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ALMOST FLAT MANIFOLDS

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1. Introduction

1.1. We denote by V a connected n-dimensional complete Riemannian manifold, by $d = d(V)$ the diameter of V, and by $c^+ = c^+(V)$ and $c^- = c^-(V)$, respectively, the upper and lower bounds of the sectional curvature of V. We set $c = c(V) = \max(|c^+|, |c^-|)$.

We say that V is ε-flat, $\varepsilon \geq 0$, if $cd^2 \leq \varepsilon$.

1.2. Examples.

a. Every compact flat manifold is ε-flat for any $\varepsilon \geq 0$.

b. Every compact nil-manifold possesses an ε-flat metric for any $\varepsilon \geq 0$. (*A* manifold is called a nil-manifold if it admits a transitive action of a nilpotent Lie group; see 4.5.)

The second example shows that for $n \geq 3$, $\varepsilon > 0$ there are infinitely many ε-flat n-dimensional manifolds with different fundamental groups.

1.3. Define inductively $ex_i(x) = \exp(ex_{i-1}(x))$, $ex_0(x) = x$, and set $\hat{\varepsilon}(n) = \exp(-ex_j(n))$, where $j = 200$. (We are generous everywhere in this paper because the true value of the constants is unknown.)

1.4. Main Theorem. *Let V be a compact $\hat{\varepsilon}(n)$-flat manifold, and π its fundamental group. Then*:

(a) *There exists a maximal nilpotent normal divisor $N \subset \pi$*;

(b) $\operatorname{ord}(\pi/N) \leq ex_3(n)$;

(c) *the finite covering of V corresponding to N is diffeomorphic to a nil-manifold.*

Corollary. *If V is $\hat{\varepsilon}(n)$-flat, then its universal covering is diffeomorphic to R^n. If V is $\hat{\varepsilon}(n)$-flat and π is commutative, then V is diffeomorphic to a torus.*

1.5. Manifolds of positive and almost positive curvature. For such manifolds one expects the properties (a) and (b) from Main theorem 1.4, but we are able to prove only the following:

(i) If V is a manifold of nonnegative sectional curvature ($c^- \geq 0$), then its fundamental group π and every subgroup of π can be generated by 3^n elements.

(ii) If $d(V) \leq \mathscr{D}$, $c^-(V) \geq -K$, $K \geq 0$, then π can be generated by $N \leq 3^n\, ex_2(nK\mathscr{D}^2)$ elements; if π is a free group and $K\mathscr{D}^2 \leq \hat{\varepsilon}(n)$, then π is generated by one element.

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1.6. Manifolds of almost negative curvature. The universal coverings of such manifolds are expected to be contractable. If $n = 2$, it is so for V with $c^+(V) \leq 1$, $d(V) < \frac{1}{4}\pi$ (S. Mayers, see [4]), but for $n = 3$ we have

Counterexample. For given $\varepsilon > 0$ there exists a manifold V diffeomorphic to the sphere S^3 such that $d(V) \leq \varepsilon$, $c^+(V) \leq \varepsilon$. (See [5].)

1.7. The volume and the injectivity radius. A slight modification of Cheeger's arguments from [1], [2] shows that the lower bound on the volume vol (V) or on the injectivity radius reduces drastically the number of almost flat manifolds (compare with Examples 1.2):

(a) The number of distinct up to diffeomorphism manifolds with $d(V) \leq 1$, vol $(V) \geq K^{-1}$, $c(V) \leq K$, $K \geq 0$, is less than $ex_6(n + K)$, Cheeger [1].

(b) If $d(V) \leq 1$, vol $(V) \geq K^{-1}$, $K \geq 0$ and $c(V) \leq \hat{\varepsilon}(n + K)$, then V is diffeomorphic to a flat manifold.

1.8. The second statement is a weak pinching theorem. For positive curvature there is much better result:

If $c^+(V) \leq 1$, $c(V) \geq 0.97$, then V is diffeomorphic to a manifold of a constant positive curvature (Grove, Karcher, Ruh [7]).

The following is known for the negative case:

If $c^+(V) \leq -1$, $c^-(V) \geq -1 - \kappa$, $\kappa \geq 0$, then in the following three cases V is diffeomorphic to a manifold of constant negative curvature:

(a) $\kappa \leq (ex_7(n + d(V)))^{-1}$; (E. Heintze, see [8]).

(b) $\kappa \leq (ex_7(n + \mathrm{vol}(V)))^{-1}$ and $n \neq 3$ (for $n = 3$ it is unknown).

(c) n is even and $\kappa \leq (ex_9(n + |\chi(V)|))^{-1}$, where $\chi(V)$ is the Euler characteristic.

Proof. In view of the Margulis-Heintze theorem (see the next section) one can apply to (a) Cheeger's arguments as in the previous section. About (b) see [6]. The case (c) follows from "b" and the Gauss-Bonnet theorem.

1.9. About the proof of the main theorem. Our arguments imitate the proof of the Bieberbach theorem (see [9]). The first application of the discrete group technique to geometry is due to Margulis who proved (but has never published) the following analog of the Kazdan-Margulis theorem (see [9]):

If V is compact, $c^+(V) < 0$, $c^-(V) > -1$, then $\mathrm{vol}(V) \geq C_n^{-1}, C_n \leq ex_4(n)$. (Margulis is not responsible for that particular C_n.)

This fact was independently discovered by Ernst Heintze (see [8]).

