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COMPLETE CONVEX HYPERSURFACES OF A HILBERT SPACE

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1. Statement of the results

A complete *convex hypersurface* M of a Hilbert space H is a one-codimensional C^∞ submanifold of H, which is complete as a metric subspace of H such that $M = \partial K$ is the boundary of a closed convex set K with nonvoid interior. For each $p \in M$ let $\nu(p)$ be the unit normal vector which points to the interior of K. In this way we define the *Gauss map* $\nu : M \to \Sigma$ from M into the unit sphere Σ of H. This is a C^∞ map and its derivative at each point $p \in M$ is self-adjoint. We say that M *bounds a half-line* if there exists a half-line $\{p + tv ; t \geq 0\}$ contained in the interior of K.

In the case where M is a complete convex hypersurface of a Euclidean n-space R^n, the condition for M to bound a half-line is equivalent to that for M to be unbounded. In § 2 we give an example of an unbounded, positively curved, convex hypersurface which does not bound any half-line. In Theorem A we characterize the three possible cases of boundness (bounded, unbounded and bounding a half-line, unbounded and bounding no half-line) in terms of the Gauss map of M. In [5] H. H. Wu proved that if M is an unbounded complete convex hypersurface of R^n such that at a point $p \in M$ the sectional curvatures are all positive, then M is a pseudograph over one of its tangent hyperplanes (see definition below). Our example shows that this is not true in the infinite dimensional case. Theorem B gives a necessary and sufficient condition for M to be a pseudograph over one of its tangent hyperplanes. Theorem C is the Bonnet-Myers theorem for hypersurfaces of a Hilbert space.

In what follows, by a Hilbert space we mean a separable Hilbert space. As usual, int (A) denotes the interior of A and cl (A) its closure.

Theorem A. *Let M be a complete convex hypersurface of a Hilbert space H. Then:*

(1) *M is bounded if and only if the Gauss map $\nu : M : \to \Sigma$ is onto,*

(2) *M is unbounded and bounds a half-line if and only if the image of the Gauss map is contained in a hemisphere,*

(3) *M is unbounded and does not bound any half-line if and only if the image of the Gauss map is dense and has void interior.*

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Before stating Theorem B, we define what means a pseudograph (cf. [5]). A hypersurface M of a Hilbert space H is a *pseudograph* over the tangent hyperplane F when:

(a) M lies in one of the closed half-spaces determined by F,

(b) M is the graph of a C^∞ function over the int (A), where $A = \pi(M)$, $\pi : H \to F$ being the orthogonal projection,

(c) for every $x \in A -$ int (A), $M \cap \pi^{-1}(x)$ is a closed half-line,

(d) for every hyperplane L above F, $M \cap L$ is bounded.

Theorem B. *Let M be a complete convex hypersurface of a Hilbert space H. Then M is unbounded and* int $(\nu(M)) \neq \emptyset$ *if and only if M is a pseudograph over one of its tangent hyperplanes $TM_p \neq M$.*

Theorem C (*The Bonnet-Myers theorem for Hilbert hypersurface*). *Let M be a complete connected hypersurface of a Hilbert space H. If the sectional curvatures of M are all bounded away from zero (i.e., there exists a $\delta > 0$ such that $K(\sigma) \geq \delta$ for every $p \in M$ and every two-plane section $\sigma \subset TM_p$), then M is bounded, the diameter ρ of M satisfies $\rho \leq \pi\sqrt{\delta}$ and the Gauss map is a diffeomorphism.*

Remark. We can show that if at a point $p \in M$ the sectional curvatures are all bounded away from zero, then the Gauss map is a diffeomorphism on a neighborhood of p. So by combining Theorem B with a result proved by Leo Jonker [4] we have that if M is an unbounded complete hypersurface of a Hilbert space H such that the sectional curvatures of M are nonnegative and all bounded away from zero at a point, then M is a pseudograph over one of its tangent hyperplanes. It also follows that if M is an unbounded complete convex hypersurface which does not bound any half-line, then the sectional curvatures of M are not bounded away from zero at any point of M.

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2. Examples

In this section we give an example of an unbounded, positively curved, convex hypersurface which does not bound any half-line.

(1) Let $A : H \to H$ be a self-adjoint, continous, positive semi-definite operator on the Hilbert space H. Set $f(x) = \langle A(x), x\rangle$, $M = \{x \in H; f(x) = 1\}$ and $K = \{x \in H; f(x) \leq 1\}$. It is clear that $M = \partial K$. The derivative $f'(x)$ is given by $f'(x)\cdot v = 2\langle A(x), v\rangle$, and 1 is a regular value of f. It follows from this that M is a C^∞ (complete) hypersurface. To prove that M is convex take two points x, y in K and consider the segment $\{tx + (1-t)y; 0 \leq t \leq 1\}$. Then

$$\begin{aligned} g(t) &= \langle A(tx + (1-t)y, tx + (1-t)y\rangle \\ &= t^2\langle A(x-y), x-y\rangle + 2t\langle A(x-y), y\rangle + \langle A(y), y\rangle . \end{aligned}$$

