $\gamma_l(a) = i^*\Gamma_l(a)$, the pull-backs of the $\Gamma_l(a)$. Then the $\gamma_l(a)$ are $O(n_1 - n) \times O(n) \times O(n_2 - n) \times SO(n)$ invariant polynomial maps from the collection $\mathcal{C}$ of tensors (*described in Theorem* 3.3) *to n-forms on N.*

Proof. Similar to the proofs of Theorems 2.1, 3.3, and 4.1.

One may also generalize Lemma 4.2 to obtain

Lemma 6.2.

$$\sum_{N \in \Omega} \int_N \gamma_{l+n/2}(a) = 0, \qquad l \neq 0,$$

$$\sum_{N \in \Omega} \int_N \gamma_{l+n/2}(a) = [N_1] \cap [N_2], \quad l = 0,$$

where γ_l are the invariants of Theorem 6.1, *and $[N_1] \cap [N_2]$ is the intersection number of N_1 and N_2.*

Proof. Similar to the proof of Lemma 4.2.

Now it is well-known [6, p. 196] that

$$[N_1] \cap [N_2] = \sum_{N \in \Omega} \int_N \chi^{\perp}(\Omega),$$

where $\chi^{\perp}(\Omega)$ is the Euler form of $TN_{12}^{\perp}$. This leads one to the following local version of Lemma 6.2.

Theorem 6.3. *Let N_1, $N_2 \subset M$ be totally geodesic oriented compact submanifolds of the compact oriented manifold M. Suppose N, of dimension n, is some oriented component of the intersection $N_1 \cap N_2$. Then for the expansion*

$$i^* \int_{F_a} K_t^{n_1}(x)[*N_2] \sim (4\pi t)^{-n/2} \sum_{l=0}^{\infty} t^l \gamma_l(a),$$

one has

$$\gamma_l(a) = 0, \qquad l < n/2,$$
$$\gamma_l(a) = \chi^{\perp}(\Omega), \quad l = n/2,$$

where $\chi^{\perp}(\Omega)$ *is the Euler form of the normal bundle* $TN_{12}^{\perp}$.

Proof. If we scale the metric on M: $g \to c^2 g$, then ${}^*yK^{n_1}(t, x, y) \to {}^*yK^{n_1}(c^{-2}t, x, y)$. Therefore $\gamma_l(a) \to c^{n-2l}\gamma_l(a)$.

From Lemma 5.3 it follows that $\gamma_l = 0$ for $l < n/2$. Now observe that if we reverse the orientation on $TN_{12}^{\perp}$, then

$$i^* \int_{F_a} K_t^{n_1}(x)[{}^*N_2] \to -i^* \int_{F_a} K_t^{n_1}(x)[{}^*N_2].$$

Therefore $\gamma_l \to -\gamma_l$ if the orientation on $TN_{12}^{\perp}$ is reversed. By Lemma 5.4 we conclude that if n is even and $l = n/2$, then γ_l is a multiple of $\chi^{\perp}(\Omega)$. However, $\int_N \gamma_l = \int_N \chi^{\perp}(\Omega)$. So $\gamma_l = \chi^{\perp}(\Omega)$ if $l = n/2$, n even.

Let f: $X \to X$ be an isometry of the compact oriented Riemannian manifold X of dimension m. The fixed point set Ω of f is the disjoint union of compact totally geodesic submanifolds N of dimension n. Suppose that the components N of Ω are oriented. Consider the product manifold $M = X \times X$. The submanifolds $N_1 = G_f$, the graph of f, and $N_2 = \mathfrak{D}$, the diagonal in $X \times X$, are totally geodesic. The components N of the intersection $N_1 \cap N_2$ may be identified with the components of the fixed point set Ω of f.

Now let $e^p(t, x, y)$ be the heat kernel for p-forms on X. Then, as is well-known [8],

$$L(f) = \int_X \sum_p (-1)^p \operatorname{Tr}(f^* e^p(t, x, f(x)))^*1,$$

where $*$ is the Hodge star operator. $L(f)$ is the Lefschetz number of f, the alternating sum of the traces of the maps induced by f on each of the cohomology groups $H^p(M, R)$.