To prove that theorem Margulis established the following:

The Margulis Lemma. *Let V be as above, and suppose $\alpha, \beta \in \pi = \pi_1(V, v_0)$ can be represented by loops of the length $\leq C^{-1}$. If $C \geq ex_2(n)$, there is a natural number m such that $\alpha^m, \beta^m \subset \pi$ generate a nilpotent group.*

The ideas of Margulis lying behind his lemma are crucial for our proof of the Main Theorem. I am also very much indebted to Yu. Burago, J. Cheeger, D. Gromoll, V. Eidlin, W. Meyer and J. Milnor for discussions having led to a simplification of the proof. I am essentially thankful to Professor H. Karcher for his constructive criticism and suggestions. In particular, the present versions

of statements 2.3, 2.5, 2.6, 2.8.and 7.2 are due to him.

2. Almost positive curvature

2.1. For a group Γ with a function $\gamma \to \|\gamma\| \in \boldsymbol{R}_+$ we denote the "ball" $(\| \, \|)^{-1}[0, \rho]$ by $\Gamma_\rho \subset \Gamma$. We say that Γ is discrete with respect to $\| \, \|$ if all balls are finite.

We call $\gamma_1, \gamma_2, \cdots, \gamma_s \in \Gamma$ a short basis (or short generators) and the sequence of subgroups $e = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_s = \Gamma$ a short filtration with respect to $\| \, \|$, if Γ_i is generated by $\gamma_1, \cdots, \gamma_i$ and $\|\gamma_{i+1}\|$ is minimal for all γ from the complement $\Gamma \backslash \Gamma_i$.

2.2. From now on we fix a point $v_0 \in V$, denote the tangent space at v_0 by T, and set $\pi = \pi_1(V, v_0)$. For a geodesic $\lambda : [0, l] \to V$ with $\lambda(0) = v_0$ we denote by $t(\lambda) \in T$ the corresponding tangent vector with length $(t(\lambda)) =$ length (λ). For $\alpha \in \pi$ we denote by $\|\alpha\|$ the length of the shortest loop representing α.

2.3. Let $\alpha, \beta \in \pi$, and λ, μ be the corresponding shortest loops with ϕ the angle between $t(\lambda)$ and $t(\mu)$. Put $\rho = \max(\|\alpha\|, \|\beta\|)$ and $\kappa^2 = \max(0, -c^-(V))$. If $\cos\phi \geq \cosh\kappa\rho \cdot (1 + \cosh\kappa\rho)^{-1}$ (i.e., for $\kappa = 0$ if $\phi \leq \frac{1}{3}\pi$), then $\|\alpha^{-1}\beta\| \leq \max(\|\alpha\|, \|\beta\|)$.

Proof. Apply the Toponogov comparison theorem to the universal covering $\tilde{V}$.

2.4. Proof of 1.5 (i). Take the short basis $\gamma_1, \cdots, \gamma_s \in \pi$ and the corresponding shortest loops $\lambda_1, \cdots, \lambda_s$. From 2.3 it follows that all angles between $t(\lambda_i)$ and $t(\lambda_j)$, $1 \leq i < j \leq s$, are at least $\pi/3$ and so $s \leq \mathrm{vol}(S^n)/\mathrm{vol}(B^n_{\pi/6}) \leq 3^n$.

2.5. If $\rho \geq 2d(V)$, then the ball $\pi_\rho \subset \pi$ generates π, since every loop strictly longer than ρ can be decomposed into two shorter ones.

2.6. Therefore we can estimate the number of generators in 1.5 (ii) by using ϕ from $\cos\phi = \cosh(2\kappa\mathscr{D}) \cdot (1 + \cosh 2\kappa\mathscr{D})^{-1}$ by

$$s \leq \mathrm{vol}(S)/\mathrm{vol}(B^n_{\phi/2}) \leq 3^n \cdot \cosh^n(\kappa\mathscr{D}) \, .$$

For the last statement we need an algebraic fact.

2.7. For a group Γ with generators $\gamma_1, \cdots, \gamma_s$ we denote by $N^k(\gamma_1, \cdots, \gamma_s)$ the smallest number N such that every subgroup in Γ generated by words of length $\leq k$ admits a system of N generators. Denote by $N^k(\Gamma)$ the minimum of all $N^k(\gamma_1, \cdots, \gamma_s)$ with respect to all systems of generators of Γ.

If Γ is free and noncommutative, then $N^k(\Gamma) \geq k$. This is obvious and in fact $N^k(\Gamma)$ grows exponentially.

2.8. End of the proof of 1.5. For a short basis $\gamma_1, \cdots, \gamma_s \subset \pi$ we conclude as before $N^k(\gamma_1, \cdots, \gamma_s) \leq 3^n \cdot \cosh^n(\kappa \cdot k\mathscr{D})$. Now, if $\kappa \cdot \mathscr{D} < 3^{-2n}$, then this upper bound for $N^k(\gamma_1, \cdots, \gamma_s)$ is, for noncommutative π, incompatible with $k \leq N^k(\gamma_1, \cdots, \gamma_s)$, (e.g., at $k = 3^n$).

3. q-isometries

3.1. A set in a metric space X is said to be δ-dense if it intersects every ball of radius δ. A discrete set $\Delta \subset X$ is said to be σ-uniformly δ-dense if for any two balls $A, B \subset X$ of radius δ the numbers i, j of points in $A \cap \Delta$, $B \cap \Delta$ satisfy

$$\sigma^{-1} \leq i/j \leq \sigma .$$

A map f from one metric space to another is called a R-restricted q-isometry if for any two points x, y with dist $(x, y) \leq R$ we have

$$q^{-1} \leq \frac{\operatorname{dist}(f(x), f(y))}{\operatorname{dist}(x, y)} \leq q .$$

3.2. For a complete Riemannian C^∞-manifold X, a discrete set $\Delta \subset X$ and a finite C^∞-function $\psi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ we construct a map $\varphi: X \to H = l^2(\Delta)$ (= the space of l^2-functions on Δ): $(\varphi(x))\,(y) = \psi\,(\operatorname{dist}(x, y))$, $x \in X$, $y \in \Delta$. Further we fix ψ with properties: ψ is supported in the interval $[0.1, 1]$ if $x \in [\frac{1}{3}, \frac{2}{3}]$, then $\psi(x) = x$ and $\psi(x) + |\psi'(x)| + |\psi''(x)| \leq 100$, $x \in [0, 1]$.

