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DERIVATIVES OF SECONDARY CHARACTERISTIC CLASSES

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Introduction

Secondary characteristic classes have been studied extensively in recent years, particularly with regard to foliations. One of the most interesting properties of these classes is their ability to vary continuously with a continuous deformation of the foliation. In this paper we construct the derivatives of these secondary classes for a given foliation.

Let F be a foliation of codimension q on a manifold M. Let Φ be the sheaf of germs of vector fields on M which preserve F. Then $H^1(M; \Phi)$ is the space of infinitesimal deformations of F. There are a graded differential complex WO_q and a natural map

$$\alpha_F^*: H^*(WO_q) \to H^*(M; R)$$

depending only on F, which gives characteristic classes for the foliation. We construct a natural map

$$D_F: H^1(M; \Phi) \times H^*(WO_q) \to H^*(M; R)$$

which depends only on F. This map gives the derivatives of the characteristic classes for the foliation in the sense that if $\beta \in H^1(M; \Phi)$ is the infinitesimal deformation associated to an actual deformation F_s, $s \in R$, $F_0 = F$, then for $f \in H^*(WO_q)$

$$D_F(\beta, f) = \frac{\partial}{\partial s} \alpha_{F_s}^*(f)\Big|_{s=0} .$$

In a crude sense, if one views $\alpha^*(f)$ as a map from the space of foliations on M to the cohomology of M, one may think of $D(.,f)$ as the induced map on the tangent space of the space of foliations. The point of this construction is that it allows one to compute derivatives of characteristic classes corresponding to deformations of a fixed foliation F using only information provided by the foliation, i.e. one does *not* need to know what the deformation is in order to

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construct its associated derivatives. All the information is contained in F and $H^1(M; \Phi)$.

This construction may also be applied to the secondary classes of Simons [7].

The paper is organized as follows. In section 1 we review known results and gather information required later. Section 2 contains the construction of the map D_F. In section 3 we show how to partially extend this construction to the case of complex foliations. In section 4 we compute the derivatives of two important examples, the horocyclic flow on a surface of constant negative curvature and the Hopf fibrations, and show that these derivatives are zero.

1. Review of needed results

For a more thorough treatment of the material presented in this section the reader should consult the references.

Throughout the paper we treat only smooth (C^∞) objects and observe the conventions that M is a manifold, F is a foliation of codimension q on M with tangent bundle τ, normal bundle ν and dual normal bundle ν^*. $C^\infty(M)$ is the space of smooth real valued functions on M. If ξ is a bundle over M, $C^\infty(\xi)$ denotes the space of smooth sections of ξ. The cohomology class determined by a closed form ω on M is denoted by $[\omega]$. Finally we observe the Einstein convention of summing over repeated indices in any expression.

We begin by briefly recalling the Chern-Weil construction of characteristic classes. See [17, Chapter XII].

Let gl_q be the Lie algebra of the real general linear group GL_q, and denote by $I^k(GL_q)$ the set of all symmetric multilinear maps

$$f\colon \underset{k}{\times}\, gl_q \to R$$

such that for $a \in GL_q$, $X_1, \cdots, X_k \in gl_q$

$$f(aX_1a^{-1}, \cdots, aX_ka^{-1}) = f(X_1, \cdots, X_k)\ .$$

Such a map is called an invariant polynomial of degree k. If we define

$$I(GL_q) = \sum_{k=0}^{\infty} I^k(GL_q)\ ,$$

then $I(GL_q)$ has the structure of a graded ring and is given by

$$I(GL_q) = R[c_1, \cdots, c_q]\ ,$$

where c_k is the kth Chern polynomial and degree $c_k = k$. We observe the Chern convention for invariant polynomials, that is, if $f \in I^k(GL_q)$ and f contains fewer than k arguments, the last one is repeated a number of times to make f a function of k arguments. Thus

$$f(X_1, X_2) = f(X_1, \underbrace{X_2, \cdots, X_2}_{k-1}) , \qquad X_1, X_2 \in gl_q .$$

Let $\Pi\colon P \to M$ be a principal GL_q bundle over M. If $a \in GL_q$, then a acts freely on P on the left by L_a. A gl_q-valued r-form Λ on P is called tensorial of type ad if

(i) Λ is horizontal, i.e., if $X_1, \cdots, X_r \in TP$ and $\Pi_*(X_i) = 0$ some i then $\Lambda(X_1, \cdots, X_r) = 0$,

(ii) $L_a^*\Lambda = a\Lambda a^{-1}$.

Observe that if $\Lambda_1, \cdots, \Lambda_k$ are tensorial of type ad and $f \in I^k(GL_q)$, then $f(\Lambda_1, \cdots, \Lambda_k)$ is a well defined form on P which projects to M. In particular the curvature Ω of a connection θ on P is a gl_q-valued 2-form which is tensorial of type ad and the $2k$-form $f(\Omega)$ is a closed form on M.

If θ_0 and θ_1 are two connections on P, we define

$$\Delta_f(\theta_1, \theta_0) = k \int_0^1 f(\theta_1 - \theta_0, \Omega_t) dt ,$$

where Ω_t is the curvature of the connection $t\theta_1 + (1 - t)\theta_0$, $t \in R$. As $\theta_1 - \theta_0$ is tensorial of type ad, $\Delta_f(\theta_1, \theta_0)$ is a well defined $(2k - 1)$-form on M, and

$$d(\Delta_f(\theta_1, \theta_0)) = f(\Omega_1) - f(\Omega_0) .$$

Thus the cohomology class $[f(\Omega)]$ does not depend on θ and in fact depends only on P. The resulting map

$$W\colon I(GL_q) \to H^*(M; R)$$

is called the Chern-Weil homomorphism.

Now denote by $R[c_1, \cdots, c_q]$ the polynomial ring over R in the indicated variables with degree $c_i = 2i$. Let $R_q[c_1, \cdots, c_q]$ be the ring $R[c_1, \cdots, c_q]$/(elements of degree $> 2q$), and let $\Lambda(h_1, h_3, \cdots, h_{2k+1})$ be an exterior algebra on the h's where degree $h_i = 2i - 1$, and $2k + 1$ is the largest odd integer $\leq q$. WO_q is the differential complex

$$\Lambda(h_1, \cdots, h_{2k+1}) \otimes R_q[c_1, \cdots, c_q] ,$$

where $d(1 \otimes c_i) = 0$, $d(h_i \otimes 1) = c_i \otimes 1$.

Denote the ring of differential forms on M by $A(M)$. Let θ_0 be a Riemannian connection on the principal dual normal bundle P of F. If Ω_0 is the curvature of θ_0, we remark that for any odd i the form $c_i(\Omega_0)$ is identically zero. Let θ_1 be a basic connection [4] on P with curvature Ω_1. Define

$$\alpha_F\colon WO_q \to A(M)$$

by

$$\alpha_F(c_i) = c_i(\Omega_1)\,, \qquad \alpha_F(h_i) = \Delta_{c_i}(\theta_1, \theta_0)\,,$$

and extend by linearity. The c_i on the right are the Chern polynomials in $I(GL_q)$. This map commutes with the differential operators and induces the map

$$\alpha_F^*\colon H^*(WO_q) \to H^*(M; R)\,.$$

This map does not depend on the choices made.

This is one of several independently discovered and essentially equivalent approaches to secondary classes due to Berstein-Rozenfel'd [2], Bott-Haefliger [6], Malgrange (unpublished), and Kamber-Tondeur [15]. We have adopted here the method given in [4].

J. Vey [11] has given a basis of $H^*(WO_q)$ by elements of the form

$$h_{i_1} \cdots h_{i_k} \otimes c_{j_1} \cdots c_{j_l}\,, \quad i_1 < \cdots < i_k\,, \quad j_1 \leq \cdots \leq j_l\,,$$

satisfying the auxiliary conditions

(i) if no h's appear, each c_j must have j even and $j_1 + \cdots + j_l \leq q$. (These give the Pontrjagin classes of ν^*.)

(ii) if some h's appear, then $i_1 \leq$ smallest odd j appearing and $i_1 + j_1 + \cdots + j_l > q$. (These give the secondary characteristic classes of F.)

A Γ vector field on a manifold M with a codimension-q foliation F is a vector field Y such that the one-parameter family of local diffeomorphisms generated by Y maps leaves of F to leaves of F. We identify two Γ vector fields if their difference is a vector field tangent to F. If Φ is the sheaf of germs of local Γ vector fields on M, then $H^1(M; \Phi)$ may be viewed as infinitesimal deformations of F. The cohomology groups $H^*(M; \Phi)$ can be computed as follows.

Recall τ is the tangent bundle of F, ν the normal bundle, and let ∇ be the covariant derivative determined by θ, a basic connection on ν^*. $H^*(M; \Phi)$ is the homology of the differential complex

$$C^\infty(\nu) \xrightarrow{\hat{d}} C^\infty(\tau^* \otimes \nu) \xrightarrow{\hat{d}} C^\infty(\Lambda^2\tau^* \otimes \nu) \xrightarrow{\hat{d}}$$
$$\cdots \xrightarrow{\hat{d}} C^\infty(\Lambda^{n-q}\tau^* \otimes \nu) \longrightarrow 0\,.$$

If $\sigma \in C^\infty(\Lambda^k\tau^* \otimes \nu)$ and $X_0, \cdots, X_k \in C^\infty(\tau)$, then

$$(1.4)\qquad \begin{aligned} \hat{d}\sigma(X_0, \cdots, X_k) &= \sum_{0\leq i\leq k} (-1)^i \nabla_{X_i}\sigma(X_0, \cdots, \hat{X}_i, \cdots, X_k) \\ &+ \sum_{0\leq i<j\leq k} (-1)^{i+j}\sigma([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k)\,. \end{aligned}$$

The $\hat{}$ over X_i or X_j means that entry is deleted. Thus each element of $H^1(M; \Phi)$ is represented by a section σ of $\tau^* \otimes \nu$ with $\hat{d}\sigma = 0$.

For our point of view on this see [14]. See also [10], [18], [16], [19], [20], and [21].

