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UMBILICAL SUBMANIFOLDS OF SASAKIAN SPACE FORMS

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1. The purpose of this note is to prove the following theorem:

Theorem. *Let N^n, $n \geq 3$, be an umbilical submanifold of a Sasakian space form $M^{2n+1}(c)$. If the mean curvature vector is parallel in the normal bundle, then N^n is one of the following:*

(i) *N^n is a real space form immersed as an integral submanifold of the contact distribution, and N^n is totally geodesic when $n = m$.*

(ii) *The characteristic vector field of the contact structure is tangent to N^n, N^n is totally geodesic and N^n is a Sasakian space form with the same ϕ-sectional curvature.*

(iii) *$c = 1$ and N^n is a real space form.*

If the mean curvature vector is not parallel, then

(iv) *N^n is an anti-invariant submanifold, and if N^n has constant mean curvature, then $c < -3$ and N^n admits a codimension* 1 *foliation by umbilical submanifolds of type* (i).

The four cases of the theorem do occur. In fact, the first three can occur in the odd-dimensional sphere $S^{2m+1}(1)$; for example $S^{2m+1}(1)$ admits a great m-sphere which is an integral submanifold of the usual contact structure [1] and a codimension 2 great sphere such that the characteristic vector field is tangent and the sphere inherits the contact structure of S^{2m+1}. Sasakian submanifolds of Sasakian manifolds have been studied quite extensively; see e.g. [2], [4]. In $\boldsymbol{R}^{2m+1}$ with coordinates (x^i, y^i, z), the usual contact form $\eta = \frac{1}{2}(dz - \sum y^i dx^i)$ together with the Riemannian metric $G = \eta \otimes \eta + \frac{1}{4} \sum ((dx^i)^2 + (dy^i)^2)$ is a Sasakian structure with constant ϕ-sectional curvature equal to -3. The vector fields $\partial/\partial y^i$ span an integrable distribution whose leaves are integral submanifolds of the contact distribution $\eta = 0$. Moreover these submanifolds are totally geodesic (see e.g. [1]) and G restricted to these submanifolds is just the Euclidean metric. Hence taking an $(n - 1)$-sphere $\sum (y^i)^2 =$ constant we have an umbilical submanifold in $\boldsymbol{R}^{2m+1}(-3)$. We devote § 5 to an example of type (iv).

2. Let M be a $(2m + 1)$-dimensional contact manifold with contact form η, i.e., $\eta \wedge (d\eta)^m \neq 0$. It is well known that a contact manifold admits a vector field ξ, called the *characteristic vector field*, such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$.

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Moreover M admits a Riemannian metric G and a tensor field ϕ of type (1, 1) such that

$$\phi^2 = -I + \xi \otimes \eta , \qquad G(\phi X, \phi Y) = G(X, Y) - \eta(X)\eta(Y) ,$$
$$\Phi(X, Y) \overset{\text{def}}{=\!=} G(X, \phi Y) = d\eta(X, Y) .$$

We then say that (ϕ, ξ, η, G) is a *contact metric structure.*

Let $\tilde{\nabla}$ denote the Riemannian connection of G. Then M is a normal contact metric (Sasakian) manifold if

$$(\tilde{\nabla}_X \phi)Y = G(X, Y)\xi - \eta(Y)X ,$$

in which case we have

$$\tilde{\nabla}_X \xi = -\phi X .$$

A plane section of the tangent space $T_m M$ at $m \in M$ is called a *ϕ-section* if it is spanned by vectors X and ϕX orthogonal to ξ.

The sectional curvature $\tilde{K}(X, \phi X)$ of a ϕ-section is called a *ϕ-sectional curvature*. A Sasakian manifold is called a *Sasakian space form*, and denoted $M(c)$ if it has constant ϕ-sectional curvature equal to c; in this case the curvature transformation $\tilde{R}_{XY} = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]}$ is given by

$$\begin{aligned} \tilde{R}_{XY}Z = {} & \tfrac{1}{4}(c + 3)\{G(Y, Z)X - G(X, Z)Y\} + \tfrac{1}{4}(c - 1)\{\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + G(X, Z)\eta(Y)\xi - G(Y, Z)\eta(X)\xi \\ & + \Phi(Z, Y)\phi X - \Phi(Z, X)\phi Y + 2\Phi(X, Y)\phi Z\} . \end{aligned} \tag{2.1}$$