3.3. Let X_1 and X_2 be manifolds as above of dimension n, and $\Delta_1 \subset X, \Delta_2 \subset X$ be σ-uniformly δ-dense sets. Denote by R_0 the minimum of the injectivity radii Rad (X_1), Rad (X_2), and by K the maximum of the curvatures $c(X_1)$ and $c(X_2)$. Let $f: \Delta_1 \to \Delta_2$ be a bijective R-restricted q-isometry. If $\sigma \leq 2, \delta \leq \exp(-10n)$, R, $R_0 \geq 10$, $q \leq 1 + \exp(-10n)$, $K \leq \exp(-10n)$, then there exists a diffeomorphism $F: X_1 \to X_2$.

Proof. Using $f: \Delta_1 \to \Delta_2$ we identify Δ_1 with Δ_2, and set $H = l^2(\Delta_1) = l^2(\Delta_2)$. It is easy to see that the maps $\varphi_1: X_1 \to H$ and $\varphi_2: X_2 \to H$ are smooth imbeddings, the image X_1' of the first map is contained in a normal tubular neighborhood of the image X_2' of the second map, and the normal projection $X_1' \to X_2'$ is a diffeomorphism.

3.4. Remark. Our construction for F is metrically invariant. So if f commutes with an isometrical action of a group in Δ_1 and Δ_2, then so does F. (We suppose here that a group acts isometrically on X_1 and X_2, and Δ_1, Δ_2 are invariant sets.)

3.5. Notice that 1.7 (a) immediately follows from 3.3 and the Cheeger inequality: If $d(V) \leq 1$, then Rad $(V) \geq$ vol $(V)\,(ex_2(n + K)^{-1}$; see [1].

4. Lie groups

4.1. The group of motions. We normalize the biinvariant metric in $O(n)$ by the condition $d(O(n))$ (= diam $(O(n)) = 1$, and denote by $M(n)$ the group of rigid motions of $\boldsymbol{R}^n$ with the metric induced by the decomposition $M(n) = O(n) \times \boldsymbol{R}^n$. We denote the projections $M(n) \to O(n)$ and $M(n) \to \boldsymbol{R}^n$ by "rot" and "trans" respectively. In all three groups we denote by $\|\alpha\|$ the distance from α to the identity element, and by B_a, $a \geq 0$, the ball of radius a centered at the identity element.

By $[\alpha, \beta]$ we denote the commutator of α and β. For $A \in O(n)$ by $E_{\max}(A) \subset \boldsymbol{R}^n$ we denote the eigenspace corresponding to the (complex) eigenvalue λ maximizing the distance: dist $(\lambda, 1)$.

4.2. The following properties of the commutators are obvious and well known (see [9]):

(a) $\|[\alpha, \beta]\| \leq C_n \|\alpha\| \cdot \|\beta\|$, where α, β from $O(n)$ or $M(n)$, $\|\alpha\|$, $\|\beta\| \leq 1$ and $C_n \leq ex_2(n)$;

(b) Let $A \in O(n)$, $b \in E_{\max}(A)$, and $\alpha\colon x \mapsto Ax$, $\beta\colon x \mapsto x + b$, $x \in R^n$ be the motions from $M(n)$. Set $\alpha_1 = [\alpha, \beta]$, $\alpha_i = [\alpha, \alpha_{i-1}]$. Then $\|\alpha_i\| \geq n^{-i} \|A\|^i \|b\|$.

Nilpotent groups

4.3. Let L be an n-dimensional simply connected nilpotent Lie group, and l its Lie algebra. Equip l with an Euclidean structure, and L with the corresponding left invariant metric. Expressing curvature of L in terms of l we have

4.4. If $\|[x, y]\| \leq c \|x\| \|y\|$, $x, y \in l$, $c \geq 0$, then the curvature c (L) satisfies $c(L) \leq 100c^2$.

4.5. Take a triangular basis $x_1, \cdots, x_n \in l$ (i.e., $[x, x_i] \in l_{i-1}$, $x \in l$, and l_{i-1} is spanned by $x_1, \cdots, x_{i-1}$), and for $x = \sum_{i=1}^n a_i x_i$ set $\|x\|^2 = \sum_{i=1}^n \mu_i a_i^2$, $\mu_i \geq 0$.

If $\mu_{i-1} \leq \mu_i^n$ and μ_n is small, then the curvature $c(L)$ is small because of 4.4, and for given uniform discrete subgroup $\Gamma \subset L$ the diameter $d(L/\Gamma)$ is also small. This provides the second example in 1.2.

4.6. For vectors $x_1, \cdots, x_k \in \boldsymbol{R}^n$, $k \leq n$ we denote by $\mathscr{D}(x_1, \cdots, x_k)$ the volume of the k-dimensional parallelepiped spanned by $x_1, \cdots, x_k$. We say that a system of independent vectors $x_1, \cdots, x_k$ is regular if $\|x_i\| \leq 3^{i-1} \|x_j\|$, $1 \leq i < j \leq k$, and $\mathscr{D}(x_1, \cdots, x_k) \geq A_n \prod_{i=1}^k \|x_i\|$, $A_n^{-1} = ex_2(n)$.

