THE POINCARÉ LEMMA FOR $d\omega = F(x, \omega)$

HOWARD JACOBOWITZ

Introduction

One can locally solve the equation $d\alpha = \beta$ only if the $(p + 1)$-form β satisfies $d\beta = 0$. Poincaré's lemma states that this condition is also sufficient. We wish to consider a nonlinear version of this and to relate it to the Frobenius theorem. This theorem, in its classical formulation concerns a system of partial differential equations $\partial f_i/\partial x_j = F_{ij}(x, f)$ and asserts that one can solve these equations and have f take on a given value at any point in a region if and only if $\partial F_{ij}/\partial x_k = \partial F_{ik}/\partial x_j$ when the derivatives $\partial f_i/\partial x_j$ which occur in the use of the chain rule are replaced by $F_{ij}(x, f)$. In this paper we consider the equation $d\omega = F(x, \omega)$ where ω is a p-form, and $F(x, \omega)$ a $(p + 1)$-form. We discuss the analogue of the condition of Frobenius and show it is a sufficient condition for local solvability (§ 1). Both the Poincaré lemma and the Frobenius theorem are included in our formulation. In § 2 we consider various geometric applications. Finally in the last section we return to the analogy with the Frobenius theorem and show that our sufficient condition is also necessary for the existence of solutions to a certain special initial value problem.

It might be interesting to try to obtain our results, in the real analytic case, using the Cartan-Kaehler theorem. Note the proposition in § 1 does not quite state that $\{d\omega - F(\omega), dF(\omega)\}$ generates under the wedge product a differential ideal, when the coefficients of ω are admitted as new independent variables. However, it may be that various generalizations of the Cartan-Kaehler theorem, for instance the work of Goldschmit [3], do include as a very special case the present results for real analytic data. See also our comments in § 2 on a paper by Gasqui.

All our discussion will be local. Let M^N be an open subset of $\boldsymbol{R}^N$. Let Λ_x^p be the space of p-forms at the point $x \in M^N$, $\Lambda_x^p T(M^N)$ the dual space of p-vectors, and Γ_x^p the space of germs of p-forms at x. Recall that forms α and β define the same element in Γ_x^p if there is some open neighborhood U of x with $\alpha = \beta$ on U. For sanity's sake, we identify α with the element in Γ_x^p which it defines. Often we delete the base point x. We will sometimes consider submanifolds $M \subset M^N$ and the associated spaces $\Lambda_x^p(M)$ and $\Gamma_x^p(M)$. We introduce the germs in order to avoid specifying on what neighborhood of a given point each of

our results is valid. It will be important to distinguish between the statements
"$\alpha|_M = \beta|_M$" and "$\alpha = \beta$ on M." In the former case we require that the forms
α and β agree when acting on $\Lambda^p T(M)$, while in the latter $\alpha(v) = \beta(v)$ for
$v \in \Lambda^p T(M^N)_x$ when $x \in M$. As an example $\alpha|_x = 0$ for any form but in general
$\alpha \neq 0$ at (or on)x. For each x in some domain $F(x, \beta)$ is a map of $\Lambda^p_x \to \Lambda^{p+1}_x$.
In particular, F does not depend on the derivatives of the coefficients of the
form β. We will usually surpress the x-dependence and write $F(\beta)$.

For $X \in \Lambda^q T(M^N)$ and $\omega \in \Lambda^p$, $p \geq q$, we denote interior multiplication by
$X \lrcorner \omega$. This is defined by the equation $X \lrcorner \omega(Y_1, \cdots, Y_{p-q}) = \omega(X_1, \cdots, X_q,$
$Y_1, \cdots, Y_{p-q})$ when $X = X_1 \wedge \cdots \wedge X_q$ and then defined by linearity for any
$X \in \Lambda^q T(M^N)$. Now for $X \in T(M^N)$ let $\mathscr{L}_X$ denote the Lie derivative. Recall
$\mathscr{L}_X \omega = X \lrcorner (d\omega) + d(X \lrcorner \omega)$.

Finally we use the following multi-index and summation conventions. If I is
the multi-index $I = (i_1, \cdots, i_p)$, then $|I| = p$ and $\{I\} = \{i_1, \cdots, i_p\}$. Summa-
tions will be indicated by Σ, and the sum goes over any index which occurs
more than once. We denote by $[L : K]$ the parity of the permutation which takes
the sequence L to the sequence K if $\{L\} = \{K\}$ and zero otherwise. By $(\alpha)_K$ we
mean the coefficient in the form α of $dx_{k_1} \wedge \cdots \wedge dx_{k_p}$. This is well defined
under the usual convention that $(\alpha)_K = [L : K](\alpha)_L$ when $\{L\} = \{K\}$.