Since $g(0) \leq 1$ and $g(1) \leq 1$, we obtain that $g(t) \leq 1$ for $0 \leq t \leq 1$. We can easily see that if A is positive definite, then the equation $g(t) = 1$ has exactly two distincts roots. From this we conclude that *if A is positive definite, then M does not bound any half-line.*

(2) We shall now show that if A is positive definite, then M is positively curved. The gradient of f and x is given by $2A(x)$, and from this it follows that $N(x) = A(x)/\|A(x)\|$, $x \in M$, is a unit normal vector field on M. Let $\{v, w\}$ be an orthogonal set in TM_x. By the Gauss egregium theorem, the sectional curvature on the plane σ generated by $\{v, w\}$ is given by

$$\begin{aligned} K(\sigma) &= \langle N'(x)\cdot v, v\rangle\langle N'(x)\cdot w, w\rangle - \langle N'(x)\cdot v, w\rangle^2 \\ &= (1/\|A(x)\|)^2(\langle A(v), v\rangle\langle A(w), w\rangle - \langle A(v), w\rangle^2) \ . \end{aligned}$$

Since A is self-adjoint and positive definite, by Schwarz inequality this last expression is positive, and we conclude that M is positively curved.

(3) Suppose now that A is positive definite and compact. That is, there exist a complete orthonormal set $\{e_i\}$ and positive real numbers $\{\alpha_i\}$, where $\Sigma\alpha_i < \infty$, such that $A(\Sigma x_i e_i) = \Sigma\alpha_i x_i e_i$ for every $x = \Sigma x_i e_i$ in H. We have that M is unbounded, because the points $(1/\alpha_i)e_i$ belong to M. Since A is posi-tive definite, M is positively curved and does not bound any half-line.

3. Proof of Theorem A

Let M be a convex hypersurface of a Hilbert space. Having the normal vector $\nu(p)$ pointing to the interior of the convex body K of M is equivalent to $\langle\nu(p), x - p\rangle \geq 0$ for every x in K. From this it easily follows that a point $v \in \Sigma$ is a point of $\nu(M)$ if and only if the height function $h_v(x) = \langle v, x\rangle$ assumes its minimum on K at a point $p \in M$. In this case, we have $\nu(p) = v$.

A subset A of Σ is said to be convex if: (a) given two points $x, y \in A, x \neq -y$ implies that the minimal geodesic segment joining x and y is contained in A, (b) given x and $-x$ in A, at least one of the minimal geodesic segment joining x and $-x$ is contained in A. It is not difficult to prove that if A is convex (and closed) in Σ, then the cone $C = \{tx; x \in A, t \geq 0\}$ is convex (and closed) in H. From this and the well known fact that if C is a closed convex set of H then the distance function $\|a - x\|$ (where a is a fixed point in H) assume its minimum on C, we can prove that a closed convex set of Σ is either Σ itself or contained in a hemisphere.

A point $v \in \Sigma$ is called a *pole* if $\nu(M)$ is contained in the hemisphere $E_v = \{x \in \Sigma; \langle v, x\rangle \geq 0\}$. Note that in the above definition we may substitute $\mathrm{cl}(\nu(M))$ by $\nu(M)$.

Lemma 1. *Let M be a convex hypersurface in a Hilbert space H, and K its convex body. A point $v \in \Sigma$ is a pole if and only if given $p \in$* int (K) *the half-line $\{p + tv; t \geq 0\}$ is contained in* int (K).

Proof. First suppose that $\nu(M) \subset E_v$. Let $p \in \operatorname{int}(K)$, and suppose that there exists $t_0 > 0$ such that $q = p + t_0 v \in M$. Since $p \in \operatorname{int}(K)$, we have that $\langle p - q, \nu(q)\rangle > 0$. From this we get that $\langle \nu(q), v\rangle < 0$, which is a contradiction. Suppose now that $\{p + tv;\, t \geq 0\}$ is contained in $\operatorname{int}(K)$ and that there exists a $q \in M$ such that $\langle \nu(q), v\rangle < 0$. Let P be the two-dimensional plane determined by p, q, v. Clearly $P \cap TM_q$ is a line containing q. Let $\{r(s) = q + sw;\, s \in \boldsymbol{R}\}$, $\|w\| = 1$, be this line. Consider the equation $r(s) - v(t) = 0$, where $v(t) = p + tv$. We assert that the above equation has a unique solution (s_0, t_0) with $t_0 > 0$. Indeed, since

$$\langle \nu(q), v\rangle < 0\ ,$$

$\langle \nu(q), w\rangle = 0$, and $p - q$ is in the plane generated by $\{v, w\}$, $\{v, w\}$ are linearly independent and thus there exists a unique (s_0, t_0) such that $p - q = s_0 w - t_0 v$. Moreover, $-t_0 = \langle p - q, \nu(q)\rangle / \langle \nu(q), v\rangle$. This implies $t_0 > 0$, $r(s_0) = v(t_0)$. Since M is convex, we have that $v(t) \notin K$ for $t > t_0$. This contradicts the hypothesis that $v(t) \in \operatorname{int}(K)$ for $t > 0$, and hence the lemma is proved.