We will be working with R^q-and gl_q-valued forms such as $\omega = (\omega_1, \cdots, \omega_q)$, $\theta = (\theta^i_j)$, etc. To avoid becoming swamped in sub and superscripts we will often abbreviate. For example, (2.13) written with scripts reads

$$(\theta'_i - \theta'_0)^i_j = d\delta^i_j - \sum_{k=1}^{q} (\theta^i_k \delta^k_j - \delta^i_k \theta^k_j) , \qquad i, j = 1, \cdots, q .$$

2. Construction of the derivatives

In this section we construct, for a foliated manifold M, the natural map

$$D_F \colon H^1(M; \Phi) \times H^*(WO_q) \to H^*(M; R)$$

referred to in the introduction. This map gives the derivative of the map

$$\alpha^*_F \colon H^*(WO_q) \to H^*(M; R) .$$

We first construct for a given basic connection θ and a section σ representing $\beta \in H^1(M; \Phi)$ an infinitesimal derivative θ' of θ.

Let F be a codimension-q foliation on an n-dimensional manifold M, and denote by T^*M the cotangent bundle of M, by τ the tangent bundle to F. The dual normal bundle ν^* of F is the subbundle of T^*M consisting of those elements which restrict to zero on τ. Let P be the principal bundle associated to ν^* and $\Pi \colon P \to M$ the projection. A point $\omega \in P$ consists of a q-tuple $\omega = (\omega_1, \cdots, \omega_q)$ where the ω_i are linearly independent elements of ν^* at the point $\Pi(\omega)$. We will also denote by ω the canonical R^q-valued one form on P, $\omega = (\omega_1, \cdots, \omega_q)$, given by

$$(2.1) \qquad \omega(X) = (\omega_1(\Pi_* X), \cdots, \omega_q(\Pi_* X)) , \qquad X \in TP_{(\omega_1, \cdots, \omega_q)} .$$

If $a \in GL_q$, it acts freely on the left on P by L_a and

$$(2.2) \qquad L_a{}^*\omega = a\omega .$$

The canonical forms $\omega_1, \cdots, \omega_q$ define the foliation Π^*F on P, with tangent bundle $\Pi^{-1}(\tau)$, and generate an ideal $I(\omega)$ in the ring of differential forms on P which is closed under exterior differentiation. Also note that the product of any $q + 1$ elements in $I(\omega)$ must be zero so $I(\omega)^{q+1} \equiv 0$. If θ is a basic connection on P, then

$$(2.3) \qquad d\omega = \theta \wedge \omega .$$

That is,

$$d\omega_i = \theta^i_j \wedge \omega_j .$$

It follows easily that the curvature

$$\Omega = d\theta - \theta \wedge \theta \tag{2.4}$$

of θ satisfies

$$\Omega \wedge \omega = 0 ,$$

and so $\Omega \in I(\omega)$. Thus we may write

$$\Omega^i_j = \Gamma^i_{jk} \wedge \omega_k , \qquad \Gamma^i_{jk} = \Gamma^i_{kj} , \tag{2.5}$$

where the Γ^i_{jk} are one-forms on P.

We now translate some of the results of previous section into statements about P. The space $C^\infty(\Lambda\tau^* \otimes \nu)$ consists of equivalence classes of sections of $\Lambda^k T^*M \otimes \nu$ where two sections are identified if their restrictions to τ coincide. Let $\hat{\nu}$ be the normal bundle of the foliation Π^*F on P. An element $\sigma \in C^\infty(\Lambda^k T^*P \otimes \hat{\nu})$ may be viewed as an R^q-valued k-form on P by composing with the canonical one-form ω. We will *always* think of such σ as R^q-valued forms in this way. An element $\sigma \in C^\infty(\Lambda^k T^*P \otimes \hat{\nu})$ projects to an element $\bar{\sigma} \in C^\infty(\Lambda^k T^*M \otimes \nu)$ if and only if

$$\begin{aligned} &\text{(i)} \quad \sigma \text{ is horizontal} , \\ &\text{(ii)} \quad L_a{}^*\sigma = a\sigma , \qquad a \in GL_q . \end{aligned} \tag{2.6}$$

If $\bar{\sigma} \in C^\infty(\Lambda^k\tau^* \otimes \nu)$, then $\bar{\sigma}$ may be represented by an element $\sigma \in C^\infty(\Lambda^k T^*P \otimes \hat{\nu})$ satisfying (2.6). It follows directly from (1.4) that $d\bar{\sigma}$ may be represented by $d\sigma - \theta \wedge \sigma$. Thus, if $\beta \in H^1(M; \Phi)$, it may be represented by an R^q-valued one-form σ on P satisfying

$$\begin{aligned} &\text{(i)} \quad \sigma \text{ is horizontal} , \\ &\text{(ii)} \quad L_a^*\sigma = a\sigma , \\ &\text{(iii)} \quad d\sigma - \theta \wedge \sigma|_{\Pi^{-1}(\tau)} \equiv 0 . \end{aligned} \tag{2.7}$$

Definition (2.8). Let $\beta \in H^1(M; \Phi)$ be represented by the R^q-valued one-form σ on P. *The derivative ω' of the canonical one-form ω with respect to σ* is given by

$$\omega' = -\sigma .$$

Equation (iii) of (2.7) means that

$$d\omega' - \theta \wedge \omega' \in I(\omega) .$$

Thus there is a gl_q-valued one-form θ' on P sat*isfying*

$$d\omega' - \theta \wedge \omega' = \theta' \wedge \omega . \tag{2.9}$$

Definition (2.10). Any gl_q-valued one-form θ' satisfying (2.9) is called an *infinitesimal derivative of θ with respect to σ.*

If θ'_0 and θ'_1 are two infinitesimal derivatives of θ with respect to σ, then (2.9) implies $(\theta'_1 - \theta'_0) \wedge \omega = 0$. Thus

$$(2.11) \qquad \theta'_1 - \theta'_0 = \lambda\omega \ , \qquad \lambda^i_{jk} = \lambda^i_{kj} \ ,$$

where $\lambda = (\lambda^i_{jk})$, and the λ^i_{jk} are functions on P.

Lemma (2.12). *If θ' is an infinitesimal derivative of θ, then*

(i) *θ' is horizontal,*

(ii) *θ' is tensorial of type* ad *modulo* ω *(i.e., $L_a{}^*\theta' - a\theta' a^{-1} \in I(\omega)$).*

Proof. (i) Let $\nu \subset TM$ be a complementary bundle to τ, and let $\hat{\nu}$ be its horizontal lift. Choose a representative σ_0 of β such that $\sigma_0|_{\hat{\nu}} = 0$. Let $X \in TP_\omega$ be such that $\Pi_* X = 0$, and choose an equivariant vector field $Y \in \hat{\nu}$ such that $\omega_k(Y_\omega) = \delta^j_k$. Now $\omega'(Y) = 0$, $\omega'(X) = 0$, and $\omega(X) = 0$. Since $\Pi_*([X, Y]_\omega) = 0$, we have $\omega'_i([X, Y]) = 0$ and

$$\begin{aligned} \theta'^i_j(X) &= \theta'^i_k(X)\omega_k(Y) = (\theta'^i_k \wedge \omega_k)(X, Y) \\ &= (d\omega'_i - \theta^i_k \wedge \omega'_k)(X, Y) = -\omega'_i([X, Y]) = 0 \ . \end{aligned}$$

If σ_1 is another representative of β whose restriction to $\Pi^{-1}(\tau)$ is the same as σ_0, then we have

$$\sigma_1 - \sigma_0 = \delta\omega \ , \qquad \delta = (\delta^i_j) \ .$$

(2.2) and (2.7) (ii) imply that

$$L^*_a\delta = a\delta a^{-1} \ .$$

Let θ'_1, θ'_0 be the derivatives determined by σ_1 and σ_0 respectively. A straightforward computation using (2.9) shows that modulo ω,

$$(2.13) \qquad \theta'_1 - \theta'_0 = d\delta - [\theta, \delta] \ .$$

By [3, Theorem 6, p. 86], the right-hand side of (2.13) is horizontal. As any one-form in $I(\omega)$ is horizontal, θ' is always horizontal.

(ii) Since θ is a connection we have

$$L^*_a\theta = a\theta a^{-1}$$

and also

$$L^*_a\omega = a\omega \ , \qquad L^*_a\omega' = a\omega' \ .$$

Applying L^*_a to (2.9) we have

$$L^*_a\theta' \wedge a\omega = a\theta' a^{-1} \wedge a\omega \ ,$$

and modulo ω

$$L_a^*\theta' = a\theta' a^{-1} .$$

We can now construct the derivatives of the secondary characteristic classes of the foliation F on M. Let θ^r and θ be connections on P, θ^r Riemannian and θ basic with curvature Ω. Let σ be an R^q-valued one-form on P which represents $\beta \in H^1(M; \Phi)$. Choose an infinitesimal derivative θ' of θ with respect to σ. For each element $h_{i_1} \cdots h_{i_k} \otimes c_{j_1} \cdots c_{j_l}$ of the Vey basis of $H^*(WO_q)$, we set $u + 1 = i_1 + j_1 + \cdots + j_l$ and define a differential form on P by

$$\begin{aligned} (2.14) \quad & D_\sigma(h_{i_1} \cdots h_{i_k} \otimes c_{j_1} \cdots c_{j_l}) \\ & = (-1)^{k-1}(u+1)\Delta_{c_{i_2}}(\theta, \theta^r) \cdots \Delta_{c_{i_k}}(\theta, \theta^r)(c_{i_1}c_{j_1} \cdots c_{j_l}(\theta', \Omega)) , \end{aligned}$$

and we extend to all of $H^*(WO_q)$ by linearity.

We remark that if $i_1 + j_1 + \cdots + j_l > q + 1$ (i.e., if $h_{i_1} \cdots h_{i_k} \otimes c_{j_1} \cdots c_{j_l}$ is a rigid element of $H^*(WO_q)$ [13]), then (2.5), which is essentially the Bott Vanishing Theorem [4], implies that the form $c_{i_1}c_{j_1} \cdots c_{j_l}(\theta', \Omega)$ is identically zero. Thus D_σ applied to such an element will be zero.