If U_N is a tubular neighborhood of N, we may identify U_N with a neighborhood of the zero section of the normal bundle to N in X. This identification is obtained via the exponential map along N in X. For $a \in N$, denote F_a the fiber over a of $U_N \to N$. It is apparent that

$$\int_X \sum (-1)^p \operatorname{Tr}(f^* e^p(t, x, f(x)))^*1$$
$$= \sum_{N \in \Omega} \int_N \left[\int_{F_2} \sum (-1)^p \operatorname{Tr}(f^* e^p(t, x, f(x)))^*1 \right] + O(e^{-c/t}),$$

where $c > 0$ is a constant. The integral inside brackets, on the right, denotes integration over the fiber.

As a corollary of Theorem 6.3 we may rederive the local form of the Lefschetz theorem which was given in [9]:

Corollary 6.4. *Let* f: $X \to X$ *be an isometry of the compact oriented*

Riemannian manifold X. If N is an oriented component of the fixed point set, then one has the local Lefschetz formula:

$$\int_{F_a} \sum (-1)^p \operatorname{Tr}(f^* e^p(t, x, f(x)))^* 1 = \chi(\Omega) + O(t)$$

for $a \in N$, where the integration on the left is integration over the fiber, and $\chi(\Omega)$ is the Euler form of N.

Proof. The heat kernel on m-forms for the product manifold $X \times X$ is given by

$$K^m(t, (x_1, x_2), (y_1, y_2)) = \sum_{p+q=m} e^p(t, x_1, y_1) \wedge e^q(t, x_2, y_2).$$

Moreover, as in the proof of Corollary 4.4 we may write

$$\begin{aligned} &{}^* y K^m(t, (x, f(x)), (y, y)) \\ &= \sum_{p+q=n} (-1)^q \sum_{i,j} e^{-t\lambda_{i,p} - t\lambda_{j,q}} \phi_{i,p}(x) \wedge {}^*\phi_{j,p}(f(x)) \otimes \phi_{j,p}(y) \wedge {}^*\phi_{i,p}(y), \end{aligned}$$

so that

$$K_t^m(x, f(x))[{}^*N_2] = \sum_{p+q=n} (-1)^q \sum_i e^{-2t\lambda_{i,p}} \phi_{i,p}(x) \wedge {}^*\phi_{i,p}(f(x)).$$

Thus

$$\begin{aligned} \int_{F_a} \sum (-1)^p \operatorname{Tr}(f^* e_p(t, x, f(s)))^* 1 &= \int_{F_a} K_t^m(x, f(x))[{}^*N_2] \\ &= \chi^{\perp}(\Omega) + O(t) \end{aligned}$$

by Theorem 6.3, where $\chi^{\perp}(\Omega)$ is the Euler form of the normal bundle TN_{12}, consisting of all vectors in $TM|N$ which are normal to both $TN_1|N$ and $TN_2|N$. However, $TN_{12}^{\perp}$ is isomorphic to TN via the isomorphism $(v, v) \to (v, -v)$. Consequently $\chi^{\perp}(\Omega) = \chi(\Omega)$, the Euler form of N.

Thus

$$\int_{F_a} \sum (-1)^p \operatorname{Tr}(f^* e_p(t, x, f(x)))^* 1 = \chi(\Omega)(a) + O(t).$$

Remark. If χ, N are not necessarily orientable, then one may still derive a local Lefschetz formula

$$\int_{F_a} \sum (-1)^p \operatorname{Tr}(f^* e_p(t, x, f(x))) \psi(x) dvol_{F_a}(x) = {}^*\chi(\Omega)(a) + O(t),$$

where $dvol_{F_a}(x)$ is the measure induced on F_a by the Riemannian metric of M, and $\psi(x)$ is defined by $dvol_M(x) = \psi(x)[\pi^* dvol_N(x)] dvol_{F_a}(x)$.

This follows from Corollary 6.4 by observing that the statement is local in nature, and that the manifolds N, X are always locally orientable.

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