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HARMONIC AND RELATIVELY AFFINE MAPPINGS

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The theory of harmonic mappings of a Riemannian space into another has been initiated by Eells and Sampson [2] and studied by Chern [1], Goldberg [1], [3], T. Ishihara [3], [5] and others.

In this paper, we study projective and affine mappings of a manifold with symmetric affine connection into another and harmonic and relatively affine mappings of a Riemannian space into another.

1. Differentiable mappings of a manifold with symmetric affine connection into another

Let (M, ∇) be a manifold of dimension n with symmetric affine connection ∇, and $(N, \bar{\nabla})$ a manifold of dimension p with symmetric affine connection $\bar{\nabla}$, where $n, p \geq 2$. Let there be given a differentiable mapping $f: M \to N$ which we denote sometimes by $f: (M, \nabla) \to (N, \bar{\nabla})$. Manifolds, mappings and geometric objects which we discuss in this paper are assumed to be of differentiability class C^∞. Take coordinate neighborhoods $\{U; x^h\}$ of M and $\{\bar{U}, y^\alpha\}$ of N in such a way that $f(U) \subset \bar{U}$, where $(x^h) = (x^1, x^2, \cdots, x^n)$ and $(y^\alpha) = (y^{\bar{1}}, y^{\bar{2}}, \cdots, y^{\bar{p}})$ are local coordinates of M and N respectively. The indices $h, i, j, k, l, m, r, s, t$ run over the range $\{1, 2, \cdots, n\}$, and the indices $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu$ the range $\{\bar{1}, \bar{2}, \cdots, \bar{p}\}$. The summation convention will be used with respect to these two systems of indices. Suppose that $f: (M, \nabla) \to (N, \bar{\nabla})$ is represented by equations

$$y^\alpha = y^\alpha(x^1, x^2, \cdots, x^n) \tag{1.1}$$

with respect to $\{U, x^h\}$ and $\{\bar{U}, y^\alpha\}$. We put

$$A_i^{\ \alpha} = \partial_i y^\alpha(x^1, x^2, \cdots, x^n) \ , \tag{1.2}$$

where $\partial_i = \partial/\partial x^i$. Then the differential df of the mapping f is represented by the matrix $(A_i^{\ \alpha})$ with respect to the local coordinates (x^h) and (y^α) of M and N.

When a function ρ, local or global, is given in N, throughout the paper we shall identify ρ with the function $\rho \circ f$ induced in M. We denote by Γ_{ji}^h the

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components of the affine connection ∇ in M, and by $\Gamma^{\alpha}_{\gamma\beta}$ those of the affine connection $\bar{\nabla}$ in N.

In this and the next sections, X, Y and Z denote arbitrary vector fields in M with local expressions $X = X^h\partial_h$, $Y = Y^h\partial_h$ and $Z = Z^h\partial_h$ respectively. Then $(A_i{}^\alpha X^i)\partial_\alpha$, where $\partial_\alpha = \partial/\partial y^\alpha$, is the local expression of the vector field $(df)X$ defined along $f(M)$. If we put in U

$$(1.3) \qquad A_{ji}{}^\alpha = \nabla_j A_i{}^\alpha ,$$

where

$$(1.4) \qquad \nabla_j A_i{}^\alpha = \partial_j A_i{}^\alpha + \Gamma^\alpha_{\gamma\beta} A_j{}^\gamma A_i{}^\beta - \Gamma^h_{ji} A_h{}^\alpha ,$$

then $(A_{ji}{}^\alpha X^j Y^i)\partial_\alpha$ is the local expression of a vector field B defined along $f(M)$, and $A_{ji}{}^\alpha = A_{ij}{}^\alpha$.

Consider a curve $\gamma: I \to M$ in M, I being an interval, and denote by $\bar{\gamma} = f \circ \gamma: I \to N$ the image of γ by f. When γ is locally represented by $x^h = x^h(t)$, t being a parameter belonging to I, $\bar{\gamma}$ is so by $y^\alpha = y^\alpha(x^h(t))$. If γ satisfies

$$\frac{d^2x^h}{dt^2} + \Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt} = \alpha(t)\frac{dx^h}{dt}$$

with a certain function $\alpha(t)$ of t, then γ is called a *path* of (M, ∇). It is easily seen that the above equations can be reduced to

$$\frac{d^2x^h}{dt^2} + \Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt} = 0$$

by a suitable change of the parameter t. In this case γ is called a path with *affine parameter* t. A path in N and the affine parameter on this path will be similarly defined.