4.7. Consider an n-dimensional lattice $\Lambda \subset \boldsymbol{R}^n$ equipped additionally with the structure of a nilpotent group without torsion. Let $\lambda_1, \cdots, \lambda_n \subset \Lambda$ be a basis in Λ such that the sublattices $\Lambda_i = \{\sum_{j=1}^i m_j \lambda_j\}$ are also invariant subgroups with respect to the nilpotent group structure, $[\Lambda, \Lambda_i] \subset \Lambda_{i-1}$, $i = 1, \cdots, n$, and $\lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_n^{m_n} = \sum_{i=1}^n m_i \lambda_i$, $m_i = \cdots, -1, 0, 1, \cdots$, $i = 1, \cdots, n$.

Realize Λ now (see [9]) as a uniform discrete subgroup in a nilpotent group L and associate with the basis $\lambda_1, \cdots, \lambda_n \in \Lambda \subset R^n$ a left invariant metric in L as follows: take $x_1, \cdots, x_n \in l$ with $\exp(x_i) = \lambda_i \in \Lambda \subset L$, equip l with the Euclidean structure induced by the isomorphism $\boldsymbol{R}^n \to l$ extending $\lambda_i \to x_i$, and take the corresponndig metric in L. For $\lambda, \mu \in \Lambda \subset L$ we denote the distance with respect to this metric by $d_L(\lambda, \mu)$.

4.8. Suppose that for $\lambda, \mu \in \Lambda \subset R^n$ with $\|\lambda\|, \|\mu\| \leq \rho > 0$ we have $\|[\lambda, \mu]\| \leq c\|\lambda\| \|\mu\|$. If the basis $\lambda_1, \cdots, \lambda_n \in \Lambda \subset R^n$ is regular and $\rho/\|\lambda_n\| \geq ex_3(n)$, then $c(L) \leq (c')^2$, $c' = c \cdot ex_5(n)$, and for $\lambda \in \Lambda \subset R^n$ with $\|\lambda\| \leq (c\, ex_6(n))^{-1}$ we have $q^{-1} \leq d_L(e, \lambda)/\|\lambda\| \leq q$, where $e \in \Lambda \subset L$ is the identity element and $q \leq \exp(c \cdot \|\lambda\| \cdot ex_6(n))$.

Proof. The product in the nilpotent group $\Lambda \subset \boldsymbol{R}^n$ is given by a polyinomial $P\colon \Lambda \times \Lambda \to \Lambda$ of degree $\leq n$. Extending this polynomial to $\boldsymbol{R}^n \times \boldsymbol{R}^n$ provides on $\boldsymbol{R}^n$ the structure of a nilpotent Lie group isomorphic to L. The bracket in the Lie algebra may be expressed in terms of the coefficients of P and so by an obvious interpolation argument inequalities $\|[\lambda, \mu]\| \leq c\|\lambda\|\|\mu\|$ in the ball in Λ yield the analogous inequality for l:

$$\|[x, y]\| \leq 10^{-2}c'\|x\|\|y\|\,, \qquad x, y \in l\,.$$

This, together with **4.4**, proves the first statement of the lemma and the same interpolation arguments prove the second.

5. Pseudogroups

5.1. A pseudogroup is by definition a set Γ with a product $\alpha \cdot \beta \in \Gamma$ defined for some pairs $\alpha, \beta \in \Gamma$ and having the following properties:

There is the unique identity element $e \in \Gamma$, and every $\gamma \in \Gamma$ has a unique inverse.

If the products $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ are defined, they are equal and are written as $\alpha\beta\gamma$. Generally, the notation $\gamma_1\gamma_2 \cdots \gamma_k$ means that the product is defined for any setting of brackets.

5.2. Example. A symmetric subset of a group, containing the identity element, is a pseudogroup.

5.3. Any pseudogroup Γ can be viewed as a presentation (by generators and relations) of a group $\pi = \pi(\Gamma)$. If the natural map $\Gamma \to \pi$ is injective, we say that Γ is injective. The pseudogroups from the above example are injective.

5.4. A symmetric subset of a pseudogroup containing the identity is again a pseudogroup, but we use the term "subpseudogroup" only for sets closed with respect to the multiplication.

5.5. A function $\gamma \mapsto \|\gamma\| \in \boldsymbol{R}_+$, $\gamma \in \Gamma$, is called a norm if it is symmetric ($\|\gamma^{-1}\| = \|\gamma\|$), positive outside the identity element, and $\|\alpha\beta\| \leq \|\alpha\| + \|\beta\|$.

We introduce the radius $\operatorname{rad}(\Gamma) = \max_{\gamma \in \Gamma} \|\gamma\|$, and say that Γ is radial if for $\alpha, \beta \in \Gamma$ with $\|\alpha\| + \|\beta\| \leq \operatorname{rad}(\Gamma)$ the product $\alpha \cdot \beta$ is defined.

5.6. Example: the local fundamental pseudogroup. Denote by Ω the H-space of all piecewise smooth loops in V based at $v_0 \in V$ with the composition denoted by $\varphi \circ \psi$ for $\varphi, \psi \in \Omega$. Denote by Ω_ρ, $\rho > 0$ the set of loops of length less than or equal to ρ and by $\Gamma = \pi_\rho$ the set of all geodesic loops in Ω_ρ. We denote by $\|\gamma\|$, $\gamma \in \Gamma$, the length of γ. If $\rho^2 c^+(V) \leq 0.1$ we define for $\alpha, \beta \in \Gamma$ with $\alpha \circ \beta \in \Omega_\rho$ the product $\alpha\beta \in \Gamma$: $\alpha\beta$ is the shortest loop homotopic in Ω_ρ to $\alpha \circ \beta$. The pseudogroup Γ so defined is discrete (see 2.1) and radial, and if $\rho > 4d(V)$ then $\pi(\Gamma)$ is canonically isomorphic to $\pi_1(V, v_0)$; but it may be not injective (see 1.6).

Our major concern is the injectivity for the almost flat case. To prove that we shall later need the following two facts. For their proof note that a pseudogroup is trivially injective if it can be described as a pseudogroup of transformations of some set.