Let $\iota \colon N \to M$ be an immersed submanifold, and g the induced metric. The Gauss equation for the induced connection ∇ and the second fundamental form $\sigma(X, Y)$ is

$$\tilde{\nabla}_{\iota_* X} \iota_* Y = \iota_* \nabla_X Y + \sigma(X, Y) .$$

For simplicity we shall henceforth not distinguish notationally between X and $\iota_* X$. Let R denote the curvature of ∇. Then the Gauss equation for the curvature of N is

$$g(R_{XY}Z, W) = G(\tilde{R}_{XY}Z, W) + G(\sigma(X, W), \sigma(Y, Z)) - G(\sigma(X, Z), \sigma(Y, W)) .$$

We denote by $\nabla^\perp$ the connection in the normal bundle, and for the second fundamental form σ we define the covariant derivative $'\nabla$ with respect to the connection in the (tangent bundle) $\oplus$ (normal bundle), by

$$('\nabla_X \sigma)(Y, Z) = \nabla^\perp_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) .$$

Finally, the tangential and normal parts of a tensor field will be denoted by the superscripts t and $\perp$ respectively.

For a contact manifold M it is well known that the (tangent) subbundle D defined by $\eta = 0$ admits integral submanifolds up to and including dimension n but of no higher dimension. D is generally referred to as the *contact distribution* of the contact structure η. A more general class of submanifolds than the integral submanifolds of D are those which satisfy $d\eta(X, Y) = 0$; these are called *anti-invariant submanifolds* [3] since ϕ maps the tangent space into the normal space.

3. We now consider an umbilical submanifold N with $n = \dim N \geq 3$ immersed in a Sasakian space form $M(c)$ of dimension $2m + 1$. The second fundamental form σ is then given by $\sigma(X, Y) = g(X, Y)H$ where H is the mean curvature vector and the Codazzi equation becomes

$$(\tilde{R}_{XY}Z)^{\perp} = ('\nabla_X\sigma)(Y, Z) - ('\nabla_Y\sigma)(X, Z) = g(Y, Z)\nabla_X^{\perp}H - g(X, Z)\nabla_Y^{\perp}H .$$

Since $n \geq 3$, for any X tangent to N we can choose a unit tangent vector field Y such that Y is orthogonal to X and ϕX. Then

$$(\tilde{R}_{XY}Y)^{\perp} = \nabla_X^{\perp}H ,$$

but from (2.1)

$$R_{XY}Y = \tfrac{1}{4}(c + 3)X + \tfrac{1}{4}(c - 1)(\eta(X)\eta(Y)Y - \eta(Y)^2X - \eta(X)\xi) ,$$

and hence

$$\nabla_X^{\perp}H = -\tfrac{1}{4}(c - 1)\eta(X)\xi^{\perp} . \tag{3.1}$$

Thus if H is parallel in the normal bundle, we have either (i) N is an integral submanifold of the Sasakian space form, (ii) ξ is tangent to N, or (iii) $c = 1$.

Case (i). From the Gauss equation we see that for an integral submanifold of $M(c)$ and an orthonormal pair $\{X, Y\}$

$$g(R_{XY}Y, X) = \tfrac{1}{4}(c + 3) + \mu^2 ,$$

where μ is the mean curvature, and hence that N is a real space form.

If ζ_1 and ζ_2 are normal vector fields, and A_1 and A_2 the corresponding Weingarten maps, then the equation of Ricci-Kühn is

$$G(\tilde{R}_{XY}\zeta_1, \zeta_2) = G(R_{XY}^{\perp}\zeta_1, \zeta_2) - g([A_1, A_2]X, Y) .$$

Since N is umbilical, $[A_1, A_2] = 0$ and since $\nabla^{\perp}H = 0$ we have

$$G(\tilde{R}_{XY}H, \phi Y) = 0 .$$

(2.1) then gives

$$G(H, \phi Y)G(\phi X, \phi Y) - G(H, \phi X)G(\phi Y, \phi Y) = 0 .$$

Choosing Y orthogonal to X, we have

$$G(H, \phi X) = 0 .$$

Thus either $m > n$ or H is in the direction of ξ. But if N is not totally geodesic, H cannot be in the direction of ξ, for if $\sigma(X, Y) = g(X, Y)\mu\xi$, $\mu \neq 0$, then

$$g(X, Y)\mu = G(\tilde{\nabla}_X Y, \xi) = -G(Y, \tilde{\nabla}_X \xi) = G(Y, \phi X) = 0 .$$

Therefore if $m = n$, N is totally geodesic.