Now, using $y^\alpha = y^\alpha(x^h(t))$, (1.3) and (1.4), we find

$$(1.5) \quad \frac{d^2y^\alpha}{dt^2} + \Gamma^\alpha_{\gamma\beta}\frac{dy^\gamma}{dt}\frac{dy^\beta}{dt} = A_h{}^\alpha\left(\frac{d^2x^h}{dt^2} + \Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt}\right) + A_{ji}{}^\alpha\frac{dx^j}{dt}\frac{dx^i}{dt} .$$

We assume that an arbitrary path in (M, ∇) is mapped by f into a path in $(N, \bar{\nabla})$. Such a mapping f is said to be *projective*. Under this assumption, we have from (1.5)

$$\beta(t)\frac{dy^\alpha}{dt} = A_{ji}{}^\alpha\frac{dx^j}{dt}\frac{dx^i}{dt}$$

for any path $\gamma: x^h = x^h(t)$ in (M, ∇), $\beta(t)$ being a certain function of t. Thus, γ being arbitrary, we find $\beta A_h{}^\alpha \xi^h = A_{ji}{}^\alpha \xi^j \xi^i$ for any direction $\xi = \xi^h\partial_h$ at any point of M, from which we conclude that

$$(1.6) \qquad A_{ji}{}^{\alpha} = p_j A_i{}^{\alpha} + p_i A_j{}^{\alpha}$$

for some local functions p_i in U, which are the components of a 1-form in M. The converse being evident, we have

Proposition 1.1. *In order for a mapping $f: (M, \nabla) \to (N, \bar{\nabla})$ to be projective, it is necessary and sufficient that $A_{ji}{}^{\alpha}$ has the form* (1.6).

We next assume that an arbitrary path in (M, ∇) is mapped by f into a path in $(N, \bar{\nabla})$ with the affine parameter preserved. Such a mapping f is said to be *affine*. Under this assumption, we have from (1.5)

$$(1.5) \qquad A_{ji}{}^{\alpha} \frac{dx^j}{dt} \frac{dx^i}{dt} = 0$$

for any path $\gamma: x^h = x^h(t)$ in (M, ∇). Thus, γ being arbitrary, we have $A_{ji}{}^{\alpha}\xi^j\xi^i = 0$ for any direction $\xi = \xi^h \partial_h$ at any point of M, from which we conclude that $A_{ji}{}^{\alpha} = 0$. The converse being evident, we have

Proposition 1.2. *In order for a mapping $f: (M, \nabla) \to (N, \bar{\nabla})$ to be affine, it is necessary and sufficient that $A_{ji}{}^{\alpha} = 0$.*

2. Differentiable mapping of a Riemannian space into another

Let (M, g) and $(N, \bar{g})$ be Riemannian spaces of dimensions n and p respectively. Let there be given a mapping $f: M \to N$ denoted sometimes by $f: (M, g) \to (N, g)$. We denote by g_{ji} the components of the Riemannian metric g in M, and by $g_{\gamma\beta}$ those of the Riemannian metric $\bar{g}$ in N. The Christoffel symbols formed with g_{ji} and $g_{\gamma\beta}$ are denoted by $\left\{ {h \atop ji} \right\}$ and $\left\{ {\alpha \atop \gamma\beta} \right\}$ respectively. Thus, denoting by ∇ the affine connection determined by $\left\{ {h \atop ji} \right\}$ and by $\bar{\nabla}$ that determined by $\left\{ {\alpha \atop \gamma\beta} \right\}$, we can regard f as $f: (M, \nabla) \to (N, \bar{\nabla})$.

If we put

$$(2.1) \qquad g^*_{ji} = g_{\gamma\beta} A_j{}^{\gamma} A_i{}^{\beta} ,$$

then g^*_{ji} are the components of the tensor $g^* = f^*\bar{g}$ induced in M from $\bar{g}$ by f. For $g^* = \rho g$, $f: (M, g) \to (N, \bar{g})$ is said to be *conformal*, *homothetic* or *isometric* according as the function ρ is positive, constant or equal to 1.