5.7. Let Γ be discrete and radial (we use the notation from 3.1).

(a) If subpseudogroup $\Delta \subset \Gamma$ is injective and δ-dense in Γ, then the ball $\Gamma_\rho \subset \Gamma$ (with the induced pseudogroup structure) is injective for $\rho \leq 0.1 \operatorname{rad}(\Gamma) - 10\delta$.

(b) Suppose N, $A \subset \Gamma$ are injective subpseudogroups, N is invariant ($\Gamma N \Gamma^{-1} \subset N$ when the product is defined), the map $(\nu, \alpha) \mapsto \nu \cdot \alpha \in \Gamma$, $\nu \in N$, $\alpha \in A$, is injective (where it is defined) and every $\gamma \in \Gamma_\rho \subset \Gamma$, $\rho \leq \operatorname{rad} \Gamma$, admits the decomposition $\gamma = \nu\alpha$, $\nu \in N$, $\alpha \in A$. Then the ball $\Gamma_{\rho_0} \subset \Gamma$ is injective for $\rho_0 \leq 0.1\ \rho$.

5.8. Nilpotency. We say that a set $A \subset \Gamma$ is nilpotent if in the sequence $A_0 = A$, $A_i = [A, A_{i-1}]$ all commutators are defined and there exists a number d such that $A_d = \{e\}$. A minimal such d is denoted by nil (A).

A system of generators $\gamma_1 \cdots \gamma_s \in \Gamma$ is called a nilpotent basis if all commutators $[\gamma_i, \gamma_j]$, $1 \leq i, j \leq s$, are defined and $[\gamma_i, \gamma_j] \in \Gamma_{i-1}$, where by Γ_i we denote the subseudogroup generated by $\gamma_1 \cdots \gamma_i$.

Let Γ be a discrete pseudogroup of radius R, and $A \subset \Gamma_\rho \subset \Gamma$ a symmetric set containing the identity element. If A has a nilpotent basis $\alpha_1, \cdots, \alpha_s \in A$, and $R \geq \rho\, ex_2(s)$, then nil $(A) \leq s$.

This is obvious.

6. Pseudogroups of motions

6.1. A map $h\colon \Gamma \to G$ from a discrete radial pseudogroup to a Lie group G (both with the norms $\| \ \|$) is called an ε-homomorphism if

$$h(e) = e\ , \qquad h(\gamma^{-1}) = (h(\gamma))^{-1};$$

if $\alpha\beta\gamma = e$, $\alpha, \beta, \gamma \in \Gamma$, then $\|h(\alpha)h(\beta)h(\gamma)\| \leq \varepsilon \|\alpha\| \|\beta\|$.

6.2. Let $r\colon \Gamma \to O(n)$ be an ε-homomorphism (about $O(n)$ see 4.1), and let $\rho_0, \rho_1, \theta, \mu$ be given numbers with $0 \leq \rho_0 < \rho_1 \leq \operatorname{rad} \Gamma$, $0 < \theta, \mu < 1$.

If $\rho_0\rho_1^{-1} \leq \mu^N$, $N \geq (10 + \theta^{-1})^{3k}$, $k = \dim O(n) = \frac{1}{2}n(n-1)$, and $\rho_1^2\varepsilon \leq 0.1\theta$, then there exists a ρ, $\rho_0 \leq \rho \leq \rho_1$, such that the inverse image $r^{-1}(B_\theta) \subset \Gamma$ of the ball $B_\theta \subset O(n)$ is δ-dense in $\Gamma_\rho \subset \Gamma$ with $\delta \leq \mu\rho$.

Proof. This follows from the possibility of covering $0(n)$ by N balls of the radius $\frac{1}{3}\theta$.

6.3. Let $r\colon \Gamma \to 0(n)$ be an ε-homomorphism with image in the ball $B_\theta \subset O(n)$, $\theta \leq \exp(-n)$. If $\rho \leq \operatorname{rad}(\Gamma)$ and $\varepsilon \leq 0.1\,(\theta\rho^{-2})$, then $\|r(\gamma)\| \leq 10\theta\rho^{-1}\|\gamma\|$, $\gamma \in \Gamma$.

Proof. If $\alpha, \alpha^2 \cdots \alpha^i \in B_\theta$, then $\|\alpha^i\| = i\|\alpha\|$. Given this, the inequality

$\|r(\gamma^i)\| \leq \theta$, with $i = \text{ent}\,(\rho/\|\gamma\|)$, yields the proof.

6.4. For an ε-homomorphism $m: \Gamma \to M(n)$ we set $t(\gamma) = \text{trans}\,(m(\gamma)) \in \boldsymbol{R}^n$ and $r(\gamma) = \text{rot}\,(m(\gamma)) \in 0(n)$, $\gamma \in \Gamma$. We suppose that $\|t(\gamma)\| = \|\gamma\|$.

6.5. Let m be as above, and let θ, ρ be positive numbers. Denote by $N \subset \Gamma_\rho \subset \Gamma$ the pseudogroup generated in Γ_ρ by $\Gamma_\rho \cap r^{-1}(B_\theta)$, $B_\theta \subset O(n)$. If $\theta + \rho \leq \exp(-ex_2 n)$, $\text{rad}\,\Gamma \geq \rho\; ex_3(d)$, $d = 10^k$, $k = \dim M(n) = \frac{1}{2}n(n+1)$, and $\varepsilon \leq 0.01$, then nil $(N) \leq d$.

Proof. In N take a short basis $\gamma_1, \cdots, \gamma_p \in N$ with respect to the function $\gamma \to \|m(\gamma)\|$. As in 2.4 we conclude that $p \leq d$; from 4.2 (a) it follows that this basis is nilpotent, and applying 5.8 we finish the proof.