Let $f = h_{i_1} \cdots h_{i_k} \otimes c_{j_1} \cdots c_{j_l}$ be an element of the Vey basis of $H(WO_q)$, and set $g = c_{i_1}c_{j_1} \cdots c_{j_l} \in I^{u+1}(GL_q)$. Then

$$(2.15) \qquad D_\sigma(f) = (-1)^{k-1}(u+1)\Delta_{c_{i_2}}(\theta, \theta^r) \cdots \Delta_{c_{i_k}}(\theta, \theta^r) g(\theta', \Omega) .$$

Theorem 2.16. *For each f as above, $D_\sigma(f)$ is a globally well defined closed form on M depending on θ and σ.*

Proof. (*a*) $D_\sigma(f)$ is independent of the choice of θ'.

If θ'_0 and θ'_1 are two infinitesimal derivatives, we have by (2.11)

$$g(\theta'_1, \Omega) - g(\theta'_0, \Omega) = g(\lambda\omega, \Omega) \in I(\omega)^{u+1} \equiv 0 .$$

(b) $D_\sigma(f)$ projects to a form on M.

As each $\Delta_{c_{i_s}}(\theta, \theta^r)$ projects to a form on M, we check only that $g(\theta', \Omega)$ projects. By [17, Chapter XII, Lemma 1], we need only that θ', Ω are horizontal, and

$$L_a^* g(\theta', \Omega) = g(\theta', \Omega) .$$

It is well known that Ω is horizontal and θ' is horizontal by Lemma 2.12. From Lemma 2.12 (ii) and the fact that g is adjoint invariant, we have

$$\begin{aligned} L_a^* g(\theta', \Omega) - g(\theta', \Omega) &= g(L_a^*\theta', a\Omega a^{-1}) - g(a\theta' a^{-1}, a\Omega a^{-1}) \\ &= g(L_a^*\theta' - a\theta' a^{-1}, a\Omega a^{-1}) \in I(\omega)^{u+1} \equiv 0 . \end{aligned}$$

(c) $D_\sigma(f)$ is closed.

We first show $g(\theta', \Omega)$ is closed. Let Ω' be the gl_q-valued 2-form on P

$$\Omega'^i_j = \Gamma^i_{jk} \wedge \omega'_k ,$$

where

$$\Omega^i_j = \Gamma^i_{jk} \wedge \omega_k .$$

Thus

$$-\Omega \wedge \omega' = \Omega' \wedge \omega .$$

Taking the exterior derivative of (2.9) we obtain

$$-\Omega \wedge \omega' = (d\theta' - [\theta, \theta']) \wedge \omega .$$

Thus

$$\Omega' = d\theta' - [\theta, \theta']$$

modulo ω and

$$g(d\theta' - [\theta, \theta'], \Omega) = g(\Omega', \Omega) .$$

For each $s \in R$, let $I_s(\omega)$ be the ideal of forms on P generated by $\omega_1 + s\omega'_1, \cdots, \omega_q + s\omega'_q$. Note that

$$I_s(\omega)^{q+1} \equiv 0 , \qquad \Omega(s) = \Omega + s\Omega' \in I_s(\omega) .$$

The exterior derivative of (2.4) implies that

$$d\Omega = [\theta, \Omega] .$$

Using (66) of [8] we have

$$\begin{aligned} d(g(\theta', \Omega)) &= g(d\theta', \Omega) - ug(\theta', d\Omega, \Omega) \\ &= g(d\theta', \Omega) - ug(\theta', [\theta, \Omega], \Omega) \\ &= g(d\theta' - [\theta, \theta'], \Omega) = g(\Omega', \Omega) \\ &= \frac{\partial}{\partial s} \frac{1}{u+1} g(\Omega(s)) \Big|_{s=0} . \end{aligned}$$

As $g(\Omega(s)) \in I_s(\omega)^{u+1} \equiv 0$, we have $dg(\theta', \Omega) = 0$. Now each i_n is odd, and θ^r is Riemannian; thus we have

$$d(\Delta_{c_{i_n}}(\theta, \theta^r)) = c_{i_n}(\Omega) .$$

Since $\Omega \in I(\omega)$, the form $d(\Delta_{c_{i_n}}(\theta, \theta^r)g(\theta', \Omega) \in I(\omega)^{u+i_n} = 0$ and so applying d to (2.15) we see it is closed.

Theorem 2.17. *For each $f \in H^*(WO_q)$ and $\beta \in H^1(M; \Phi)$ with σ representing β, the cohomology class $[D_\sigma(f)] \in H^*(M; R)$ depends only on β and the foliation F.*

Definition 2.18. We denote $[D_\sigma(f)]$ by $D_\beta(f)$, and call it the *derivative of f in the direction of β.*

In what follows we will be doing many computations involving gl_q-valued forms $\alpha, \beta, \gamma, \cdots$ on P. The reader should note that whenever we consider a form such as $f(\alpha, \beta, \gamma, \cdots)$ on P, that form always projects to a form on M.

Proof of Theorem 2.17. (a) $D_\beta(f)$ is independent of the basic connection. Let θ_0 and θ_1 be two basic connections on P, with associated derivatives θ_0' and θ_1' with respect to σ. The connection

$$\theta_t = t\theta_1 + (1-t)\theta_0\,, \qquad t \in R$$

is basic and has associated derivative

$$\theta_t' = t\theta_1' + (1-t)\theta_0'$$

with respect to σ. Again using (66) of [8] we have

$$\begin{aligned}
&d(g(\theta_t', \theta_1 - \theta_0, \Omega_t)) \\
&\quad = g(d\theta_t', \theta_1 - \theta_0, \Omega_t) - g(\theta_t', d\theta_1 - d\theta_0, \Omega_t) \\
&\qquad + (u-1)g(\theta_t', \theta_1 - \theta_0, [\theta_t, \Omega_t], \Omega_t) \\
&\quad = g(d\theta_t' - [\theta_t, \theta_t'], \theta_1 - \theta_0, \Omega_t) - g(\theta_t', d(\theta_1 - \theta_0) - [\theta_t, \theta_1 - \theta_0], \Omega_t)\,.
\end{aligned}$$

Combining this with the equation

$$\frac{\partial}{\partial t} g(\theta_t', \Omega_t) = g(\theta_1' - \theta_0', \Omega_t) + ug(\theta_t', d(\theta_1 - \theta_0) - [\theta_t, \theta_1 - \theta_0], \Omega_t)\,,$$

we obtain

$$\begin{aligned}
(2.19) \qquad &\frac{\partial}{\partial t} g(\theta_t', \Omega_t) + udg(\theta_t', \theta_1 - \theta_0, \Omega_t) \\
&\qquad = g(\theta_1' - \theta_0', \Omega_t) + ug(\theta_1 - \theta_0, d\theta_t' - [\theta_t, \theta_t'], \Omega_t)\,.
\end{aligned}$$

Lemma 2.20.

$$g(\theta_1' - \theta_0', \Omega_t) + ug(\theta_1 - \theta_0, d\theta_t' - [\theta_t, \theta_t'], \Omega_t) \equiv 0\,.$$

Proof. If we apply (2.5) to θ_t we have that the curvature Ω_t of θ_t satisfies

$$(\Omega_t)^i_j = (\Gamma_t)^i_{jk} \wedge \omega_k\,.$$

Set

$$(\Omega_t')^i_j = (\Gamma_t)^i_{jk} \wedge \omega_k'\,.$$

Then modulo ω

$$\Omega_t' = d\theta_t' - [\theta_t, \theta_t']\,.$$

(2.3) applied to θ_1 and θ_0 implies that $\theta_1 - \theta_0 \in I(\omega)$, i.e.,

$$(\theta_1 - \theta_0)^i_j = \lambda^i_{jk}\omega_k , \qquad \lambda^i_{jk} = \lambda^i_{kj} . \tag{2.21}$$

If we now apply (2.9) and (2.20) to θ'_1 and θ'_0, we obtain that modulo ω

$$(\theta'_1 - \theta'_0)^i_j = \lambda^i_{jk}\omega'_k .$$

Set

$$\begin{aligned} \Omega(s, t) &= \Omega_t + s\Omega'_t , \\ \theta(s) &= (\theta_1 - \theta_0) + s\lambda\omega' , \\ I_s(\omega) &= I(\omega_1 + s\omega'_1, \cdots, \omega_q + s\omega'_q) . \end{aligned}$$

Then $\Omega(s, t)$, $\theta(s) \in I_s(\omega)$ so

$$g(\theta(s), \Omega(s, t)) \in I_s(\omega)^{u+1} \equiv 0 ,$$

$$\begin{aligned} g(\theta'_1 - \theta'_0, \Omega_t) &+ ug(\theta_1 - \theta_0, d\theta'_t - [\theta_t, \theta'_t], \Omega_t) \\ &= g(\partial/\partial s\theta(s), \Omega(s, t)) + ug(\theta(s), \partial/\partial s\Omega(s, t), \Omega(s, t))|_{s=0} \\ &= \partial/\partial s g(\theta(s), \Omega(s, t))|_{s=0} \equiv 0 . \end{aligned}$$

Thus (2.19) and Lemma 2.20 imply

$$\frac{\partial}{\partial t} g(\theta'_t, \Omega_t) = -ud(g(\theta'_t, \theta_1 - \theta_0, \Omega_t) .$$

By [13, Theorem 1] we have

$$\frac{\partial}{\partial t} \Delta_{c_i}(\theta_t, \theta^r) = dW_i + ic_i(\theta_1 - \theta_0, \Omega_t) ,$$

where W_i is some form on M. Thus

$$\begin{aligned} &\frac{\partial}{\partial t}(\Delta_{c_{i_2}}(\theta_t, \theta^r) \cdots \Delta_{c_{i_k}}(\theta_t, \theta^r) g(\theta'_t, \Omega_t)) \\ &\quad = \sum_{n=2}^{k} \Delta_{c_{i_2}}(\theta_t, \theta^r) \cdots (dW_{i_n} + i_n c_{i_n}(\theta_1 - \theta_0, \Omega_t)) \cdots \Delta_{c_{i_k}}(\theta_t, \theta^r) g(\theta'_t, \Omega_t) \\ &\qquad - \Delta_{c_{i_2}}(\theta_t, \theta^r) \cdots \Delta_{c_{i_k}}(\theta_t, \theta^r) ud(g(\theta'_t, \theta_1 - \theta_0, \Omega_t)) . \end{aligned}$$

Each term in the sum of the form $\Delta_{c_{i_2}} \cdots (i_n c_{i_n}) \cdots \Delta_{c_{i_k}} g$ is zero as

$$c_{i_n}(\theta_1 - \theta_0, \Omega_t) g(\theta'_t, \Omega_t) \in I(\omega)^{u+i_n} = 0 .$$

As

$$d(\Delta_{c_{i_n}}(\theta_t, \theta^r) g(\theta'_t, \Omega_t)) = c_{i_n}(\Omega_t) g(\theta'_t, \Omega_t) \in I(\omega)^{u+i_n} = 0 ,$$

each term in the sum of the form $\Delta_{c_{i_2}} \cdots (dW_{i_n}) \cdots \Delta_{c_{i_k}} g$ may be written as $d(\Delta_{c_{i_2}} \cdots W_{i_n} \cdots \Delta_{c_{i_k}} g)$. Finally as

$$d(\Delta_{c_{i_n}}(\theta_t, \theta^r))g(\theta'_t, \theta_1 - \theta_0, \Omega_t) = c_{i_n}(\Omega_t)g(\theta'_t, \theta_1 - \theta_0, \Omega_t) \in I(\omega)^{u+i_n} = 0 ,$$

we may write the last term as $-d(\Delta_{c_{i_2}} \cdots \Delta_{c_{i_k}} ug)$. Thus

$$\frac{\partial}{\partial t}(\Delta_{c_{i_2}}(\theta_t, \theta^r) \cdots \Delta_{c_{i_k}}(\theta_t, \theta^r)g(\theta'_t, \Omega_t)) = dW \tag{2.22}$$

for some form W on M. Integrating (2.22) from $t = 0$ to $t = 1$ finishes the proof of part (a).