Differentiating (2.1) covariantly, we find

$$(2.2) \qquad \nabla_k g^*_{ji} = D_{kji} + D_{kij} ,$$

where we have put

$$D_{kji} = A_{kj}{}^{\gamma} A_i{}^{\beta} g_{\gamma\beta} . \tag{2.3}$$

Changing indices in (2.2), we obtain

$$\nabla_j g^*_{ki} = D_{jki} + D_{jik} , \tag{2.4}$$

$$\nabla_i g^*_{kj} = D_{ikj} + D_{ijk} . \tag{2.5}$$

Forming (2.2) + (2.4) − (2.5), we find

$$D_{kji} = \tfrac{1}{2}(\nabla_k g^*_{ji} + \nabla_j g^*_{ki} - \nabla_i g^*_{kj}) , \tag{2.6}$$

where we have used $D_{kji} = D_{jki}$ which is a direct consequence of $A_{kj}{}^{\alpha} = A_{jk}{}^{\alpha}$. When $\nabla g^* = 0$, that is, when $\nabla_k g^*_{ji} = 0$ is satisfied, $f: (M, g) \to (N, \bar{g})$ is said to be *relatively affine* (see [4]). Since we can see from (2.2) and (2.6) that $\nabla_k g^*_{ji} = 0$ and $D_{kji} = 0$ are equivalent, we now have

Proposition 2.1. *A mapping $f: (M, g) \to (N, \bar{g})$ is relatively affine if and only if $D_{kji} = 0$, i.e., if and only if $A_{kj}{}^{\gamma} A_i{}^{\beta} g_{\gamma\beta} = 0$.*

Thus any affine mapping is relatively affine.

The conditions $\nabla g^* = 0$ and $g^* = \rho^2 g$ imply $\rho^2 =$ const. Thus we have

Proposition 2.2. *If a mapping $f: (M, g) \to (N, \bar{g})$ is relatively affine and at the same time conformal, then it is homothetic.*

It is easily seen that the rank of the mapping $f: (M, g) \to (N, \bar{g})$, i.e., the rank of $(A_i{}^{\alpha})$ is equal to the rank of (g^*_{ji}) at each point of M. If the mapping f is relatively affine, then $\nabla g^* = 0$ which implies that g^* is of constant rank m. Therefore, if f is relatively affine, then f is of constant rank m. Assume that f is relatively affine and of constant rank $m < n$, and for any point p of M put $D_p = \{X \in T_p(M) \mid (df)_p X = 0\}$, which is a subspace of dimension $n - m$ in the tangent space $T_p(M)$ of M at p. Therefore the correspondence $D: p \to D_p$ defines an $(n - m)$-dimensional distribution D in M, which is called the *vertical distribution*. It is easily verified that a vector field X belongs to the vertical distribution D if and only if $A_i{}^{\alpha} X^i = 0$, or equivalently, if and only if $g^*_{ji} X^i = 0$. By considering such a vector field X and differentiating $A_i{}^{\alpha} X^i = 0$ covariantly, we then obtain $A_{ji}{}^{\alpha} X^i + A_i{}^{\alpha} \nabla_j X^i = 0$. Thus transvecting $A_k{}^{\beta} g_{\beta\alpha}$ to this equation and using $D_{kji} = 0$, we have $g^*_{ki} \nabla_j X^i = 0$, i.e., $(df)(\nabla_Y X) = 0$. Consequently, we arrive at

Proposition 2.3. *Let a mapping $f: (M, g) \to (N, \bar{g})$ be relatively affine. If M is connected, then f is of constant rank m. When $0 < m < \dim M = n$, the vertical distribution D is of dimension $n - m$ and parallel.*

As a corollary to Proposition 2.3, we have

Proposition 2.4. *Let $f: (M, g) \to (N, \bar{g})$ be relatively affine. If (M, g) is a connected and irreducible Riemannian space, then f is either of rank $n(=\dim M)$ or a constant mapping.*

We now put

(2.7) $$A^\alpha = g^{ji}A_{ji}{}^\alpha\ ,$$

where $(g^{ji}) = (g_{ji})^{-1}$. Then the vector field T with components A^α defined along $f(M)$ is called the *tension field* of the mapping $f\colon (M, g) \to (N, \bar{g})$. It is well known that $f\colon (M, g) \to (N, \bar{g})$ is harmonic if and only if $T = 0$, i.e., if and only if $A^\alpha = 0$ (see [2]).