6.6. Let m be an ε-homomorphism as in 6.4, let $\Gamma_\rho \subset \Gamma$, $\rho \leq 1$ be the ball with nil $(\Gamma_\rho) \leq d$, and let $\theta', \delta', \delta, \theta \geq 0$ be real numbers with $ex_3(n + d + \theta^{-1}) \leq (\varepsilon + \theta' + (\delta + (\delta + \delta')/\rho))^{-1}$. If the set $r^{-1}(B_{\theta'}) \subset \Gamma$ is δ'-dense in Γ_ρ, and the image of $t: \Gamma \to \boldsymbol{R}^n$ is δ-dense in the ball $B_\rho \subset \boldsymbol{R}^n$, then $\|r(\gamma)\| \leq \theta$, $\gamma \in \Gamma_\rho$.

Proof. Take $x \in E_{\max}(r(\gamma))$ (see 4.1), $\gamma \in \Gamma_\rho$, with $\|x\| = \frac{1}{2}\rho$ and $\alpha \in r^{-1}(B_{\theta'})$ with $\|t(\alpha) - x\| \leq \delta + \delta' + 2\varepsilon$. Consider $\alpha_1 = [\alpha, \gamma], \cdots, \alpha_i = [\alpha_{i-1}, \gamma], \cdots$. If $\|r(\gamma)\| > \theta$, then using 4.2 (*b*) we conclude: $\|\alpha_i\| \geq h^{-i}(\theta/2)^i\,\|\alpha\|$, $i = 1, \cdots, d$, but the condition nil $(\Gamma_\rho) \leq d$ yields $\|\alpha_d\| = \|e\| = 0$, and the contradiction proves the lemma.

6.7. A discrete set $\Gamma \subset \boldsymbol{R}^n$ equipped with a pseudogroup structure is called an ε-lattice of radius $R = R(\Gamma) = \max_{\gamma \in \Gamma} \|\gamma\|$ if the origin in $\boldsymbol{R}^n$ serves as the identity element in Γ, the product $\alpha\beta$ is defined for $\alpha, \beta \in \Gamma$ with $\|\alpha\| + \|\beta\| \leq \frac{1}{2}R$, and $\|\alpha\beta - \alpha - \beta\| \leq \varepsilon\|\alpha\|\,\|\beta\|$. Here $\|\;\|$ means the norm in $\boldsymbol{R}^n$ but as a function on Γ it may not satisfy the conditions in 5.5, and we do not suppose that Γ (as a pseudogroup) has any norm at all. Notice also that $\Gamma \subset \boldsymbol{R}^n$ is not necessarily symmetric: $\gamma^{-1} \neq -\gamma$.

Example. Let $m: \Gamma \to M(n)$ be an ε-homomorphism as in 6.4 with $\|r(\gamma)\| \leq \nu\,\|\gamma\|$, $\gamma \in \Gamma$, and let the map $t: \Gamma \to \boldsymbol{R}^n$ be injective. Then its image is an ε-lattice with $\varepsilon' \leq (\varepsilon + \nu)\exp(n + 10)$.

6.8. For an ε-lattice $\Gamma \subset \boldsymbol{R}^n$ we call the system of generators $\gamma_1, \cdots, \gamma_k \in \Gamma$ a normal basis if the following conditions are satisfied:

1. If the commutator $[\gamma, \gamma_i]$, $\gamma \in \Gamma$, $i = 1, \cdots, k$, is defined, then $[\gamma, \gamma_i] \in \Gamma_{i-1}$, where Γ_i is the subpseudogroup generated by $\gamma_1, \cdots, \gamma_i$.
2. If $\|\gamma\| \leq \exp(-ex_2(n))\,R(\Gamma)$, then there exists a unique representation $\gamma = \gamma_1^{m_1}\gamma_2^{m_2}\cdots\gamma_k^{m_k}$.
3. The system of vectors $\gamma_1, \cdots, \gamma_k$ is regular (see 4.6).

6.9. Consider an ε-lattice $\Gamma \subset \boldsymbol{R}^n$ with a normal basis $\gamma_1, \cdots, \gamma_n \in \Gamma$. For $\gamma \in \Gamma$ represented as $\gamma = \gamma_1^{m_1}\cdots\gamma_n^{m_n}$ denote the sum $\gamma = \sum_{i=1}^n m_i\gamma_i$ by $\lambda = \lambda(\gamma)$. A simple calculation shows

$$q^{-1}\|\lambda\| \leq \|\gamma\| \leq q\,\|\lambda\|\,, \text{ with } 1 \leq q \leq 1 + \tau, \tau \geq 0\,,$$
$$ex_5(n + \tau^{-1}) \geq (\varepsilon\,\|\gamma\|)^{-1}\,.$$

If the commutator $[\alpha, \beta] \in \Gamma$ and $\lambda(\alpha)$, $\lambda(\beta)$, $\lambda([\alpha, \beta])$, $\alpha, \beta \in \Gamma$, are defined, then $\|\lambda([\alpha, \beta])\| \leq \varepsilon' \|\lambda(\alpha)\| \|\lambda(\beta)\|$, where $ex_7(n + (\varepsilon')^{-1}) \geq \varepsilon^{-1}$.

6.10. Let $\Gamma \subset \boldsymbol{R}^n$ be an ε-lattice of radius R. If Γ is δ-dense in the ball $B_R \subset \boldsymbol{R}^n$ and $(\varepsilon R + \delta R^{-1})^{-1} \geq ex_6(n)$, then there exists a normal basis $\gamma_1, \cdots, \gamma_n$ in Γ.