(b) $D_\beta(f)$ is independent of the choice of Riemannian connection.

As $d(g(\theta', \Omega)) = 0$, we need only show that for two Riemannian connections θ_0^r and θ_1^r on P

$$\Delta_{c_{i_2}}(\theta, \theta_1^r) \cdots \Delta_{c_{i_k}}(\theta, \theta_1^r) - \Delta_{c_{i_2}}(\theta, \theta_0^r) \cdots \Delta_{c_{i_k}}(\theta, \theta_0^r) = \text{exact on } M .$$

If we write

$$\begin{aligned}
&\Delta_{c_{i_2}}(\theta, \theta_1^r) \cdots \Delta_{c_{i_k}}(\theta, \theta_1^r) - \Delta_{c_{i_2}}(\theta, \theta_0^r) \cdots \Delta_{c_{i_k}}(\theta, \theta_0^r) \\
&\quad = \sum_{n=2}^{k} (\Delta_{c_{i_2}}(\theta, \theta_0^r) \cdots \Delta_{c_{i_{n-1}}}(\theta, \theta_0^r)\Delta_{c_{i_n}}(\theta, \theta_1^r) \cdots \Delta_{c_{i_k}}(\theta, \theta_1^r) \\
&\qquad - \Delta_{c_{i_2}}(\theta, \theta_0^r) \cdots \Delta_{c_{i_n}}(\theta, \theta_0^r)\Delta_{c_{i_{n+1}}}(\theta, \theta_1^r) \cdots \Delta_{c_{i_k}}(\theta, \theta_1^r)) ,
\end{aligned}$$

then this follows directly from the facts:

(i) If $\deg c_i$ is odd, then $\Delta_{c_i}(\theta_0^r, \theta_1^r) = $ exact on M. See [9, Proposition 4.3].

(ii) $\Delta_{c_i}(\theta, \theta_1^r) - \Delta_{c_i}(\theta, \theta_0^r) = \Delta_{c_i}(\theta_0^r, \theta_1^r) +$ exact on M. See [13, Theorem 1].

(c) $D_\beta(f)$ is independent of the choice of representative of β.

Suppose that σ_0 and σ_1 are two R^q-valued one-forms on P representing β, whose restrictions to $\Pi^{-1}(\tau)$ are the same. Let θ'_0 and θ'_1 be the corresponding derivatives of θ. From the proof of Lemma 2.3 (i) we have

$$\theta'_1 - \theta'_0 = d\lambda - [\theta, \lambda] .$$

Thus

$$\begin{aligned}
g(\theta'_1, \Omega) - g(\theta'_0, \Omega) &= g(d\lambda - [\theta, \lambda], \Omega) \\
&= g(d\lambda, \Omega) + ug(\lambda, [\theta, \Omega], \Omega) = dg(\lambda, \Omega) .
\end{aligned}$$

Now

$$\begin{aligned}
&\Delta_{c_{i_2}}(\theta, \theta^r) \cdots \Delta_{c_{i_k}}(\theta, \theta^r)g(\theta'_1, \Omega) - \Delta_{c_{i_2}}(\theta, \theta^r) \cdots \Delta_{c_{i_k}}(\theta, \theta^r)g(\theta'_0, \Omega) \\
&\quad = \Delta_{c_{i_2}}(\theta, \theta^r) \cdots \Delta_{c_{i_k}}(\theta, \theta^r)d(g(\lambda, \Omega)) \\
&\quad = d(\Delta_{c_{i_2}}(\theta, \theta^r) \cdots \Delta_{c_{i_k}}(\theta, \theta^r)g(\lambda, \Omega)) .
\end{aligned}$$

The last equality follows from the fact that for each i_n, $n \geq 2$

$$d(\Delta_{c_{i_n}}(\theta, \theta^r))g(\lambda, \Omega) = c_{i_n}(\Omega)g(\lambda, \Omega) \in I(\omega)^{u+i_n} = 0 .$$

Thus $D_\beta(f)$ does not depend upon the extension to T^*M of the section $\bar{\sigma} \in C^\infty(\tau^* \otimes \nu)$ representing β.

If γ and δ are elements of $C^\infty(\tau^* \otimes \nu)$ with $\hat{d}\gamma = 0$ and $\hat{d}\delta = 0$, then for $f \in H^*(WO_q)$, $D_{\gamma+\delta}(f) = D_\gamma(f) + D_\delta(f)$. Thus it is sufficient to show that given $\lambda \in C^\infty(\nu)$, there is a form W on M such that

$$D_{\hat{d}\lambda}(f) = dW .$$

An element $\lambda \in C^\infty(\nu)$ corresponds to an R^q-valued function h on P which satisfies

$$h(a\omega) = ah(\omega) , \qquad a \in GL_q .$$

The correspondence is given by choosing a representative

$$Y \in C^\infty(TM)$$

for λ and letting $\hat{Y}$ be its horizontal lift. Then

$$h(\omega) = \omega(\hat{Y}) , \qquad \text{i.e., } h_k(\omega) = \omega_k(\hat{Y}) .$$

The element $\hat{d}\lambda$ is represented by the R^q-valued one-form

$$dh - \theta h .$$

Then

$$\begin{aligned} \omega' &= -dh + \theta h , \\ \theta' \wedge \omega = d\omega' - \theta \wedge \omega' & \\ &= d\theta \wedge h - \theta \wedge dh + \theta \wedge dh - \theta \wedge \theta h = \Omega h . \end{aligned}$$

By (2.5), $\Omega h = \Gamma \wedge \omega h$. Now as

$$\begin{aligned} \Gamma^i_{jk} &= \Gamma^i_{kj} , \\ \theta'^i_j \wedge \omega_j = \Gamma^i_{jk} \wedge \omega_k h_j &= \Gamma^i_{jk} h_k \wedge \omega_j , \end{aligned}$$

and so

$$\theta' = \Gamma h .$$

Note that (2.5) implies that modulo ω

$$\Gamma h = -i(\hat{Y})\Omega ,$$

where $i(\ )$ is interior product. Thus we may use

$$\theta' = -i(\hat{Y})\Omega\ ,$$

and for $g \in I^{u+1}(GL_q)$ we have $g(\Omega) \in I(\omega)^{u+1} = 0$, so

$$g(\theta', \Omega) = -g(i(\hat{Y})\Omega, \Omega) = \frac{-1}{u+1} i(\hat{Y})g(\Omega) = 0\ .$$

Thus $D_{\hat{d}\lambda}(f) = 0$.

This completes the proof of Theorem 2.17.

Theorem 2.23. *Let F_s, $s \in R$ be a differentiable family of foliations on M, and let β be the infinitesimal deformation of F_0 determined by F_s. Then for all $f \in H^*(WO_q)$*

$$D_\beta(f) = \frac{\partial}{\partial s}(\alpha^*_{F_s}(f))\Big|_{s=0}\ .$$

That is, the infinitesimal derivative gives the actual derivative of the characteristic classes for the foliation F_0.

Proof. Let ω^s, θ_s be respectively the canonical R^q-valued one-form and a basic connection on P_s the principal dual normal bundle of F_s. We choose θ_s to vary differentiably in s. For small s, P_s is canonically isomorphic to P_0. We use these isomorphisms to obtain two families on P_0, ω^s and θ_s satisfying

(i) ω^s is an R^q-valued one-form,

(ii) θ_s is a connection form,

(iii) $d\omega^s = \theta_s \wedge \omega^s$.

Indicating derivatives in s evaluated at $s = 0$ by $\cdot$, and writing ω for ω^0, θ for θ_0, $\dot{\omega}$ for $\dot{\omega}_s$, and $\dot{\theta}$ for $\dot{\theta}_s$, we have

$$d\dot{\omega} = \dot{\theta} \wedge \omega + \theta \wedge \dot{\omega}\ .$$

If σ is a representative of the class β, we have likewise

$$d\omega' = \theta' \wedge \omega + \theta \wedge \omega'\ ,$$

where $\omega' = -\sigma$. If we can choose a σ so that $\dot{\omega} = \omega'$, we would have modulo ω

$$\dot{\theta} = \theta'\ .$$

Let θ^r be a Riemannian connection on P_0. Then by Lemma 2.25 below we would have for $f = h_{i_1} \cdots h_{i_k} \otimes c_{j_1} \cdots c_{j_l}$, $i_1 + j_1 + \cdots + j_l = u + 1$

$$\begin{aligned}
&\frac{\partial}{\partial s}\alpha^*_{F_s}(f)\Big|_{s=0} \\
&\quad = (-1)^{k-1}(u+1)\Delta_{c_{i_2}}(\theta, \theta^r) \cdots \Delta_{c_{i_k}}(\theta, \theta^r)c_{i_1}c_{j_1} \cdots c_{j_l}(\dot{\theta}, \Omega) \\
&\quad = (-1)^{k-1}(u+1)\Delta_{c_{i_2}}(\theta, \theta^r) \cdots \Delta_{c_{i_k}}(\theta, \theta^r)c_{i_1}c_{j_1} \cdots c_{j_l}(\theta', \Omega) \\
&\quad = D_\beta(f)\ ,
\end{aligned}$$

and we will be done.