Consider the divergence of the vector field with local expression $(g^{hi}A_i{}^\gamma A^\beta g_{\gamma\beta})\partial_h$ in M. We then obtain

$$\nabla_l(g^{li}A_i{}^\gamma A^\beta g_{\gamma\beta}) = A^\gamma A^\beta g_{\gamma\beta} + A_i{}^\gamma(\nabla_l A^\beta)g^{li}g_{\gamma\beta}\ ,$$

where we have put

(2.8) $$\nabla_i A^\alpha = \partial_i A^\alpha + \left\{ {\alpha \atop \gamma\beta} \right\} A_i{}^\gamma A^\beta\ .$$

Thus we have

Proposition 2.5. *A mapping $f\colon (M, g) \to (N, \bar{g})$ is harmonic if M is compact and $\nabla T = 0$ which means $\nabla_i A^\alpha = 0$.*

3. Laplacian of $\|df\|^2$

We shall compute Laplacian of $\|df\|^2$ for later use. We now put in U

(3.1) $$\nabla_k A_{ji}{}^\alpha = \partial_k A_{ji}{}^\alpha + \left\{ {\alpha \atop \gamma\beta} \right\} A_k{}^\gamma A_{ji}{}^\beta - \left\{ {m \atop kj} \right\} A_{mi}{}^\alpha - \left\{ {m \atop ki} \right\} A_{jm}{}^\alpha\ .$$

Then $(\nabla_k A_{ji}{}^\alpha X^k Y^j Z^i)\partial_\alpha$ is the local expression of a vector field defined along $f(M)$. Taking account of (1.3), (1.4) and (3.1), we obtain the following formula of Ricci-type:

(3.2) $$\nabla_k\nabla_j A_i{}^\alpha - \nabla_j\nabla_k A_i{}^\alpha = R_{\delta\gamma\beta}{}^\alpha A_k{}^\delta A_j{}^\gamma A_i{}^\beta - R_{kji}{}^h A_h{}^\alpha\ ,$$

where $R_{\delta\gamma\beta}{}^\alpha$ and $R_{kji}{}^h$ are the components of the curvature tensors of $\bar{g}$ and g respectively. We are now going to compute Laplacian of $\|df\|^2$. We then have

(3.3) $$\begin{aligned} \tfrac{1}{2}\Delta\|df\|^2 &= \tfrac{1}{2}g^{lk}\nabla_l\nabla_k(A_j{}^\beta A_i{}^\alpha g^{ji}g_{\beta\alpha}) \\ &= g^{lk}(\nabla_l\nabla_k A_j{}^\beta)A_i{}^\alpha g^{ji}g_{\beta\alpha} + \|B\|^2\ , \end{aligned}$$

where

(3.4) $$\|B\|^2 = A_{lk}{}^\beta A_{ji}{}^\alpha g^{lj}g^{ki}g_{\beta\alpha}\ .$$

Thus using (3.2) and putting $R_{\delta\gamma\beta\alpha} = R_{\delta\gamma\beta}{}^\lambda g_{\lambda\alpha}$, from (3.3) we obtain

$$
\begin{aligned}
(3.5) \qquad \tfrac{1}{2}\Delta\|df\|^2 &= (\nabla_j A^\beta)A_i{}^\alpha g^{ji} g_{\beta\alpha} + \|B\|^2 \\
&\quad + R_{\delta\gamma\beta\alpha} A_l{}^\delta A_i{}^\gamma A_k{}^\beta A_j{}^\alpha g^{lk} g^{ji} + R_i{}^h g^*_{hj} g^{ij} ,
\end{aligned}
$$

where $R_j{}^h = R_{jk} g^{hk}$ are the mixed components of the Ricci tensor of (M, g) and $\nabla_j A^\alpha$ are defined by (2.8). Thus taking account of (3.5) we have

Lemma 3.1. *For a harmonic mapping $f\colon (M, g) \to (N, \bar{g})$, we have*

$$(3.6) \qquad \tfrac{1}{2}\Delta\|df\|^2 = \|B\|^2 + R_{\delta\gamma\beta\alpha} A_l{}^\delta A_i{}^\gamma A_k{}^\beta A_j{}^\alpha g^{lk} g^{ji} + R_j{}^h g^*_{hi} g^{ji} .$$