Proof. Take a nontrivial $\gamma_1 \in \Gamma \subset \boldsymbol{R}^n$ with minimal norm, and consider $\boldsymbol{R}^{n-1} \subset \boldsymbol{R}^n$ orthogonal to γ_1. Obviously (compare with 6.5) γ_1 belongs to the "center" of Γ. For $\gamma \in \Gamma$ with $\|\gamma\| \leq \frac{1}{3}R$ consider the trajectory $\{\gamma_1^i\gamma\}$, $i = \cdots, -1, 0, 1, \cdots$, as far as it is defined, and take $\tilde{\gamma} \in \{\gamma_1^i\gamma\}$ with the properties: $\langle\tilde{\gamma}, \gamma_1\rangle \geq 0$, $\langle\gamma_1^{-1}\tilde{\gamma}, \gamma_1\rangle < 0$. Such a $\tilde{\gamma}$ exists and it is unique. Denote by $\gamma' \subset \boldsymbol{R}^{n-1}$ the orthogonal projection of $\tilde{\gamma}$ to $\boldsymbol{R}^{n-1}$, and by $\Gamma' \subset \boldsymbol{R}^{n-1}$ the set of all such $\gamma' \in \boldsymbol{R}^{n-1}$. Setting $\gamma_1'\gamma_2' = (\tilde{\gamma}_1\beta_2)'$ we equip Γ' with a pseudogroup structure. It is easy to see that Γ' is ε'-pseudogroup of radius R' where $\varepsilon' \leq 20\varepsilon$, $R' \geq \frac{1}{4}R$.

Now, by induction having constructed the normal basis $\gamma_2', \cdots, \gamma_n' \in \Gamma'$, we take $\gamma_1, \tilde{\gamma}_2, \cdots, \tilde{\gamma}_n$ for the normal basis in Γ, and verfy the properties 1–3 in 6.8 again by an obvious induction.

6.11. Consider an ε-homomorphism $m: \Gamma \to M(n)$ as in 6.4. If $\varepsilon^{-1} \geq ex_2(n + 1)$, rad $\Gamma \geq 10$, then the restriction of $t: \Gamma \to \boldsymbol{R}^n$ to the unit ball $\Gamma_{\rho=1} = \Gamma_1 \subset \Gamma$ is injective, and we identify Γ_1 with the image of that restriction $t: \Gamma_1 \to \boldsymbol{R}^n$.

Let $\Gamma_1 \subset \boldsymbol{R}^n$ be δ-dense in the unit ball $B_1 \subset \boldsymbol{R}^n$ where $(\delta + \varepsilon)^{-1} \geq ex_{80}(n)$. Then there exists a subpseudogroup $N_1 \subset \Gamma_1$ with the following properties:

1. N_1 is δ'-dense in B, with $\delta' \leq ex_4(n)\delta$.
2. If $\gamma \in N$, then $\|r(\gamma)\| \leq \nu \|\gamma\|$ where $\exp_7(n + \nu^{-1}) = (\varepsilon + \delta)^{-1}$.
3. If $\|r(\gamma)\| \leq \exp(-ex_4(n))$, $\gamma \in \Gamma_1$, then $\gamma \in N$. (Notice that $\theta > \nu$.)
4. Both pseudogroups Γ_1 and N_1 are injective; the group $\pi(N_1) \subset \pi(\Gamma_1)$ is a maximal nilpotent subgroup and the maximal invariant nilpotent subgroup at the same time; $\pi(N_1)$ has no torsion, rank $(\pi(N_1)) = n$ and ord $(\pi(\Gamma_1)/\pi(N_1)) \leq ex_3(n)$.

Proof. Take the ball $\Gamma_\rho \subset \Gamma$ with $\rho = \exp(-ex_{40}(n))$, and generate N_1 by the intersection $\Gamma_\rho \cap r^{-1}(B_\theta)$, $B_\theta \in O(n)$, $\theta = \exp(-ex_4(n))$.

From 6.2 it follows that N_1 is δ''-dense in Γ_1 with $\delta'' = \exp(-ex_{20}(n))$, and properties 2 and 3 for $\gamma \in \Gamma_\rho$ follow from 6.3, 6.5, 6.6. Property 2 shows that $\Gamma_\rho \subset \boldsymbol{R}^n$ is an ε'-lattice, and ε' is small enough to apply 6.10 (see the example in 6.7) and to construct a normal basis in N_ρ. The existence of the normal basis, together with 6.6, 5.7 and properties 2, 3, yields property 4 with the exception of the last inequality, but that inequality is reduced now to the following obvious fact:

If a maximal nilpotent subgroup $N \subset \pi$ is invariant and has no torsion, rank $(N) = n$ and the group $G = \pi/N$ is finite, then ord $(G) \leq ex_3(n)$.

Noticing that $\pi(\Gamma_\rho) = \pi(\Gamma_1)$ and $\pi(N_\rho) = \pi(N_1)$ we extend all properties of Γ_ρ to Γ_1, again using 6.6. Notice in the end that the inequality ord $(\pi(\Gamma_1)/\pi(N))$ $\leq ex_3(n)$ yields property 1 with $\delta' < \delta''$.

7. The proof of the main theorem

7.1. We return now to the manifold V with a fixed point $v_0 \in V$ (see 5.6). We identify the tangent space of V at v_0 with $\boldsymbol{R}^n$, and denote the linear and the affine holonomy maps by $r: \Omega \to O(n)$ and $m: \Omega \to M(n)$ respectively

Consider a contractable loop $w \in \Omega$, $w: [0, 1] \to V$ and a deformation $w_t: [0, 1] \to V$, with $w_t \in \Omega$, $t \in [0, 1]$, $w_{t=0} = w$ and $w_{t=1}$ the constant map. The family w_t can be viewed as a map of a 2-dimensional disk to V. Denote by S the area of that map and denote by L the maximum of the lengths of w_t, $t \in [0, 1]$.