Choose a Riemannian metric on M, and denote by $\Pi_s^\perp$ the projection of TM onto the subbundle ν_s normal to the foliation F_s. Then (see [14]) β is represented by the element $\sigma \in C^\infty(\tau_0^* \otimes \nu_0)$ given by

$$\sigma(X) = -\frac{\partial}{\partial s}\Pi_s^\perp(X)\Big|_{s=0}, \qquad \text{for } X \in \tau_0 . \tag{2.24}$$

The normal subbundle ν_0 of F_0 pulls back to a horizontal sub-bundle $\hat{\nu}$ of TP which is complementary to $\Pi^{-1}(\tau_0)$, and $\hat{\nu}$ is trivial. For small s, ω^s is a non-singular R_q-valued one-form on $\hat{\nu}$. Let

$$X_s = (X_s^1, \cdots, X_s^q)$$

be a global framing of $\hat{\nu}$ which is dual to ω^s, i.e.,

$$\omega_j^s(X_s^i) = \delta_j^i .$$

It follows easily that σ is represented by the R^q-valued one-form $\hat{\sigma}$ on $\Pi^{-1}(\tau)$ given by

$$\hat{\sigma}(X) = -\omega\left(\frac{\partial}{\partial s}(\omega_i^s(X)\cdot X_s^i)\Big|_{s=0}\right) .$$

Since

$$\begin{aligned} \omega_i^0(X) &= 0 \qquad \text{for } X \in \Pi^{-1}(\tau_0) , \\ \sigma(X) &= -\omega\left(\frac{\partial}{\partial s}\omega_i^s(X)\Big|_{s=0}\cdot X_0^i\right) \\ &= \left(\frac{\partial}{\partial s}\omega_1^s\Big|_{s=0}, \cdots, \frac{\partial}{\partial s}\omega_q^s\Big|_{s=0}\right)(X) = -\dot{\omega}(X) . \end{aligned}$$

Thus $\omega' = -\sigma = \dot{\omega}$ and we are done.

Lemma 2.25. *Let $f = h_{i_1} \cdots h_{i_k} \otimes c_{j_1} \cdots c_{j_l} \in H^*(WO_q)$ be an element of the Vey basis. Set $u + 1 = i_1 + j_1 + \cdots + j_l$, and let F_s be a smooth family of codimension-q foliations on M. Let θ_s be a family of connections on P_0 as in the proof of Theorem* 2.23, *and let θ^r be a Riemannian connection on P_0. Then*

$$\begin{aligned} &\frac{\partial}{\partial s}\alpha_{F_s}^*(f) \\ &\quad = (-1)^{k-1}(u+1)\left[\Delta_{c_{i_2}}(\theta_s, \theta^r) \cdots \Delta_{c_{i_k}}(\theta_s, \theta^r)c_{i_1}c_{j_1} \cdots c_{j_l}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right)\right] . \end{aligned}$$

Proof. Again we use [13, Theorem 1] which states

$$\frac{\partial}{\partial s} c_i(\Omega_s) = \mathrm{id}\, c_i\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right),$$

$$\frac{\partial}{\partial s}\Delta_{c_i}(\theta_s, \theta^r) = dW_i + ic_i\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right),$$

where W_i is some form on M. Thus

$$\begin{aligned}
&\frac{\partial}{\partial s}(\alpha^*_{F_s}(f)) \\
&\quad = \frac{\partial}{\partial s}[\Delta_{c_{i_1}}(\theta_s, \theta^r) \cdots \Delta_{c_{i_k}}(\theta_s, \theta^r) c_{j_1} \cdots c_{j_l}(\Omega_s)] \\
&\quad = \Bigg[\sum_{n=1}^{k} \Delta_{c_{i_1}}(\theta_s, \theta^r) \cdots \left(dW_{i_n} + i_n c_{i_n}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right)\right) \\
&\qquad\qquad\qquad\qquad \cdots \Delta_{c_{i_k}}(\theta_s, \theta^r) c_{j_1} \cdots c_{j_l}(\Omega_s) \\
&\qquad\qquad + \sum_{m=1}^{l} \Delta_{c_{i_1}}(\theta_s, \theta^r) \cdots \Delta_{c_{i_k}}(\theta_s, \theta^r) c_{j_1}(\Omega_s) \\
&\qquad\qquad\qquad\qquad \cdots j_m dc_{j_m}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) \cdots c_{j_l}(\Omega_s)\Bigg].
\end{aligned}$$

Since

$$\begin{aligned}
&d(\Delta_{c_{i_n}}(\theta_s, \theta^r) c_{j_1} \cdots c_{j_l}(\Omega_s)) \\
&\quad = c_{i_n}(\Omega_s) c_{j_1} \cdots c_{j_l}(\Omega_s) \in I(\omega_s)^{u+1+(i_n-i_1)} = 0,
\end{aligned}$$

all the terms in the first sum involving the dW_i's are exact. As

$$c_{i_n}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) c_{j_1} \cdots c_{j_l}(\Omega_s) \in I(\omega_s)^{u+(i_n-i_1)},$$

all the other terms except the first are zero; for if $n \geq 2$, $u + (i_n - i_1) \geq q + 1$. In the second sum we use the fact that for $n \geq 2$

$$\begin{aligned}
&d(\Delta_{c_{i_n}}(\theta_s, \theta^r)) c_{j_1}(\Omega_s) \cdots j_m c_{j_m}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) \cdots c_{j_l}(\Omega_s) \\
&\quad = c_{i_n}(\Omega_s) c_{j_1}(\Omega_s) \cdots j_m c_{j_m}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) \cdots c_{j_l}(\Omega_s) \in I(\omega)^{u+i_n-i_1} = 0
\end{aligned}$$

to show that modulo exact terms

$$\Delta_{c_{i_1}}(\theta_s, \theta^r) \cdots \Delta_{c_{i_k}}(\theta_s, \theta^r) c_{j_1}(\Omega_s) \cdots j_m dc_{j_m}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) \cdots c_{j_l}(\Omega_s)$$

$$= (-1)^{k-1} c_{i_1}(\Omega_s) \Delta_{c_{i_2}}(\theta_s, \theta^r) \cdots \Delta_{c_{i_k}}(\theta_s, \theta^r) c_{j_1}(\Omega_s) \cdots j_m c_{j_m}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) \cdots c_{j_l}(\Omega_s) .$$

Thus we obtain

$$\begin{aligned} &\frac{\partial}{\partial s}\alpha^{*(f)}_{F_s} \\ &= \Big[i_1 c_{i_1}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) \Delta_{c_{i_2}}(\theta_s, \theta^r) \cdots \Delta_{c_{i_k}}(\theta_s, \theta^r) c_{j_1} \cdots c_{j_l}(\Omega_s) \\ &\quad + \sum_{m=1}^{l} (-1)^{k-1} c_{i_1}(\Omega_s) \Delta_{c_{i_2}}(\theta_s, \theta^r) \cdots \Delta_{c_{i_k}}(\theta_s, \theta^r) c_{j_1}(\Omega_s) \\ &\qquad \cdots j_m c_{j_m}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) \cdots c_{j_l}(\Omega_s) \Big] \\ &= (-1)^{k-1} \Big[\Delta_{c_{i_2}}(\theta_s, \theta^r) \cdots \Delta_{c_{i_k}}(\theta_s, \theta^r) \Big\{ i_1 c_{i_1}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) c_{j_1} \cdots c_{j_l}(\Omega_s) \\ &\quad + c_{i_1}(\Omega_s) \sum_{m=1}^{l} c_{j_1}(\Omega_s) \cdots j_m c_{j_m}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) \cdots c_{j_l}(\Omega_s) \Big\} \Big] . \end{aligned} \tag{2.26}$$

Now if $f \in I^k(GL_q)$ and $g \in I^l(GL_q)$, by definition (see [17])

$$f \cdot g(X_1, \cdots, X_{k+l}) = \frac{k!l!}{(k+l)!} \sum_{\pi} f(X_{\pi(1)}, \cdots, X_{\pi(k)}) g(X_{\pi(k+1)}, \cdots, X_{\pi(k+l)}) ,$$

where the sum is taken over all k, l shuffles π. It is easy to check that

$$(k + l)(fg)(X_1, X_2) = kf(X_1, X_2)g(X_2) + lf(X_2)g(X_1, X_2) , \tag{2.27}$$

and so the sum inside the { }in (2.26) is equal to

$$(u + 1)c_{i_1}c_{j_1} \cdots c_{j_l}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) ,$$

finishing the proof.

Note that if $u > q$, then we have both $(\partial/\partial s)\alpha^*_F(f)$ and $D_\beta(f)$ are zero.

3. Extension to complex foliations

We now show how to partially extend the construction of the previous section to the case of complex foliations. The numbering in this section was done so that objects corresponding to things in § 2 have corresponding numbers, i.e., (3.3) corresponds to (2.3), etc.

We begin by remarking that if GL_qC denotes the complex general linear group, then

$$I(GL_qC) = I(GL_q) \otimes C ,$$

and all the comments in § 1 concerning the Chern-Weil homomorphism hold for GL_qC.

Throughout this section we adopt that notation that F is complex codimension-q complex analytic foliation on a complex manifold M. We denote the holomorphic tangent bundle of M by TM, and remark that the tangent bundle τ of F is a subbundle of TM. We denote the holomorphic cotangent bundle by T^*M. Similarly the antiholomorphic tangent and cotangent bundles are denoted by $\bar{T}M$ and $\bar{T}^*M$ respectively. We write the normal bundle TM/τ of F as ν, and its dual bundle as ν^*. A section of ν^* is then a section of T^*M, i.e., a (1, 0)-form on M, whose restriction to τ is zero.