Let $e_{(1)}, \cdots, e_{(n)}$ be n orthonormal vectors at each point of (M, g) such that

$$(3.7) \qquad g_{ji} = e_{(1)j} e_{(1)i} + \cdots + e_{(n)j} e_{(n)i} ,$$

$$(3.8) \qquad g^*_{ji} = \lambda_1 e_{(1)j} e_{(1)i} + \cdots + \lambda_n e_{(n)j} e_{(n)i} ,$$

where $e_{(s)}{}^h$ are the components of $e_{(r)}$, and $e_{(r)i} = e_{(r)}{}^h g_{hi}$. Then we find

$$(3.9) \qquad \lambda_1, \cdots, \lambda_n \geq 0 .$$

If we now put $\bar{e}_{(s)} = (df)e_{(s)}$, then $\bar{e}_{(s)}$ has components of the form $e_{(s)}{}^\alpha = A_i{}^\alpha e_{(s)}{}^i$. Therefore we get

$$R_{\delta\gamma\beta\alpha} A_l{}^\delta A_i{}^\gamma A_k{}^\beta A_j{}^\alpha g^{lk} g^{ji} = \sum_{r \neq s} R_{\delta\gamma\beta\alpha} e_{(r)}{}^\delta e_{(s)}{}^\gamma e_{(r)}{}^\beta e_{(s)}{}^\alpha$$

and hence

$$(3.10) \qquad R_{\delta\gamma\beta\alpha} A_l{}^j A_i{}^\gamma A_k{}^\beta A_j{}^\alpha g^{lk} g^{ji} = -\sum_{r \neq s} \bar{\sigma}(\bar{e}_{(r)}, \bar{e}_{(s)}) \lambda_r \lambda_s ,$$

where $\bar{\sigma}(\bar{X}, \bar{Y})$ denotes the sectional curvature of $(N, \bar{g})$, $\bar{X}$ and $\bar{Y}$ being any two linear independent vectors at any point of $(N, \bar{g})$.

On the other hand, we can easily find

$$(3.11) \qquad \sum_{r \neq s} \lambda_r \lambda_s = -\sum_s (\lambda_s - \tilde{\lambda})^2 + n(n-1)\tilde{\lambda}^2 ,$$

where we have put

$$(3.12) \qquad \tilde{\lambda} = \frac{1}{n}(\lambda_1 + \cdots + \lambda_n) \geq 0 .$$

$n\tilde{\lambda}$ is sometimes denoted by

$$(3.13) \qquad \text{Trace}\, g^* = n\tilde{\lambda} = g^*_{ji} g^{ji} \geq 0 .$$

We here consider the following condition:

(C) *There is a constant c such that* $c \geq \bar{\sigma}(\bar{X}, \bar{Y})$ *for any two linearly independent vectors* $\bar{X}$ *and* $\bar{Y}$ *at any point of* $(N, \bar{g})$.

Then using (3.10) and (3.11) we obtain

$$R_{\delta\gamma\beta\alpha}A_l{}^\delta A_i{}^\gamma A_k{}^\beta A_j{}^\alpha g^{lk}g^{ji} \geq c \sum_s (\lambda_s - \tilde{\lambda})^2 - n(n-1)c\tilde{\lambda}^2 \,, \tag{3.14}$$

when condition (C) is satisfied.

Next, using (3.7) and (3.8), we have

$$R_j{}^h g^*_{hi} g^{ji} = \lambda_1(R_{ji}e_{(1)}{}^j e_{(1)}{}^i) + \cdots + \lambda_n(R_{ji}e_{(n)}{}^j e_{(n)}{}^i) \,, \tag{3.15}$$

where $R_{ji} = R_j{}^h g_{hi}$ are the components of the Ricci tensor of (M, g). Assume M to be compact and put

$$\frac{r}{n} = \min R_{ji}A^j A^i \,, \tag{3.16}$$

where $A = A^h\partial_h$ runs over the unit sphere bundle over (M, g). Then by (3.15) and (3.16) we find

$$R_j{}^h g^*_{hi} g^{ji} \geq r\tilde{\lambda} \,, \tag{3.17}$$

and use of (3.14), (3.17) and Lemma 3.1 thus gives

Lemma 3.2. *For a harmonic mapping* $f\colon (M, g) \to (N, \bar{g})$ *we have*

$$\tfrac{1}{2}\Delta \|df\|^2 \geq \|B\|^2 + c \sum_s (\lambda_s - \tilde{\lambda})^2 + n(n-1)c\tilde{\lambda}^2 + r\tilde{\lambda} \,, \tag{3.18}$$

when M is compact and condition (C) *is satisfied.*

4. Theorems

First we shall give some remarks. If $\|B\|^2 = 0$, then we have $B = 0$ which means that $f\colon (M, g) \to (N, \bar{g})$ is affine. If $\lambda_1 = \cdots = \lambda_n = \tilde{\lambda}$, then $g^* = \tilde{\lambda}g$, which means that $f\colon (M, g) \to (N, \bar{g})$ is conformal when $\tilde{\lambda} \neq 0$ everywhere and that f is a constant mapping when $\tilde{\lambda} = 0$ everywhere and M is connected. Thus, if $\|B\|^2 = 0$ and $\lambda_1 = \cdots = \lambda_n$, and M is connected, then f is a homothetic or constant mapping, because of Proposition 2.2. Consequently from Lemma 3.2 we have