Case (ii). If ξ is tangent to N, $\tilde{\nabla}_\xi \xi = 0$ implies $\nabla_\xi \xi + H = 0$ and hence $H = 0$. Now since N is totally geodesic

$$-\phi X = \tilde{\nabla}_X \xi = \nabla_X \xi ,$$

that is, ϕX is tangent to N. Setting $\phi' = \phi|_N$,

$$\begin{aligned} g(X, Y)\xi - \eta(Y)X = (\tilde{\nabla}_X \phi)Y &= \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y \\ &= \nabla_X \phi' Y - \phi' \nabla_X Y = (\nabla_X \phi')Y , \end{aligned}$$

and therefore N is Sasakian. Now by the Gauss equation we see that N is a Sasakian space form with constant ϕ-sectional curvature equal to c.

Case (iii). If $c = 1$, M is a real space form and hence its umbilical submanifolds are space forms of constant curvature $1 + \mu^2$.

4. Let $\alpha = G(\xi, H)$, and let μ be the mean curvature. Then by (3.1)

$$X\mu^2 = XG(H, H) = -2G(\tfrac{1}{4}(c - 1)\eta(X)\xi^\perp, H) = -\tfrac{1}{2}(c - 1)\alpha\eta(X) . \tag{4.1}$$

Differentiating α twice we have

$$X\alpha = -G(\phi X, H) - G(\xi, \mu^2 X + \tfrac{1}{4}(c - 1)\eta(X)\xi^\perp) , \tag{4.2}$$

$$\begin{aligned} YX\alpha - (\nabla_Y X)\alpha = &-\alpha(1 + \mu^2 + \tfrac{1}{4}(c - 1)|\xi^\perp|^2)g(X, Y) \\ &+ \tfrac{1}{4}(c - 1)(\eta(Y)G(\phi X, \xi^\perp) + 2\alpha\eta(X)\eta(Y) \\ &+ G(\phi Y, X)|\xi^\perp|^2 - \eta(X)Y|\xi^\perp|^2) . \end{aligned}$$

Interchanging X and Y and subtracting ($c \neq 1$) we have

$$\begin{aligned} \eta(X)G(\phi Y, \xi^\perp) - \eta(Y)G(\phi X, \xi^\perp) &+ 2G(\phi X, Y)|\xi^\perp|^2 \\ &- \eta(Y)X|\xi^\perp|^2 + \eta(X)Y|\xi^\perp|^2 = 0 . \end{aligned}$$

Taking X and Y orthogonal to ξ^t we see that for ξ not tangent to N, $G(\phi X, Y) = 0$. $Y = \xi^t$, and X orthogonal to ξ^t yields

$$|\xi^t|^2 X|\xi^t|^2 = (2 - |\xi^t|^2)G(\phi\xi^t, X) . \tag{4.3}$$

On the other hand,

$$XG(\xi^t, \xi^t) = 2G(-\phi X - \tilde{\nabla}_X \xi^\perp, \xi^t) = 2(G(\phi\xi^t, X) + G(\xi^\perp, \tilde{\nabla}_X \xi^t))$$
$$= 2(G(\phi\xi^t, X) + \alpha\eta(X)) .$$

Comparing this with (4.3) we have for X orthogonal to ξ^t

$$2|\xi^t|^2 G(\phi\xi^t, X) = (2 - |\xi^t|^2)G(\phi\xi^t, X) ,$$

and hence $G(\phi\xi^t, X) = 0$ or $|\xi^t|^2 = 2/3$ which also implies by virtue of (4.3) that $G(\phi\xi^t, X) = 0$. Therefore, $G(\phi X, Y) = 0$ for all tangent vectors X and Y, i.e., N is an anti-invariant submanifold of M.

Now if N has constant mean curvature, then (4.1) gives $\alpha = 0$, that is, $\sigma(X, Y) = g(X, Y)H$ is orthogonal to ξ and hence the Weingarten map for the normal $\xi^\perp$ vanishes. Therefore $\tilde{\nabla}_X \xi^\perp = \nabla_X^\perp \xi^\perp$, but

$$\tilde{\nabla}_X \xi^\perp = \tilde{\nabla}_X(\xi - \xi^t) = -\phi X - \nabla_X \xi^t - g(X, \xi^t)H .$$