Let $R_q[\bar{c}_1, \cdots, \bar{c}_q]$ be a truncated polynomial ring isomorphic to $R_q[c_1, \cdots, c_q]$ of § 1, and denote by $\Lambda(h_1, \cdots, h_q)$ an exterior algebra on the h_i where degree $h_i = 2i - 1$. The graded differential complex WU_q is defined to be

$$WU_q = \Lambda(h_1, \cdots, h_q) \otimes R_q[c_1, \cdots, c_q] \otimes R_q[\bar{c}_1, \cdots, \bar{c}_q] ,$$

where the differential is given by

$$d(h_i \otimes 1 \otimes 1) = 1 \otimes c_i \otimes 1 - 1 \otimes 1 \otimes \bar{c}_i ,$$
$$d(1 \otimes c_i \otimes 1) = d(1 \otimes 1 \otimes \bar{c}_i) = 0 ,$$

or more informally

$$d(h_i) = c_i - \bar{c}_i , \qquad d(c_i) = d(\bar{c}_i) = 0 .$$

Denote the ring of complex valued differential forms on M by $A(M)$. Let θ_0 be a Hermitian connection, and θ_1 a basic connection [1] (with curvatures Ω_0 and Ω_1 respectively) on P, the principal bundle associated to ν^*.

Define $\alpha_F\colon WU_q \to A(M)$ by

$$\alpha_F(c_i) = c_i(\Omega_1) , \quad \alpha_F(\bar{c}_i) = \overline{c_i(\Omega_1)} , \quad \alpha_F(h_i) = \Delta_{c_i}(\theta_1, \theta_0) - \Delta_{\bar{c}_i}(\theta_1, \theta_0) ,$$

and extend linearly. Since θ_0 is Hermitian, the form $c_i(\Omega_0)$ is totally real, i.e., $c_i(\Omega_0) = \overline{c_i(\Omega_0)}$, and thus α_F commutes with the differentials and induces

$$\alpha_F^*\colon H^*(WU_q) \to H^*(M; C) .$$

As in the real case, α_F^* does not depend on the choices made. See the references in § 1 to the construction of α_F^* in the real case and [5].

A basis of $I^{q+1}(GL_qC)$ is given by elements of the form

$$c_I = c_{i_1} \cdots c_{i_k}\,, \qquad (i_1 + \cdots + i_k = q + 1,\, i_1 \leq \cdots \leq i_k)\,.$$

Each element c_I determines an element $hc_I \in H^{2q+1}(WU_q)$ given by

$$hc_I = [h_{i_1}c_{i_2} \cdots c_{i_k} + \bar{c}_{i_1}h_{i_2}c_{i_3} \cdots c_{i_k} + \cdots + \bar{c}_{i_1} \cdots \bar{c}_{i_{k-1}}h_{i_k}]\,.$$

These elements are known to vary linearly independently [5].

A Γ vector field on M for F is a holomorphic section $\langle Y\rangle \in C^\infty(\nu)$, where $Y \in C^\infty(TM \oplus \bar{T}M)$ is a vector field whose associated real part preserves F in the sense that the local diffeomorphisms which it generates map leaves to leaves. Let Φ be the sheaf of germs of local Γ vector fields for F. Then $H^1(M;\Phi)$ may be interpreted as infinitesimal deformations of F. The groups $H^*(M;\Phi)$ can be computed using the complex

$$C^\infty(\nu) \xrightarrow{\hat{d}} C^\infty(\Lambda^1(\tau^* \oplus \bar{T}^*M) \otimes \nu) \xrightarrow{\hat{d}} C^\infty(\Lambda^2(\tau^* \oplus \bar{T}^*M) \otimes \nu) \xrightarrow{\hat{d}} \cdots$$

where $\hat{d}$ is defined by (1.4) using a basic connection on ν. Thus each $\beta \in H^1(M;\Phi)$ can be represented by an element $\sigma \in C^\infty(\Lambda^1(\tau^* \oplus \bar{T}^*M) \otimes \nu)$ with $\hat{d}\sigma = 0$, and any two such representatives differ by an element $\hat{d}\gamma$, $\gamma \in C^\infty(\nu)$. See the references after (1.4).

The construction of the derivatives for the elements hc_I, $c_I \in I^{q+1}(GL_qC)$, now proceeds in a fashion nearly identical to the real case. We will outline it indicating the necessary changes.

Denote by $\Pi\colon P \to M$ the principal bundle associated to ν^*. Then, as in the real case, P has a canonical C^q-valued one-form ω, and if θ is a basic connection then

$$d\omega = \theta \wedge \omega\,. \tag{3.3}$$

This immediately implies that the curvature Ω of θ satisfies

$$\Omega \wedge \omega = 0\,, \tag{3.4}$$

so we may write

$$\Omega^i_j = \Gamma^i_{jk} \wedge \omega_k\,, \tag{3.5}$$

$$\Gamma^i_{jk} = \Gamma^i_{kj}\,. \tag{3.6}$$

Just as in the real case, $\beta \in H^1(M;\Phi)$ can be represented by a C^q-valued one-form σ on P such that

$$\begin{aligned} &\text{(i)} && \sigma \text{ is horizontal}\,,\\ &\text{(ii)} && L_a^*\sigma = a\sigma\,,\\ &\text{(iii)} && d\sigma - \theta \wedge \sigma|_{\Pi^{-1}(\tau\oplus\bar{T}M)} = 0\,. \end{aligned} \tag{3.7}$$

Definition 3.8. Let $\beta \in H^1(M; \Phi)$ be represented by the C^q-valued one-form σ on P. The derivative ω' of ω with respect to σ is given by

$$\omega' = -\sigma .$$

Equation (3.7) (iii) implies that there is a gl_qC-valued one-form θ' on P such that

$$d\omega' - \theta \wedge \omega' = \theta' \wedge \omega . \tag{3.9}$$

Definition 3.10. Any gl_qC-valued one-form θ' satisfying (3.9) is called an infinitesimal derivative of θ with respect to σ.

It is easy to check that (2.11) and Lemma 2.12 hold in the complex case. In the proof of Lemma 2.12 we note that TM is now the holomorphic tangent bundle of M, and we must replace $\Pi^{-1}(\tau)$ by $\Pi^{-1}(\tau \oplus \bar{T}M)$.

Definition 3.14. Let $f \in I^{q+1}(GL_qC)$, and denote by hf the element which f determines in $H^{2q+1}(WU_q)$. Let $\beta \in H^1(M; \Phi)$ be represented by σ. Let Ω be the curvature of a basic connection θ on P, and let θ' be an infinitesimal derivative of θ with respect to σ. Define

$$D_\beta(hf) = [2(q+1)\mathscr{I} f(\theta', \Omega)] .$$

Here $\mathscr{I}$ denotes the imaginary part of a form.

Theorem 3.17. *Let f, θ, θ', Ω be as in Definition* 3.14. *Then* $2(q+1)\mathscr{I} f(\theta', \Omega)$ *is a globally well defined closed form on M whose cohomology class depends only on β and f.*

Proof. To prove this we merely repeat the proof of Theorem 2.16, parts (a), (b) and (c) ignoring the parts pertaining to the Δ_{c_i}. Then we repeat the proof of Theorem 2.17 parts (a) and (c). In part (a) we stop at the end of the proof of Lemma 2.20. In part (c) we disregard the parts pertaining to the Δ_{c_i}, and for τ^* read $\tau^* \oplus \bar{T}M$. As the proofs carry over with only minor changes we omit them.

Theorem 3.23. *Let F_s, $s \in R$ be a differential family of complex analytic foliations on M, and let $\beta \in H^1(M; \Phi)$ be the infinitesimal deformation of F_0 determined by F_s. Then for all $f \in I^{q+1}(GL_qC)$,*

$$D_\beta(hf) = \frac{\partial}{\partial s}(\alpha^*_{F_s}(hf))\Big|_{s=0} .$$

Proof. This proof proceeds identically to the proof of Theorem 2.23. In particular we choose a family of connections θ_s on P_0 such that θ_s is basic for F_s and varies differentiably in s. Then we need to choose a representative σ of β so that modulo ω_0

$$\frac{\partial}{\partial s}\theta_s\Big|_{s=0} = \theta' ,$$

where θ' is the infinitesimal derivative determined by θ_0 and σ. To do this choose a Hermitian metric on M obtaining a family of projection operators $\Pi_s^\perp$. Then σ is given by (2.24). See [14]. In the proof of Theorem 2.23 we must replace τ_0^* by $\tau_0^* \oplus \bar{T}^*M$, and R^q by C^q. The proof of Theorem 3.23 is then completed by

Lemma 3.25. *Let $f \in I^{q+1}(GL_qC)$ and F_s, θ_s as in the proof of Theorem 3.23. Denote the curvature of θ_s by Ω_s, Then*

$$\frac{\partial}{\partial s}\alpha_{F_s}^*(hf) = 2(q+1)\mathscr{I}f\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right).$$

Proof. By linearity we may assume $f = c_{i_1} \cdots c_{i_k}$. Let θ^h be a Hermitian connection on P_0. Then

$$\begin{aligned}\alpha_{F_s}^*(hf) = [(\Delta_{c_{i_1}}(\theta_s, \theta^h) - \Delta_{\bar{c}_{i_1}}(\theta_s, \theta^h))c_{i_2}(\Omega_s) \cdots c_{i_k}(\Omega_s) + \cdots \\ + \bar{c}_{i_1}(\Omega_s) \cdots \bar{c}_{i_{k-1}}(\Omega_s)(\Delta_{c_{i_k}}(\theta_s,\theta^h) - \Delta_{\bar{c}_{i_k}}(\theta_s, \theta^h))].\end{aligned}$$

Again [13, Theorem 1] implies

$$\begin{aligned}\frac{\partial}{\partial s}c_i(\Omega_s) &= d\left(ic_i\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right)\right), \\ \frac{\partial}{\partial s}\bar{c}_i(\Omega_s) &= d\left(i\bar{c}_i\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right)\right), \\ \frac{\partial}{\partial s}\Delta_{c_i}(\theta_s, \theta^h) &= dW_i + ic_i\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right), \\ \frac{\partial}{\partial s}\Delta_{\bar{c}_i}(\theta_s, \theta^h) &= d\bar{W}_i + i\bar{c}_i\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right).\end{aligned}$$

A straightforward computation using these facts, (2.27) and the fact that $dc_i(\Omega_s) = d\bar{c}_i(\Omega_s) = 0$ shows that modulo exact terms

$$\begin{aligned}&\frac{\partial}{\partial s}(\bar{c}_{i_1}(\Omega_s) \cdots \bar{c}_{i_{r-1}}(\Omega_s)(\Delta_{c_{i_r}}(\theta_s, \theta^h) - \Delta_{\bar{c}_{i_r}}(\theta_s, \theta^h))c_{i_{r+1}}(\Omega_s) \cdots c_{i_k}(\Omega_s)) \\ &= (q+1)\Big(\bar{c}_{i_1} \cdots \bar{c}_{i_{r-1}}c_{i_r} \cdots c_{i_k}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) \\ &\qquad - \bar{c}_{i_1} \cdots \bar{c}_{i_r}c_{i_{r+1}} \cdots c_{i_k}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right)\Big).\end{aligned}$$

Summing over r we obtain

$$\frac{\partial}{\partial s}\alpha_{F_s}^*(hf) = (q+1)\left(f\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right) - \bar{f}\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right)\right)$$

$$= 2(q+1)\mathscr{I}f\left(\frac{\partial}{\partial s}\theta_s, \Omega_s\right).$$

In the next section we will compute θ' for specific foliations. We do the computations locally on M, not globally on P. Specifically we do the following.