7.2. From $|R(x, y)z| \leq 2 \cdot c(V) \cdot |x \wedge y| \cdot |z|$ for the curvature tensor and assuming $c(V) \leq \varepsilon$ we have

$$\|r(w)\| \leq 2 \cdot \varepsilon \cdot S ,$$
$$\|m(w)\| \leq L \cdot (e^{2\varepsilon S} - 1) + 2\varepsilon S .$$

Together with simple comparison arguments (see [3]) it yields:

7.3. If $c(V) \leq 10^{-10}\varepsilon$, $0 \leq \varepsilon \leq 1$, then the restrictions of the maps r and m to the local fundamental pseudogroup $\Gamma = \pi_\rho$, $\rho \leq 10$ (see 5.6) are ε-homomorphisms, m enjoys the properties from 6.4, and the image of $t: \Gamma \to \boldsymbol{R}^n$ is δ-dense in $B_\rho \subset \boldsymbol{R}^n$ with $\delta \leq 2d(V)$.

7.4. Now everything is ready for the proof of 1.4. We can suppose that $(d(V) + c(V))^{-1} \geq ex_{199}(n)$, and can apply 6.11 to $\Gamma = \pi_\rho$ because of 7.3. This gives (a) and (b) of 1.4.

Take N_1 as in 6.11, and realize $\pi(N_1)$ as a uniform discrete subgroup in a nilpotent Lie group L. Take in $N_1 \subset \Gamma_1 \subset \boldsymbol{R}^n$ (see 6.11) (viewed as an ε-lattice) a normal basis $\gamma_1, \cdots, \gamma_n$, and identify $\pi(N_1)$ with the lattice $\Lambda \subset \boldsymbol{R}^n$ spanned by $\gamma_1, \cdots, \gamma_n$, matching $\gamma = \gamma_1^{m_1} \cdots \gamma_n^{m_n}$ to $\lambda = \sum_{i=1}^n m_i \gamma_i$.

Now equip L with the metric associated with that basis (see 4.7), and consider the map f from $N = \pi(N_1) \subset L$ to the universal covering $(\tilde{V}, \tilde{v}_0)$ of (V, v_0), given by $f(\gamma) = \gamma(\tilde{v}_0)$. ($N$ lies in $\pi_1(V, v_0)$ and so acts in $\tilde{V}$.) Applying 4.8 and 6.9 we conclude that f is an R-restricted q-isometry satisfying all properties of 3.3 (L corresponds to X_1 in 3.3, $\tilde{V}$ to X_2, N to Δ_1, and $\operatorname{Im}(f)$ to Δ_2), and applying 3.3, 3.4 we construct the diffeomorphism $F: L \to \tilde{V}$ commuting with the action of N and so inducing the diffeomorphism of L/N to $\tilde{V}/N$.

8. Appendix: The proof of the Margulis theorem

8.1. The Margulis lemma follows (up to ex_i-nonsense) from 7.3, 6.2, and 6.5. To prove the theorem we need two obvious facts about $\pi = \pi(V, v_0)$ for $c^+(V) < 0$.

8.2. A. Every nilpotent subgroup of π is cyclic.

B. For every cyclic subgroup $N \subset \pi$ there exists an $\alpha \in \pi$ such that $\|\alpha\nu\alpha^{-1}\| \geq 1$, $\nu \in N$, (about $\|\ \|$ see 2.2).

8.3. Now take the shortest $\gamma \in \pi_1(V, v_0)$. If $\|\gamma\|^{-1} \leq ex_2(n)$, then the injectivity radius at $v_0 \in V$ satisfies $\operatorname{Rad}(V, v_0) \geq \frac{1}{2}(ex_2(n))^{-1}$. This yields the Margulis theorem. Otherwise we take the maximal cyclic subgroup $N \subset \pi$ with $\gamma \in N$ and $\alpha \in \pi$ as in 8.2B. Realize α by a loop: $w\colon [0, 1] \to V$, and for $\nu \in N$ denote by ν_t, $t \in [0, 1]$, the shortest loop at the point $w(t) \in V$ homotopic to the loop $w_{|[0,t]}^{-1} \circ \tilde{\nu} \circ w_{|[0,t]}$, where $w_{[0,t]}\colon [0, t] \to V$ is the restriction of w and $\tilde{\nu}$ is the geodesic loop at v_0 realizing ν. By continuity there is a $t_0 \in [0, 1]$ such that $\min_{\nu \in N} (\|\nu_{t_0}\|) = (ex_2(n))^{-1}$. Using the Margulis lemma and 8.2A we conclude that at the point $w(t_0) \in V$ the length of any geodesic loop is at least $(ex_2(n))^{-1}$, and the proof is finished.

Those arguments (up to minor details) are due to Margulis, and for the homogenous case to Kazdan and Margulis (see [9]).

Bibliography

[1] J. Cheeger, *Finiteness theorems for Riemannian manifolds,* Amer. J. Math. **92** (1970) 61–74.

[2] J. Cheeger, *Pinching theorems for a certain class of Riemannian manifolds,* Amer. J. Math. **91** (1969) 807–834.

[3] J. Cheeger & D. Ebin, *Comparison theorems in Riemannian geometry,* North-Holland, Amsterdam, 1975.

[4] D. Gromoll, W. Klingenberg & W. Meyer, R*iemannsche geometries im Grossen,* Lecture Notes in Math. Vol. 55, Springer, Berlin, 1968.

[5] M. Gromov, *The geometrical effects of the codimension* 2 *surgery,* to appear.

[6] ——, *Manifolds of negative curvature*, J. Differential Geometry **13** (1978) 223–230.

[7] K. Grove, H. Karcher & E. Ruh, *Group actions and curvature,* Invent. Math. **23** (1974) 31–48.

[8] E. Heintze, *Mannigfaltigkeiten negativer Krummung,* Preprint, Bonn, 1976.

[9] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Ergebnisse der Math., Springer, Berlin, 1972.

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