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1. We say that $d\omega = F(\omega)$ is solvable in a region D if for each $x \in D$ and
each $\omega_0 \in \Lambda^p_x$ there is some $\omega \in \Gamma^p_x$ with $d\omega = F(\omega)$ and $\omega = \omega_0$ at x. If $d\omega = F(\omega)$,
as germs, then $dF(\omega) = 0$. Consider the following restriction on F for some x:

(*) $dF(\beta) = 0$ at x for any $\beta \in \Gamma^p_x$ satisfying $d\beta = F(\beta)$ at x .

Theorem. *The equation $d\omega = F(\omega)$ is solvable in a domain D if F satisfies*
(*) *for each $x \in D$.*

In § 3 we show that the converse of this theorem is false but that there is a
restricted definition of solvability which is equivalent to (*).

We now prove that (*) implies solvability by showing that one can even solve
the following broader class of initial value problems.

Theorem 1. *Let F satisfy* (*) *at each point in a neighborhood of some point*
$x_0 \in M^N$. *Let M' be some submanifold of M^N containing x_0. If $\omega_0 \in \Gamma^p_{x_0}$ satisfies*
$d\omega_0|_{M'} = F(\omega_0)|_{M'}$, *then there exists some $\omega \in \Gamma^p_x$ satisfying $d\omega = F(\omega)$ and $\omega =$
ω_0 on M'.*

Remarks. It F does not depend on ω so that $F(\omega) \equiv \beta$ for some $(p + 1)$-
form β, then we have Poincaré's lemma with some added initial condition. If
dim $M' = 0$ then the condition for $d\omega_0$ is trivially satisfied for all $\omega_0 \in \Lambda^p_{x_0}$, and
the conclusion is that one may always solve $d\omega = F(\omega)$, $\omega = \omega_0$ at x_0. So, if we
also take $p = 0$, then we recover the Frobenius theorem. If dim M' is arbitrary
and $p = M - 1$, then any F satisfies (*) and $d\omega = F(\omega)$, with initial data as in

the theorem, may always be solved.

Our proof of this theorem generalizes to forms the usual proof of the Frobenius theorem. That is, we first find a solution to those differential equations corresponding to a given direction and then use the compatibility condition to verify that this solution automatically also satisfies the remaining equations. Let us start by discussing the compatibility condition (*). It is natural to expect that if F satisfies (*), then $dF(\beta)$ can be written for any $\beta \in \Gamma^p$ as a sum of terms each containing as a factor some component of $d\beta - F(\beta)$. We are able to give an explicit factorization. Write $\beta = \sum_{|I|=p} \beta_I dx_I$ and $F = \sum_{|J|=p+1} F_J(\beta)dx_J$. Each F_J is a function of the coefficients of the form β and also of the point $x \in M^N$. We assume $F_I = [I:J]F_J$ if $[I:J] \neq 0$. We make the dependence on the variables explicit by setting $F_J(x_1, \cdots, x_N, y_{K_1}, \cdots, y_{K_T}) = F_J(\sum y_K dx_K)$. From the fact that each F_J is a function of the differential form $\sum y_K dx_K$ rather than just of the coefficients $\{y_K\}$, it follows that $\partial F_J/\partial y_K = [K:L](\partial F_J/\partial y_L)$ when $\{K\} = \{L\}$. It then also follows that $\partial F_{iJ}/\partial y_J = \partial F_{iK}/\partial y_K$ when $\{J\} = \{K\}$.

Let $X_i = \partial/\partial x_i$ and for $K = \{k_1, \cdots, k_p\}$ let $X_K = X_{k_1} \wedge \cdots \wedge X_{k_p}$.

Proposition. *Let F satisfy (*). Then for all $\beta \in \Gamma^p$, $dF(\beta) = \sum (\partial F_{iK}/\partial y_K)(X_k \wedge X_K) \lrcorner (d\beta - F(\beta))dx_K dx_k dx_i$. The sum is taken over all $i, k, 1 \leq i, k \leq N$ and all multi-indices K with $|K| = p$.*

The first step in the proof is to study certain derivatives of F_J.

Lemma. *If F satisfies (*) then*

(1) *$\partial F_J/\partial y_K = 0$ if $\{K\}$ is not a subset of $\{J\}$, and*

(2) *$\partial F_{iaQ}/\partial y_{aQ} = \partial F_{ibQ}/\partial y_{bQ}$ for all i, a, b, Q ($|Q| = p - 1$).*

Proof. We have $dF(\beta) = G(\beta) + \sum (\partial F_I(\beta)/\partial y_K)(\partial \beta_K/\partial x_k)dx_k dx_I$ where $G(\beta)$ depends on x and the form β evaluated at x but not on derivatives of β. Let β be any element in Λ_x^p. Extend β to an element of Γ_x^p so that $d\beta = F(\beta)$ at x. So $dF(\beta) = 0$ at x. Next let $\alpha \in \Gamma_x^{p-1}$ be any form with $d\alpha = 0$ at x. Thus also $dF(\beta + d\alpha) = 0$ at x. This implies $\sum (\partial F_I(\beta)/\partial y_K)(\partial(d\alpha)_K/\partial x_k)dx_k dx_I = 0$ at x. For notational convenience, take $x = 0$. The first half of the lemma follows from considering $\alpha = \frac{1}{2}x_a^2 dx_Q$, and the second from $\alpha = x_a x_b dx_Q$.