Since ϕX is normal, we see that $\nabla_X \xi^t = 0$ and hence $g(R_{X\xi^t}\xi^t, X) = 0$. Taking X to be unit and orthogonal to ξ^t, the Gauss equation yields

$$0 = G(\tilde{R}_{X\xi^t}\xi^t, X) + |\xi^t|^2 \mu^2$$
$$= \tfrac{1}{4}(c + 3)|\xi^t|^2 + \tfrac{1}{4}(c - 1)(-|\xi^t|^4) + |\xi^t|^2 \mu^2 ,$$

or assuming $\xi^t \neq 0$, in particular assuming $\nabla^\perp H \neq 0$,

$$1 + \mu^2 + \tfrac{1}{4}(c - 1)(1 - |\xi^t|^2) = 0 . \tag{4.4}$$

Clearly $c < 1$ and writing (4.4) as

$$\tfrac{1}{4}(c + 3) + \mu^2 - \tfrac{1}{4}(c - 1)|\xi^t|^2 = 0 ,$$

we see that $c + 3 < (c - 1)|\xi^t|^2 < 0$ or $c < -3$. Moreover $\nabla_X \xi^t = 0$ implies that the distribution or subbundle on N orthogonal to ξ^t is integrable with totally geodesic leaves giving the foliation of N.

5. First let us continue the analysis of the previous section. Since $\alpha = 0$, (4.2) gives $G(\phi X, H) = 0$ for X orthogonal to ξ^t, and comparison with (4.4) yields $G(\phi\xi^t, H) = |\xi^t|^2$. Thus if $n = m$, H and $\phi\xi^t$ must be collinear; so taking the inner product of $\dfrac{H}{\mu} = \dfrac{\phi\xi^t}{|\phi\xi^t|}$ with $\phi\xi^t$ we see that $H = \phi\xi^t/(1 - |\xi^t|^2)$ and

$$\mu = \frac{|\xi^t|}{\sqrt{1 - |\xi^t|^2}} .$$

Substituting this into (4.4) we have

$$|\xi^t|^2 = 1 - \sqrt{\frac{-4}{c-1}}\ .$$

Consequently the mean curvature of an umbilical submanifold N^m of type (iv) of constant mean curvature is determined exactly by c. Moreover note that

$$\tilde{\nabla}_{\xi^t}\xi^t = \frac{|\xi^t|^2}{1-|\xi^t|^2}\phi\xi^t\ . \tag{5.1}$$

We now review the notion of a C-loxodromic transformation [6]. By a *C-loxodrome* we mean a curve γ with unit tangent γ_* in an almost contact metric manifold satisfying $\tilde{\nabla}_{\gamma_*}\gamma_* = a\eta(\gamma_*)\phi\gamma_*$, $a =$ constant. Note that such a curve makes a constant angle with the characteristic vector field ξ. Since ξ^t has constant length, (5.1) shows that the integral curves of ξ^t are C-loxodromes. A local diffeomorphism $f\colon M \to M'$ is a *C-loxodromic transformation* if it maps C-loxodromes to C-loxodromes. The main result of [6] is that a Sasakian manifold M is locally C-loxodromically equivalent to Euclidean space if and only if M is a Sasakian space form. In this case the respective connections $\tilde{\nabla}$ and $\tilde{\nabla}'$ are related by

$$\tilde{\nabla}'_X Y = \tilde{\nabla}_X Y + (Xp)Y + (Yp)X - \tfrac{1}{4}(c-1)(\eta(X)\phi Y + \eta(Y)\phi X)$$

for some function p. In particular, we see that an umbilical submanifold of type (i) is mapped to an umbilical submanifold of M.

Now since an umbilical submanifold N^m of type (iv) of $M^{2m+1}(c)$ admits a foliation by umbilical submanifolds of type (i) with a normal field ξ^t of C-loxodromes, it is determined by a locus of $(m-1)$-spheres and a C-loxodrome of the appropriate curvature in Euclidean space E^{2m+1}.

References

[1] D. E. Blair & K. Ogiue, *Geometry of integral submanifolds of a contact distribution*, Illinois J. Math. **19** (1975) 269–276.

[2] K. Kenmotsu, *On Sasakian immersions*, Publ. Study Group of Geometry **4** (1970) 42–59.

[3] G. D. Ludden, M. Okumura & K. Yano, *Anti-invariant submanifolds of almost contact metric manifolds*, Math. Ann. **225** (1977) 253–261.

[4] M. Okumura, *On contact metric immersion*, Kōdai Math. Sem. Rep. **20** (1968) 389–409.

[5] S. Tanno, *The topology of contact Riemannian manifolds*, Illinois J. Math. **12** (1968) 700–717.

[6] Y. Tashiro & S. Tachibana, *On Fubinian and C-Fubinian manifolds*, Kōdai Math. Sem. Rep. **15** (1963) 176–183.

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