Let F be a codimension-q foliation on M, and choose a basic connection θ for F. Let $\beta \in H^1(M; \Phi)$, and choose a representative $\sigma \in C^\infty(T^*M \otimes \nu)$, $(C^\infty((T^*M \oplus \bar{T}^*M) \otimes \nu)$ in the complex case). Let U be a neighborhood on M on which F is defined locally by one-forms $\omega_1, \cdots, \omega_q$.

(i) Compute the local connection and curvature forms $\theta = (\theta^i_j)$, $\Omega = (\Omega^i_j)$ with respect to the local basis $\omega_1, \cdots, \omega_q$ of ν^*.

(ii) Choose a local basis $\langle X_1 \rangle, \cdots, \langle X_q \rangle$ for ν and write $\sigma|_U = \sigma_k \otimes \langle X_k \rangle$.

(iii) Set $\omega'_i = -\omega_i(X_k)\cdot\sigma_k$.

(iv) Compute $d\omega'_i - \theta^i_j \wedge \omega'_j$. This will lie in the ideal generated by $\omega_1, \cdots, \omega_q$, i.e., write $d\omega'_i - \theta^i_j \wedge \omega'_j = \theta'^i_j \wedge \omega_j$, and set $\theta' = (\theta'^i_j)$.

(v) Finally for $f \in H^{2q+1}(WO_q)$, $D_\beta(f)$ is represented by the form whose restriction to U is $(q+1)f(\theta', \Omega)$. For $f \in I^{q+1}(GL_qC)$, $D_\beta(hf)$ is represented by the form whose restriction is $2(q+1)\mathscr{I}f(\theta', \Omega)$.

4. Some interesting trivial examples

In this section we compute two examples for which all derivatives are zero.

Example 4.1. The Lie group $SL(2, R)$ has Lie algebra

$$sl(2, R) = \{A \in gl(2, R) \mid \operatorname{tr} A = 0\}.$$

We may choose a basis of left invariant one-forms on $SL(2, R)$ $\omega, \omega_1, \omega_2$ which satisfy

$$d\omega = \omega \wedge \omega_1, \quad d\omega_1 = \omega \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_2.$$

Let X, Y and Z be left invariant vector fields dual to ω, ω_1 and ω_2 respectively. Then

$$[X, Y] = -X, \quad [X, Z] = -Y, \quad [Y, Z] = -Z.$$

Consider the foliation F on $SL(2, R)$ defined by ω. The normal bundle ν to F is spanned by $\langle X \rangle$, and as $d\omega = -\omega_1 \wedge \omega$, the covariant derivative ∇ of a basic connection on ν satisfies

$$\nabla\langle X \rangle = \omega_1 \otimes \langle X \rangle.$$

The bundle ν is trivial as is τ the tangent bundle to F. Thus

$$C^\infty(\tau^* \otimes \nu) = \{(f\omega_1 + g\omega_2) \otimes \langle X \rangle \mid f, g \in C^\infty(SL(2, R))\},$$
$$C^\infty(\Lambda^2\tau^* \otimes \nu) = \{h\omega_1 \wedge \omega_2 \otimes \langle X \rangle \mid h \in C^\infty(SL(2, R))\}.$$

A simple computation using the above information and the definition of $\hat{d}$ gives

$$\hat{d}((f\omega_1 + g\omega_2) \otimes \langle X \rangle)(Y, Z) = (Yg - Zf + 2g) \otimes \langle X \rangle .$$

Let $\beta \in H^1(SL(2, R); \Phi)$, and suppose β is represented by $(f\omega_1 + g\omega_2) \otimes \langle X \rangle$. Then

$$\omega' = -(f\omega_1 + g\omega_2) , \qquad \theta = -\omega_1 .$$

Using the fact that $\hat{d}((f\omega_1 + g\omega_2) \otimes \langle X \rangle) = 0$ we have that

$$d\omega' - \theta \wedge \omega' = (df(X)\omega_1 + dg(X)\omega_2 + f\omega_2) \wedge \omega .$$

Thus

$$\theta' = df(X)\omega_1 + dg(X)\omega_2 + f\omega_2 . \tag{4.2}$$

In addition

$$\Omega = d\theta = -\omega \wedge \omega_2 . \tag{4.3}$$

The complex $H^*(WO_1)$ satisfies

$$H^*(WO_1) = \begin{cases} 0 , & * \neq 0, 3 , \\ R , & * = 0, 3 , \end{cases}$$

and $H^3(WO_1)$ is generated over R by h_1c_1, the Godbillon-Vey invariant [12], where $h_1c_1(\theta', \Omega) = \theta' \wedge \Omega$. So from (4.2) and (4.3) we have

$$h_1c_1(\theta', \Omega) = df(X)\omega \wedge \omega_1 \wedge \omega_2 = d(f\omega_1 \wedge \omega_2) . \tag{4.4}$$

In order to obtain results about foliations on compact manifolds we form the manifolds $SL(2, R)/\Gamma$ where Γ is a discrete subgroup. For the proper choice of Γ we obtain the horocyclic foliation of geodesic flow on the unit tangent bundle of any compact Riemannian surface of constant negative curvature. Roussarie has noted that the Godbillon-Vey invariant of this foliation is non-zero as it is a multiple of the volume form of the manifold.

The computation for $SL(2, R)$ extends to these foliations by restricting to sections invariant under Γ of the relevant bundles on $SL(2, R)$.

By (4.4) we have that $[h_1c_1(\theta', \Omega)] = 0$ for any infinitesimal derivative of θ, and so the following.

Theorem (4.5). *Let M be a Riemannian surface of constant negative curvature, and let F be the horocyclic foliation of the geodesic flow on the unit tangent bundle $T^\circ M$. Then*

$$D_F\colon H^1(T^\circ M; \Phi) \times H^*(WO_1)) \to H^*(T^\circ M; R)$$

is the zero map.

Theorem 4.5 says that if we consider the Godbillon-Vey invariant as a cohomology valued function on the space of foliations on $T^\circ M$, then the horocyclic foliation F is a critical point of this function. The reader should be cautioned that it is possible for a family of foliations F_s to exist on $T^\circ M$ with $F_0 = F$ satisfying $h_1c_1(F_s)$ varies continuously. Theorem 4.5 implies only that $(\partial/\partial s)(h_1c_1(F_s))|_{s=0} = 0$.

Example 2. *The Hopf fibrations.* Consider the natural fiber bundle $C^{n+1} \sim \{0\} \rightarrow CP^n$. This gives a foliation F on $C^{n+1} \sim \{0\}$ of complex dimension 1. We prove

Theorem 4.6. *Let F_s, $s \in R$ be any differential family of complex foliations of dimension* 1 *on $C^{n+1} \sim \{0\}$ such that $F_0 = F$. Then for any element $f \in I^{n+1}(GL_nC)$ the element $hf \in H^{2n+1}(WU_n)$ satisfies*

$$\frac{\partial}{\partial s}\alpha^*_{F_s}(hf)\Big|_{s=0} = 0 \, .$$

In [6] examples are given of foliations F_s as in the theorem such that the elements determined by the $c_{j_1} \cdots c_{j_l} \in I^{n+1}(GL_nC)$ vary linearly independently. Just as Theorem 4.5, Theorem 4.6 says that if we consider the elements $hf \in H^{2n+1}(WU_n)$ as functions on the space of foliations on $C^{n+1} \sim \{0\}$, the Hopf fibration is a critical point. This is so because of the tremendous symmetry inherent in its structure.

Proof of Theorem 4.6. In what follows subscripts denoted by $i, j \cdots$ run from $1, \cdots, n$, and those denoted by $A, B, \cdots$ run from $0, \cdots, n$.

Since F is given by a fiber bundle structure over CP^n, the dual normal bundle ν^* to F is isomorphic to the pull back of the holomorphic cotangent bundle T^*CP^n on CP^n. Any connection on T^*CP^n of type 1, 0 may be pulled back to give a basic connection on ν^*. See [1]. We shall use the Kähler connection. Let U be the open set of $C^{n+1} \sim \{0\}$ defined by

$$U = \{(z_0, \cdots, z_n) \,|\, z_0 \neq 0\} \, .$$

U may also be considered as a homogeneous coordinate system on CP^n. Let $\langle\ ,\ \rangle$ denote the standard Hermitian inner product on C^{n+1}, and let $\langle z, z\rangle = \|z\|^2$. The local forms on U

$$\omega_j = \frac{z_0 dz_j - z_j dz_0}{z_0^2} \, , \qquad \bar{\omega}_j = \bar{\partial}\left[\frac{z_0\bar{z}_j}{\|z\|^2}\right]$$

form a basis for the local one-forms on CP^n. As F is spanned by the vector field

$$X = z_A \partial/\partial z_A \, ,$$

we see that the ω_j define F on U. The local connection form of the Kähler connection on CP^n in the basis given by the ω_j is

$$\theta^i_j = \frac{z_0\bar{z}_j}{\|z\|^2}\omega_i + \delta^i_j \frac{z_0\bar{z}_l}{\|z\|^2}\omega_l \ ,$$

and the local curvature form is

$$\Omega^i_j = \bar{\omega}_j \wedge \omega_i + \delta^i_j \bar{\omega}_l \wedge \omega_l \ . \tag{4.7}$$

Finally note the volume form $d\,\mathrm{Vol}$ on CP^n when restricted to U is

$$d\,\mathrm{Vol} = K\bar{\omega}_1 \wedge \omega_1 \wedge \cdots \wedge \bar{\omega}_n \wedge \omega_n \ , \qquad K \text{ a constant} \ .$$

Lemma 4.8. *If $f \in I^{n+1}(GL_nC)$, then for θ, Ω as above and any infinitesimal derivative θ' of θ, $f(\theta', \Omega)$ is a multiple of the* $\mathrm{tr}\,\theta' \wedge d\,\mathrm{Vol}$.