To prove the proposition we first note that

$$X_k \wedge X_K \lrcorner d\beta = \sum [K, k: R, r]\frac{\partial \beta_R}{\partial x_r}.$$

This together with (2) of the lemma and the fact that $\partial F_{iK}/\partial y_K = \partial F_{iJ}/\partial y_J$ when $\{K\} = \{J\}$ imply

$$\sum \frac{\partial F_{i,K}}{\partial y_K}((X_k \wedge X_K) \lrcorner d\beta)dx_K dx_k$$

$$= c \sum \frac{\partial F_{i,R}}{\partial y_R} \frac{\partial \beta_R}{\partial x_r} dx_R dx_r, \qquad (c = (p+1)!).$$

Thus

$$\sum \frac{\partial F_{i,K}}{\partial y_K}((X_k \wedge X_K) \rfloor d\beta)dx_K dx_k dx_i$$

$$= c \sum \frac{\partial F_{i,R}}{\partial y_R} \frac{\partial \beta_R}{\partial x_r} dx_R dx_r dx_i$$

$$= \sum \frac{\partial F_i}{\partial y_R} \frac{\partial \beta_R}{\partial x_r} dx_r dx_I .$$

The last equality is justified by the fact that the extra terms coming from indices R and I with $\{R\} \not\subset \{I\}$ are all zero. Thus

$$dF(\beta) = G(\beta) + \sum \frac{\partial F_{i,K}}{\partial y_K}((X_k \wedge X_K) \rfloor d\beta)dx_K dx_k dx_i .$$

Finally $df(\beta)$ must be zero if $F(\beta)$ is substituted for $d\beta$. This determines $G(\beta)$ and concludes the proof of the proposition.

We now start the proof of the theorem. Let dim $M' = m$. Choose coordinates with x_0 at the origin so that $M'\{x \mid x_{m+1} = \cdots = x_N = 0\}$. Let M^n, $n = m + 1, \cdots, N$, be the set $\{x \mid x_{n+1} = \cdots = x_N = 0\}$. We shall try to inductively solve $d\omega|_{M^n} = F(\omega)|_{M^n}$ together with an appropriate initial condition. Assume $G: \Lambda^p(M^{n+1}) \to \Lambda^{p+1}(M^{n+1})$ is any smooth bundle map.

Lemma 1. *For any $\mu \in \Gamma_x^p(M^{n+1})$ there exists some $\sigma \in \Gamma_x^p(M^{n+1})$ with $X_{n+1} \rfloor d\sigma = X_{n+1} \rfloor G(\sigma)$ and $\sigma = \mu$ on M^n.*

Proof. We wish to convert this equation to a form $\partial f/\partial x_{n+1} = G(f)$ and use the fundamental existence theorem of ordinary differential equations. Let $\sigma = \sum_I \sigma_I dx_I$ with sum taken over $|I| = p$, $I \subset \{1, \cdots, n+1\}$. Similarly

$$G(\sigma) = \sum_J G_J(\sigma)dx_J , \quad |J| = p + 1 , \quad J \subset \{1, \cdots, n+1\} .$$

For $n + 1 \in \{K\}$ set σ_K equal to an arbitrary extension of μ_K off of M^n. The equation

$$\sum \frac{\partial \sigma_I}{\partial x_{n+1}}dx_I = X_{n+1} \rfloor \sum G_J(\sigma)dx_J - \sum \frac{\partial \sigma_K}{\partial x_j}(X_{n+1} \rfloor dx_j dx_K) ,$$

where $\{I\} \subset \{1, \cdots, n\}$, $n + 1 \in \{K\}$, $j \neq n + 1$ is a system of ordinary differential equations for the unknown σ_I, $\{I\} \subset \{1, \cdots, n\}$. Solve this system with the initial values $\sigma_I = \mu_I$ on M^n. It is easy to see that $\sigma = \sum \sigma_L dx_L$, $\{L\} \subset \{1, \cdots, n+1\}$, satisfies the conclusions of the lemma.