Theorem 4.1. *Let* $f\colon (M, g) \to (N, \bar{g})$ *be a harmonic mapping of a Riemannian space* (M, g) *of dimension n into another Riemannian space* $(N, \bar{g})$, *and assume M to be compact and connected. Then*

(i) $f\colon (M, g) \to (N, \bar{g})$ *is a constant or homothetic mapping of rank n everywhere, if* (M, g) *has positive definite Ricci tensor and there is a constant* $c > 0$ *such that* $c \geq \bar{\sigma}$, $\bar{\sigma}$ *being the sectional curvature of* $(N, \bar{g})$, *and the following condition is satified*:

$$(\mathrm{A}_1) \qquad \operatorname{Trace} g^* \le \frac{r}{(n-1)c}\,,$$

where r is defined by (3.16);

(ii) $f\colon (M, g) \to (N, \bar g)$ *is a constant mapping, if the following condition is satisfied*:

$$(\mathrm{A}_2) \qquad \bar\sigma \le 0 \text{ and } (M, g) \text{ has positive definite Ricci tensor.}$$

In case (i) of Theorem 4.1, if $\dim M = n = \dim N$, then f is a regular and homothetic mapping of (M, g) onto a connected component of $(N, \bar g)$; if $\dim M = n < \dim N$, then $f\colon (M, g^*) \to (N, \bar g)$ is an isometric immersion, which is totally geodesic, and $g^* = \rho^2 g$ with constant $\rho^2 > 0$. Thus, in case (i) of Theorem 4.1 if $(N, \bar g)$ is a sphere $(S^p, \bar g_0)$ of constant curvature, then (M, g) is necessarily a sphere (S^n, g_0) of constant curvature.

We now assume that $r = 0$ and $\bar\sigma \le 0$. Using (3.10) and (3.17), from Lemma 3.1 we have

$$\tfrac{1}{2}\Delta \|df\|^2 \ge \|B\|^2 + R_j{}^h g^*_{hi} g^{ji} \ge \|B\|^2\,.$$

Thus, if M is compact, then $R_j{}^h g^*_{hi} g^{ji} = 0$, which and (3.15) imply

$$(4.1) \qquad \lambda_1(R_{ji}e_{(1)}{}^j e_{(1)}{}^i) + \cdots + \lambda_n(R_{ji}e_{(n)}{}^j e_{(n)}{}^i) = 0\,.$$

Hence it follows from (4.1) that

$$(4.2) \qquad \lambda_s(R_{ji}e_{(s)}{}^j e_{(s)}{}^i) = 0\,, \qquad (s = 1, 2, \cdots, n),$$

since $\lambda_s(R_{ji}e_{(s)}{}^j e_{(s)}{}^i) \ge 0$. (4.2) means that the Ricci tensor of (M, g) is of rank $\le n - m$ when f is of rank m everywhere. Consequently taking account of Proposition 2.3 we obtain

Theorem 4.2. *Let* $f\colon (M, g) \to (N, \bar g)$ *be a harmonic mapping of a Riemannian space* (M, g) *into another Riemannian space* $(N, \bar g)$, *and assume M to be compact and connected. Then either f is an affine mapping of constant rank* $m \ge 0$ *and the Ricci tensor of* (M, g) *is of rank* $\le n - m$, *or f is a constant mapping, if the following condition is satisfied*:

(A₃) $\bar\sigma \le 0$, *and* (M, g) *has positive semi-definite Ricci tensor and* $r = 0$, *where r is defined by* (3.16). *In this case,* Trace g^* *is necessarily constant.*

In Theorem 4.2, if (M, g) is connected and irreducible, then f is a constant mapping because of Proposition 2.4; if f is of rank n everywhere and $(N, \bar g)$ is a flat torus, then (M, g) is also a flat torus, and the isometric immersion $f\colon (M, g^*) \to (N, \bar g)$ is totally geodesic when $\dim M < \dim N$.

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