Proof. $I(GL_nC)$ is a polynomial algebra over C with generators c_k, $k = 1, \cdots, n$, where the degree $c_k = k$, and for $X_1, \cdots, X_k \in gl_nC$

$$c_k(X_1, \cdots, X_k) = \frac{(-1)^k}{(2\Pi\sqrt{-1})^k k!k!} \sum_{\Pi} \delta^{j_1\cdots j_k}_{i_1\cdots i_k} X_{\Pi(1)}{}^{i_1}_{j_1} \cdots X_{\Pi(k)}{}^{i_k}_{j_k} \ , \tag{4.9}$$

where the sum is over all permutations Π of $1, \cdots, k$, all ordered subsets $(i_1, \cdots, i_k)$ of k elements of $(1, \cdots, n)$ and all permutations $(j_1, \cdots, j_k)$ of $(i_1, \cdots, i_k)$, and $\delta^{j_1\cdots j_k}_{i_1\cdots i_k}$ denotes the sign of the permutation $i_1, \cdots, i_k \to j_1, \cdots, j_k$.

If $f = a_{r_1\cdots r_s} c_{r_1} \cdots c_{r_s}$ for $a_{r_1\cdots r_s} \in C$, then

$$f(\theta', \Omega) = \sum_{l=1}^{s} \frac{r_l}{n+1} a_{r_1\cdots r_s} c_{r_1}(\Omega) \cdots c_{r_l}(\theta', \Omega) \cdots c_{r_s}(\Omega) \ .$$

From (4.7) it is easy to see that $c_k(\Omega)$ is a multiple of $(\bar{\omega}_l \wedge \omega_l)^k$, i.e.,

$$c_k(\Omega) = K_k \bar{\omega}_{j_1} \wedge \omega_{j_1} \wedge \cdots \wedge \bar{\omega}_{j_k} \wedge \omega_{j_k} \ .$$

Thus a typical term of $f(\theta', \Omega)$ is a multiple of

$$c_r(\theta', \Omega) \wedge (\mathrm{tr}\,\Omega)^{n-r+1} \ .$$

From (4.9) we have

$$c_r(\theta', \Omega) = \frac{(-1)^r}{(2\pi\sqrt{-1})^r r!r} \sum_{l=1}^{r} \delta^{i_1\cdots i_r}_{j_1\cdots j_r} \Omega^{i_1}_{j_1} \wedge \cdots \wedge \theta'^{i_l}_{j_l} \wedge \cdots \wedge \Omega^{i_r}_{j_r} \ . \tag{4.10}$$

Consider a single term of $f(\theta', \Omega)$ of the form $\theta'^{i_1}_{j_1} \wedge \Omega^{i_2}_{j_2} \wedge \cdots \wedge \Omega^{i_r}_{j_r} \wedge (\mathrm{tr}\,\Omega)^{n-r+1}$ and assume $i_1 \neq j_1$. For the moment we also assume $\Omega^i_j = \bar{\omega}_j \wedge \omega_i$. As $i_1 \neq j_1$, $\Omega^{i_2}_{j_2} \wedge \cdots \wedge \Omega^{i_r}_{j_r}$ contains the form $\bar{\omega}_{i_1}$ but not the form ω_{i_1}. Each term of $(\mathrm{tr}\,\Omega)^{n-r+1}$ which contains ω_{i_1} also contains $\bar{\omega}_{i_1}$. Thus $\Omega^{i_2}_{j_2} \wedge \cdots \wedge$

$\Omega^{i_r}_{j_r} \wedge (\operatorname{tr} \Omega)^{n-r+1}$ is an (n, n)-form on CP^n each term of which either contains $\bar{\omega}^2_{i_1}$ or does not contain ω_{i_1}. In either case the form must be zero. The same argument works equally well for all other terms of (4.10) and the addition of the terms $\delta^i_j \bar{\omega}_l \wedge \omega_l$ to Ω^i_j changes nothing. Now by the symmetry of c_r we have

$$(4.11) \qquad c_r(\theta', \Omega) = Kc_1(\theta')c_{r-1}(\Omega) ,$$

K a constant. To complete the lemma we need only note that c_1 is a multiple of tr and $(\operatorname{tr} \Omega)^n$ is a multiple of d Vol.

Let F_s, $s \in R$ be a differentiable family of foliations on $C^{n+1} \sim \{0\}$ such that $F_0 = F$. As noted above F is spanned by $X = z_A\partial/\partial z_A$. Let

$$\omega = \frac{\bar{z}_A dz_A}{\|z\|^2} , \qquad \bar{\omega} = \frac{z_A d\bar{z}_A}{\|z\|^2} .$$

Then ω is dual to X, and the infinitesimal deformation σ of F associated to F_s is given by

$$\sigma = \omega \otimes \gamma_A\partial/\partial z_A$$

for some holomorphic vector field

$$\gamma_A\partial/\partial z_A \qquad \text{on } C^{n+1} \sim \{0\} .$$

Thus

$$\omega'_i = -\omega_i \circ \sigma = -\omega_i\left(\gamma_A \frac{\partial}{\partial z_A}\right)\omega = \frac{\gamma_0 z_i - \gamma_i z_0}{z_0^2} \cdot \omega .$$

We make several observations which will greatly simplify the necessary computations.

(a) ω, $\bar{\omega}$ and the $\bar{\omega}_j$, ω_j form a basis of 1-forms on U. Ω consists entirely of (1, 1)-forms in the $\bar{\omega}_j$, ω_j. If θ' is any infinitesimal derivative of θ, we may disregard all terms of type $\bar{\omega}_j$, ω_j in θ' when computing $f(\theta', \Omega)$ for $f \in I^{n+1}(GL_nC)$ as these terms will wedge to zero for dimensional reasons.

(b) We wish to show that

$$2\mathscr{I}(f(\theta', \Omega)) = 0$$

for any derivative θ'. Because of the linearity of our constructions as explained in § 2 we may assume that all the $\gamma_A = 0$ except γ_1.

(c) The function $\gamma = \gamma_1$ is holomorphic on $C^{n+1} \sim \{0\}$. If $n \geq 1$, then Hartog's Lemma implies that γ is holomorphic on C^{n+1}. Again by linearity, we may assume

$$\gamma(z_0, \cdots, z_n) = z_0^{\alpha_0} \cdots z_n^{\alpha_n} , \qquad \alpha_A \geq 0 ,$$

and we have

$$\omega'_1 = -\gamma/z_0 \cdot \omega ,$$
$$\omega'_j = 0 , \qquad j \neq 1 .$$

(d) $2\mathscr{P}[(f(\theta', \Omega))]$ is a well defined class in

$$H^{2n+1}(C^{n+1} \sim \{0\}; C) ,$$

and U is a dense subset of $C^{n+1} \sim \{0\}$. Thus, if we compute a local expression for $f(\theta', \Omega)$ on U and have

$$\int_{S^{2n+1} \cap U} f(\theta', \Omega) = 0 ,$$

then

$$2\mathscr{P}[f(\theta', \Omega)] = D_F(f, \langle\sigma\rangle) = 0 .$$

(e) $d\omega$ is the pull pack of 2-form of type 1, 1 on CP^n, and so can be ignored in the computation.

(f) The volume form on S^{2n+1} is $\omega \wedge d\,\mathrm{Vol}$.

Now a straightforward computation shows that modulo ω_j, $\bar{\omega}_j$

$$-\theta'^i_j = \delta^i_j\left[\frac{\bar{z}_1\gamma}{\|z\|^2}\omega\right] + \delta^i_1\left[\frac{\bar{z}_j\gamma}{\|z\|^2} - \frac{\partial\gamma}{\partial z_j} + \frac{\partial(\gamma/z_0)}{\partial z_A}\frac{z_A z_0 \bar{z}_j}{\|z\|^2}\right]\omega .$$

Thus

$$-\operatorname{tr}\theta' = \left((n+1)\frac{\bar{z}_1\gamma}{\|z\|^2} - \frac{\partial\gamma}{\partial z_1}\right)\omega + \frac{\partial(\gamma/z_0)}{\partial z_A}\frac{z_A z_0 \bar{z}_1}{\|z\|^2}\omega .$$

Observe that

$$(n+1)\int_{S^{2n+1}} \frac{\bar{z}_1\gamma}{\|z\|^2} = \int_{S^{2n+1}} \frac{\partial\gamma}{\partial z_1} .$$

For $\gamma \neq z_1$ both integrals are zero; if $\gamma = z_1$ they are equal since

$$\frac{z_A\bar{z}_A}{\|z\|^2} = 1 , \quad \int_{S^{2n+1}} \bar{z}_1 z_1 = \int_{S^{2n+1}} \bar{z}_k z_k \quad \text{for any } k .$$

As for the second term in $\operatorname{tr}\theta'$ we see that

$$\frac{\gamma}{z_0} = z_0^{\alpha_0 - 1} z_1^{\alpha_1} \cdots z_n^{\alpha_n} ,$$

and so

$$\frac{\partial(\gamma/z_0)}{\partial z_A} z_A \frac{z_0\bar{z}_1}{\|z\|^2} = \left[\left(\sum_A \alpha_A\right) - 1\right]\frac{\bar{z}_1\gamma}{\|z\|^2} .$$

Again the integral is zero unless $\gamma = z_1$, but in that case

$$\sum_A \alpha_A - 1 = 0 .$$

In all cases we have

$$\int_{S^{2n+1}} f(\theta', \Omega) = 0$$

for any $f \in I^{n+1}(GL_n C)$, and so the imaginary part is zero, proving the theorem.

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