Remark. There are thus many solutions to $X_{n+1} \rfloor d\sigma = X_{n+1} \rfloor G(\sigma)$, $\sigma = \mu$ on M^n for $p > 0$. We obtain a unique solution if we add the requirement that $X_{n+1} \rfloor \sigma = X_{n+1} \rfloor \mu$ in M^{n+1}.

We shall use this lemma and the compatibility condition (*) to solve $d\alpha = F(\alpha)$. It clearly suffices to just prove the following result.

Lemma 2. *Let* $\beta \in \Gamma^p(M^n)$ *with* $d\beta|_{M^n} = F(\beta)|_{M^n}$. *Assume* F *satisfies* (*). *Then there exists* $\alpha \in \Gamma^p(M^N)$ *with* $d\alpha|_{M^{n+1}} = F(\alpha)|_{M^{n+1}}$ *and* $\alpha = \beta$ *on* M^n.

Proof. We first try to find a p-form α defined on M^{n+1} (and taking values in $\Lambda^p(M^N)$) such that $X_{n+1} \lrcorner d\alpha|_{M^{n+1}} = X_{n+1} \lrcorner F(\alpha)|_{M^{n+1}}$. For $\gamma \in \Gamma^p(M^{n+1})$ and the given β set $G(\gamma) = (-d\beta + F(\gamma + \beta))|_{M^{n+1}}$. By the previous lemma we may find some γ satisfying

$$X_{n+1} \lrcorner d\gamma = X_{n+1} \lrcorner G(\gamma) \, ,$$
$$\gamma = 0 \qquad \text{on } M^n \, .$$

Let $\alpha \in \Gamma_x^p(M^N)$ be any form with $\alpha = \gamma + \beta$ on M^{n+1}. We have

$$X_{n+1} \lrcorner d\alpha|_{M^{n+1}} = X_{n+1} \lrcorner F(\alpha)|_{M^{n+1}} \, ,$$
$$\alpha = \beta \qquad \text{on } M^n \, .$$

To show that actually $d\alpha|_{M^{n+1}} = F(\alpha)|_{M^{n+1}}$ we proceed as follows, using the relation of $F(\alpha)$ to $d\alpha - F(\alpha)$. (All forms to be understood are restricted to M^{n+1}.)

$$\mathscr{L}_{X_{n+1}}(d\alpha - F(\alpha))$$
$$= X_{n+1} \lrcorner d(d\alpha - F(\alpha)) + d(X_{n+1} \lrcorner (d\alpha - F(\alpha)))$$
$$= -X_{n+1} \lrcorner dF(\alpha)$$
$$= -X_{n+1} \lrcorner \left(\sum \frac{\partial F_{iK}}{\partial y_K} (X_k \wedge X_K \lrcorner (d\alpha - F(\alpha)) dx_K dx_k dx_i \right)$$
$$= -\sum \frac{\partial F_{iK}}{\partial y_K} (X_k \wedge X_K \lrcorner (d\alpha - F(\alpha)))(X_{n+1} \lrcorner dx_K dx_k dx_i)$$

with the sum taken over $\{K\} \cup \{k, i\} \subset \{1, \cdots, n+1\}$. Thus, if we decompose $(d\alpha - F(\alpha))|_{M^{n+1}} = \sum \Omega_J dx_J$ and use that $\mathscr{L}_{X_{n+1}}(\sum \Omega_J dx_J) = \sum (\partial \Omega_J / \partial x_{n+1}) dx_J$ we see that $\partial \Omega_J / \partial x_{n+1} = \sum A_{JL} \Omega_L$. But on M^n we have

$$(d\alpha - F(\alpha))|_{M^n} = d\alpha|_{M^n} - F(\alpha)|_{M^n} = d\beta|_{M^n} - F(\beta)|_{M^n} = 0 \, .$$

Now by the uniqueness theorem of ordinary differential equations, $\Omega_J \equiv 0$ for each J. Thus $d\alpha|_{M^{n+1}} = F(\alpha)|_{M^{n+1}}$.

Remark. We needed for this proof not only that $dF(\alpha)$ is a linear combination of the coefficients of $d\alpha - F(\alpha)$ but also that $dF(\alpha)|_M$ depends only on $(d\alpha - F(\alpha))|_M$ for any submanifold.

For the applications it is important to note that all of the above also holds for systems. For convenience let $\omega_1, \cdots, \omega_R$ all be forms of the same degree p, and let $F_1, \cdots, F_R$ be maps into Λ^{p+1}. Then the equations $d\omega_i = F_i(x, \omega_1, \cdots, \omega_R)$ $= 0$ may be locally solved if each F_i satisfies $dF_i(x, \beta_1, \cdots, \beta_R) = 0$ when all the equations $d\beta_i = F_i(x, \beta_1, \cdots, \beta_R)$ are satisfied at the point x.

2. We want to use Theorem 1 to give new proofs of several known results. But first we discuss the case $p = N - 1$. For $F: \Lambda^{N-1}(M^N) \to \Lambda^N(M^N)$, Theorem 1 tells us that $d\omega = F(\omega)$ is always solvable. However, solvability in this case follows immediately from Lemma 1, and so need not be thought of as a consequence of Theorem 1.

Let K be a smooth function on a two-dimensional manifold. It is now reasonably well understood when K is the curvature of a Riemannian metric [4], [5]. One simple proof that in a neighborhood of each point of the manifold there is some metric having K as its curvature, can be found in [4, p. 217]. Here we give another proof.

Theorem. *Let K be a smooth function defined in a neighborhood of some point $p \in \mathbf{R}^2$. There exists some Riemannian metric defined near p which has K as its curvature.*

Proof. We seek 1-forms ω_1, ω_2, and ω_{12} satisfying

$$d\omega_1 = -\omega_{12}\omega_1 , \quad d\omega_2 = \omega_{12}\omega_2 , \quad d\omega_{12} = K\omega_1\omega_2 .$$

We know this system of equations may be solved locally on $\mathbf{R}^2$. Take initial data so that ω_1 and ω_2 are linearly independent at p. Let e_1 and e_2 be the dual tangent vectors of ω_1 and ω_2. Define a metric by taking these vectors orthonormal and a connection by setting $\nabla_X e_1 = -\omega_{12}(X)e_2$ and $\nabla_X e_2 = +\omega_{12}(X)e_1$. This connection is the Levi-Civita connection of the metric, and its curvature is K.

We next study the curvature tensor of a connection. We shall show that on a two-dimensional manifold any 2-form with values in Hom (TM, TM) is locally the curvature tensor of some connection. This raises several related questions. Can one characterize those Hom (TM, TM)-valued 2-forms on manifolds of higher dimension which are curvature tensors? The first Bianchi identity is necessary but as we shall show not sufficient. Are there any obstructions to a Hom (TM, TM)-valued 2-form on a surface being the curvature tensor of a globally defined connection?

For any connection on a manifold M set $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. R is a 2-form with values in Hom (TM, TM), the group of linear bundle maps of TM to itself.

Theorem. *Let $R(X, Y)$ be any smooth Hom $(T\mathbf{R}^2, T\mathbf{R}^2)$-valued 2-form defined in a neighborhood of some point $p \in \mathbf{R}^2$. There exists some torsion free connection defined near p, which has $\mathbf{R}$ as its curvature tensor.*

Remark. It is not true that $\mathbf{R}$ need come from a Levi-Civita connection. For example, if R is $Idxdy$, I the identity map, then $\mathbf{R}$ cannot arise from a metric. More generally, if $R = Adxdy$ where A has a real nonzero eigenvalue, then $\mathbf{R}$ cannot come from a metric.

Proof. If ω_1 and ω_2 are linearly independent one-forms on $\mathbf{R}^2$, then the dual tangent vectors e_1 and e_2 are well defined. Let $\varphi_{ij}(X, Y) = \omega_i(R(X, Y)e_j)$, and consider the system

$$d\omega_i = -\sum_j \omega_{ij}\omega_j , \qquad d\omega_{ij} = -\sum_k \omega_{ik}\omega_{kj} + \phi_{ij}$$

for $1 \leq i, j, k \leq 2$. At p choose initial values so that ω_1 and ω_2 are linearly independent. Again the equations may be solved. In terms of the now known vectors e_1 and e_2 we may write $R(X, Y)e_j = \sum_i R_{ij}(X, Y)e_i$ and this defines the 2-forms R_{ij}. Finally, note $\varphi_{ij} \equiv R_{ij}$. Thus $\{\omega_{ij}\}$ are the connection forms of a connection whose curvature tensor is R.

It would be very interesting to know the analogue of the theorem for manifolds of dimension greater than two. A Hom (TM, TM)-valued 2-form R which comes from some torsion free connection must satisfy the two Bianchi identities, namely,

$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0 ,$$
$$d\varphi_{ij} = \sum \omega_{ik}\omega_{kj} - \sum \omega_{ik}\omega_{kj}$$

with φ_{ij} defined as above. Both identities are trivial in two dimensions. For higher dimensions, the first gives a necessary condition that R comes from some torsion-free connection. The second identity might be thought to provide no restriction since it involves the unknown connection. However, the requirement that for given 2-forms $\{\varphi_{ij}\}$ there is some set of forms $\{\omega_{ij}\}$ such that this second identity is satisfied, is indeed a nontrivial restriction on φ_{ij}. This can already be seen by considering forms φ_{ij} which all vanish at some point but at least one of which has a nonzero differential at that point.

Gasqui has used Goldschmidt's generalization [3] of the Cartan Kaehler theorem to prove that any real analytic bilinear form is locally the Ricci curvature of some torsion-free connection [2]. For dimension two this implies the above theorem when R is real analytic. Can Gasqui's result be derived using a generalization of Theorem 1? That is, using ordinary differential equations? Such a derivation would eliminate the need for analyticity and might illuminate the relation of Theorem 1 to the Cartan-Kaehler theorem.

We next present two simple results about vector bundles. Let B be a vector bundle over a manifold M in a small neighborhood of some point $p \in M$. Let M' be a submanifold of M which contains p. Choose sections $\xi_1, \cdots, \xi_N$ which span B. Assume $\nabla_X\xi_i = \sum \omega_{ji}(X)\xi_j$ is a connection in the bundle, and $\langle \xi_i, \xi_j \rangle = g_{ij}$ is a symmetric metric on $B|_{M'}$ with $\nabla_X g \equiv 0$. This means $X(g_{ij}) = \sum (g_{ij}\omega_{kj} + g_{kj}\omega_{ki})(X)$, all $X \in TM'$. Or, in terms of forms, $dg_{ij}|_{M'} = \sum (g_{ik}\omega_{kj} + g_{kj}\omega_{ki})|_{M'}$.

Theorem. *If the curvature of ∇ is zero, then there exists some metric h defined on $B|_U$ for some open neighborhood U of p with $\nabla h \equiv 0$ and $h = g$ on M'.*

Remark. We may take M' to be the point p itself, and g to be an arbitrary inner product on $B|_p$. In particular, if $B = TM$ then it follows that any torsion-free flat connection is locally the Levi-Civita connection of some Riemannian metric.

Proof. We take as unknowns the functions h_{ij} for $1 \leq i \leq j \leq N$, and we define $h_{ij} = h_{ji}$ for $i > j$. We want to solve the initial value problem

$$dh_{ij} = \sum (h_{ik}\omega_{kj} + h_{kj}\omega_{ki}) ,$$
$$h_{ij} = g_{ij} \qquad \text{on } M'$$

for $1 \leq i \leq j \leq N$. When restricted to M', g_{ij} satisfies the equation, so we need only verify (*).

Since

$$\nabla_X \nabla_Y \xi_i - \nabla_Y \nabla_X \xi_i - \nabla_{[X,Y]}\xi_i = \sum_j (d\omega_{ji} + \sum_k \omega_{jk}\omega_{ki})(X, Y)\xi_j ,$$

the fact that the connection is flat is equivalent to $d\omega_{ji} = -\sum \omega_{jk}\omega_{ki}$. Now assume some set of functions $\{h_{ij}\}$ satisfies $dh_{ij} = \sum (h_{ik}\omega_{kj} + h_{kj}\omega_{ki})$ at some point. Then at this point

$$
\begin{aligned}
d \sum (h_{ik}\omega_{kj} + h_{kj}\omega_{ki}) &= \sum (h_{ir}\omega_{rk} + h_{rk}\omega_{ri})\omega_{kj} + \sum h_{ik}(-\omega_{kr}\omega_{rj}) \\
&\quad + \sum (h_{kr}\omega_{rj} + h_{rj}\omega_{rk})\omega_{ki} + \sum h_{kj}(-\omega_{kr}\omega_{ri}) \\
&= \sum h_{rk}\omega_{ri}\omega_{kj} + \sum h_{kr}\omega_{rj}\omega_{ki} = 0 .
\end{aligned}
$$

Thus by Theorem 1, the above initial value problem may be solved. Finally note that $\nabla h = 0$.

We now consider a variant of the previous result. Assume that a connection $\nabla_X \xi_i = \omega_{ji}(X)\xi_j$ is given on $B|_{M'}$ and that this connection is flat. Is it possible to find a flat connection on $B|_U$, where U is open in M, which extends the given connection?

Theorem. *Let ω_{ij} be 1-forms on M' satisfying $d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{kj}$. There exist 1-forms Ω_{ij} on some open set U containing the distinguished point p which agree with ω_{ij} on M' and satisfy $d\Omega_{ij} = -\sum \Omega_{ik} \wedge \Omega_{kj}$ on U.*

Proof. If $d\beta_{ij} = -\sum \beta_{ik} \wedge \beta_{kj}$ at some point q, then

$$d(\sum \beta_{ik} \wedge \beta_{kj}) = \sum (-\beta_{is}\beta_{sk}\beta_{kj} + \beta_{ik}\beta_{ks}\beta_{sj}) = 0 \qquad \text{at } q .$$

Our last example, taken somewhat randomly from the literature, shows the simplicity effected by Theorem 1. Lie's third fundamental theorem states that the left invariant vector fields of a Lie group from a Lie algebra. A converse result, that any Lie algebra is the Lie algebra for some local Lie group, is often reduced to the following.

Theorem. *Given n^3 constants c^i_{jk} with $c^i_{jk} = -c^i_{kj}$, there exist n linearly independent one-forms ω_i satisfying*

$$d\omega_i = \tfrac{1}{2} \sum c^i_{jk}\omega_j\omega_k$$

if and only if these constants satisfy

$$\sum \left(c^i_{jk}c^j_{rs} + c^i_{jr}c^j_{sk} + c^i_{js}c^j_{kr} \right) = 0$$

for all $1 \leq i, k, r, s, \leq n$.

Proof. If the equation is satisfied at one point, then $\sum c^i_{jk}c^j_{rs}\omega_r\omega_s\omega_k = 0$ at that point. This is equivalent, assuming $\{\omega_i\}$ is linearly independent, to

$$\sum \left(c^i_{jk}c^j_{rs} + c^i_{jr}c^j_{sk} + c^i_{js}c^j_{kr} \right) = 0 \; .$$

Thus this condition is necessary for a solution to exist, and is also sufficient by Theorem 1. Further if at some point the set of initial values for $\{\omega_i\}$ is linearly independent, the same is true for the set of solutions in a neighborhood of this point.

This proof may be compared with that in [1, pp. 108–111].

3. We have seen that (*) is a sufficient condition for the solvability of $d\omega = F(\omega)$. That it is not necessary can be seen from a simple example. Let $f(\alpha_1, \alpha_2, \alpha_3)$ be a smooth function of three variables with nowhere vanishing gradient. Define $F \colon \Lambda^1(\mathbf{R}^3) \to \Lambda^2(\mathbf{R}^3)$ by $F(\alpha_1 dx + \alpha_2 dy + \alpha_3 dz) = -f(\alpha_1, \alpha_2, \alpha_3)dxdy$. Pick some point $p \in \mathbf{R}^3$ and some $\omega \in \Lambda^1_p$. We claim there exists $\alpha \in \Gamma^1_p$ satisfying

$$d\alpha = F(\alpha) \qquad \text{in } U \; ,$$
$$\alpha = \omega \qquad \text{at } p \; .$$

Thus $d\alpha = F(\alpha)$ is solvable. But we also claim that there exists $\beta \in \Gamma^1_p$ with

$$d\beta = F(\beta) \; , \quad \beta = \omega \; , \quad dF(\beta) \neq 0 \quad \text{at } p \; .$$

Thus F does not satisfy (*).

Let $\omega = (\omega_1, \omega_2, \omega_3)$ and $p = (p_1, p_2, p_3)$. To prove the first claim, define α_1 to be the solution of $d\alpha_1/dy = -f(\alpha_1(y), \omega_2, \omega_3)$, $\alpha_1(p_1) = \omega_1$, and let $\alpha = \alpha_1(y)dx + \omega_2 dy + \omega_3 dz$. To prove the second claim assume $\partial f/\partial z \neq 0$ at ω, and let $\beta = (\beta_1, \beta_2, \beta_3)$ where $\beta_1 = \beta_1(y)$, $\beta_2 = \omega_2$, $\beta_3 = \beta_3(z)$. We take the functions β_1 and β_3 to satisfy $\beta_1(\rho) = \omega_1$, $\beta'_1(\rho) = -f(\omega_1, \omega_2, \omega_3)$, $\beta_3(p) = \omega_3$, $\beta'_3(p) \neq 0$. Then $d\beta = F(\beta)$ at p, but $dF(\beta) \neq 0$. Thus (*) is certainly not necessary for solvability.

However the classical compatibility condition for the Poincaré lemma is both necessary and sufficient. So is the condition for the Frobenius theorem if one assumes that solutions exist for all appropriate initial values at a point. In the present case too we may look at solvability for an appropriate initial value problem, and demonstrate that (*) is both necessary and sufficient for the solvability of this problem.

Theorem 2. *The following three conditions are equivalent. At each point x in some domain D we have:*

(a) *F satisfies (*) at x,*

(b) *for each $\beta \in \Gamma^p_x(M^N)$ and each n with $(d\beta - F(\beta))|_{M^n} = 0$ there exists an $\omega \in \Gamma^p_x(M^N)$ satisfying*

$$dω = F(ω) \, ,$$

$$ω = β \qquad \text{on } M^n \, ,$$

$$X_m \lrcorner ω = X_m \lrcorner β \qquad \text{on } M^m, m = n + 1, \cdots, N \, ,$$

(c) *for each $β \in Γ_x^p(M^N)$ with $dβ = F(β)$ at x there exists an $ω \in Γ_x^p(M^N)$*
satisfying

$$dω = F(ω) \, ,$$

$$ω = β \qquad \text{at } x \, ,$$

$$\mathscr{L}_X ω = \mathscr{L}_X β \qquad \text{for all } X \in TM_x^N \, .$$

Remark. It is easy to see that the $ω$ in (b) must be unique while the solution
for (c) is usually not unique.

We first show (a) $\Rightarrow$ (b). It suffices to show that if $(dα - F(α))|_{M^r} = 0$ for
some $α \in Γ_x^p(M^N)$, then there is some $γ \in Γ_x^p(M^N)$ with

$$(dγ - F(γ))|_{M^{r+1}} = 0 \, ,$$

$$γ = α \qquad \text{on } M^r \, ,$$

$$X_m \lrcorner γ = X_m \lrcorner α \qquad \text{everywhere, for each } m \geq r + 1 \, .$$

For then starting with $β$ as in (b) we may recursively construct $γ_1, \cdots, γ_{N-n}$
for which

$$(dγ_i - F(γ_i))|_{M^{n+i}} = 0 \, ,$$

$$γ_i = β \qquad \text{on } M^n \, ,$$

$$X_m \lrcorner γ_i = X_m \lrcorner β \qquad \text{on } M^m \quad \text{for } m = n + 1, \cdots, n + i - 1 \, ,$$

$$X_m \lrcorner γ_i = X_m \lrcorner β \qquad \text{everywhere for } m \geq n + i \, ,$$

and then $ω = γ_{N-n}$ satisfies the conclusion of (b).

So let $α \in Γ_x^p(M^N)$ be given. As in Lemma 1 solve $X_{r+1} \lrcorner dσ = X_{r+1} \lrcorner G(σ)$
where $σ \in Γ_x^p(M^{r+1})$, $G(σ) = (F(σ + α) - dα)|_{M^{r+1}}$, $σ = 0$ on M^r. As indicated
in the remark following that lemma we may also require $X_{r+1} \lrcorner σ = 0$ on M^{r+1}.
Thus for $γ = σ + α$ we have $X_{r+1} \lrcorner (dγ - F(γ))|_{M^{r+1}} = 0$, $γ = α$ on M^r, $X_{r+1} \lrcorner γ$
$= X_{r+1} \lrcorner α$ on M^{r+1}. Finally extend $γ$ off of M^{r+1} so that $X_m \lrcorner γ = X_m \lrcorner α$
everywhere for each $m \geq r + 1$. Now, as in the proof of Lemma 2, use that
$X_{r+1} \lrcorner (dγ - F(γ))|_{M^{r+1}} = 0$, F satisfies (*), and $γ = α$ on M^r, where $(dα -$
$F(α))|_{M^r} = 0$, to show that $(dγ - F(γ))|_{M^{r+1}} = 0$. This concludes the proof.

Next we show (b) $\Rightarrow$ (c). Recall the manifolds M^m were defined previously
with respect to some coordinate system. M^0 denotes the origin of these coordi-
nates.

Lemma. *If $Ω$ is a p-form with $dΩ = 0$ at M^0 and $X_m \lrcorner Ω = 0$ on M^m, $m =$
$1, \cdots, N$, then $\mathscr{L}_X Ω = 0$ at M^0 for each vector field X.*

For the proof write $\Omega = \sum \Omega_I dx_I$ where each multi-index satisfies $i_1 < i_2 <$ $\cdots < i_p$. For any fixed i it is clear that $\Omega_I = 0$ on M^i if $i \in \{I\}$, and it follows, in particular, that $\partial \Omega_I / \partial x_j = 0$ at M^0 if $j \leq \max \{I\}$. This enables us to write, for any L with $|L| = p$, $X_L \,\lrcorner\, d\Omega = \sum (\partial \Omega_J / \partial x_j)[L:j, J]$ with $j = \max \{L\}$ and summation over all J with $\{j, J\} = \{L\}$. But $d\Omega = 0$ at M^0. Thus $\partial \Omega_I / \partial x_j$ is also zero if $j > \max \{I\}$. This proves the lemma since any vector field is a linear combination of $\partial / \partial x_i$, $i = 1, \cdots, N$, and for $X = \partial / \partial x_j$ one has $\mathscr{L}_X \Omega = \sum (\partial \Omega_I / \partial x_j) dx_I = 0$.

Applying this lemma to $\Omega = \omega - \beta$ and setting $n = 0$ we see that (b) implies (c).

Finally, we assume (c) and derive (a). Note that $dF(\omega) = 0$ everywhere. Since $dF(\omega)$ can only depend on coefficients ω_I and their first derivatives, and since at M^0 these coincide with the coefficients and derivatives of β we see that $dF(\beta) = 0$ at M^0 and so F satisfies (*).

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