Lemma 2. *Let M be a convex hypersurface of a Hilbert space H. If* $\operatorname{int}(\nu(M)) \neq \emptyset$ *and* $v \in \operatorname{int}(\operatorname{cl}(\nu(M)))$, *then the height function h_v is bounded below on M.*

Proof. Let $\exp_v : T\Sigma_v \to \Sigma$ be the exponential map. Let $B_r(-v)$ be a closed ball in Σ of center $-v$ and radius r such that $\operatorname{int}(\nu(M)) - B_r(-v) \neq \emptyset$ and $B_r(-v) \cap \operatorname{int}(\nu(M)) \neq \emptyset$. Set $A = \{z \in S(v);\, \exp_v((\pi - r)z) \in \operatorname{int}(\nu(M))\}$, where $S(v)$ is the unit sphere of $T\Sigma_v$. It is clear that A is a nonvoid open set of $S(v)$. Since $v \in \operatorname{int}(\operatorname{cl}(\nu(M)))$, there exist a real number t, $0 < t < r$, and $z \in A$ such that $\exp_v(-tz) \in \nu(M)$. It follows from this that there exist $\alpha, \beta > 0$ and u, $w \in \nu(M)$ such that $v = \alpha u + \beta w$. By our above remarks the height functions h_u and h_w are bounded below on M, and therefore h_v is also so.

Proof of theorem A. To prove part (1) of the theorem, first suppose that M is bounded. Consider in H the weak topology, that is, the topology generated by the continous functionals of H. Since the convex body K of M is a bounded, (strongly) closed, convex set of H, K is weakly compact (see [3]). Let $v \in \Sigma$ and consider the height function h_v. Since a height function is a continuous functional of H, h_v assumes its minimum on K and, by our previous remarks, we obtain that $v \in \nu(M)$. This proves that the Gauss map is onto. Conversely, suppose that the Gauss map is onto. Then it follows that for each $v \in \Sigma$ the height function h_v assumes its minimum on M. Since $h_{-v} = -h_v$, for every $v \in \Sigma$ the height function h_v is bounded on M. From this it follows that each continuous functional of H is bounded on M. Thus by the uniform boundness theorem [3], M is bounded.

Part (2) of the theorem was proved in Lemma 1. Now we shall prove part (3). By the result proved in [2], $\operatorname{cl}(\nu(M))$ is a convex set of Σ. If $\operatorname{cl}(\nu(M)) \neq \Sigma$, then $\operatorname{cl}(\nu(M))$ is contained in a hemisphere and, by Lemma 1, M bounds a half-line. This contradicts our hypothesis so that we conclude that cl $(\nu(M)) = \Sigma$. Suppose that int $(\nu(M))$ is nonvoid. Then, by Lemma 2 we have that for each $v \in \Sigma$ the height function h_v is bounded below on M and, by the argument used in part (1), M is bounded. This is a contradiction so that we conclude that int $(\nu(M))$ is void. Conversely, if int $(\nu(M))$ is void then, by part (1), M is unbounded; if $cl(\nu(M)) = \Sigma$ then, by part (2), M does not bound any half-line. Hence the theorem is proved.

4. Proof of Theorem B

A linear submanifold of a Hilbert space H is a submanifold of the form $L = \{p + v; v \in F\} = p + F$, where p is a fixed point in H and F is a closed subspace. A hyperplane is a linear submanifold of codimension one. Let M be a hypersurface of a Hilbert space H. We say that a linear submanifold $L = p + F$ intersects M *transversally* if for each $q \in M \cap L$ we have that $TM_q + F = H$. In this case it is known that $S = M \cap L$ is a one-codimensional submanifold of L. If in addition $M = \partial K$ is convex, then $S = \partial(K \cap L)$ is a convex hypersurface of L.

Lemma 3. *Let M be a convex hypersurface of a Hilbert space H. If the set of poles $\mathscr{P}$ and* int (cl $(\nu(M))$) *are both nonvoid sets, then they have nonvoid intersection.*

Proof. Since int (cl $(\nu(M))$) is nonvoid, $\mathscr{P}$ does not contain antipodal points, as this would imply that $\nu(M)$ is contained in an equator of Σ, that is, if v and $-v$ belong to $\mathscr{P}$, then $\nu(M)$ is contained in the equator $\{w \in \Sigma; \langle v, w\rangle = 0\}$ by Lemma 1. Clearly $\mathscr{P}$ is a closed set. It is not difficult to verify that $\mathscr{P}$ is convex. To see this take $v, w \in \Sigma$. Since v is not the antipodal of w, every point ($\neq v, w$) on the minimal geodesic segment joining v and w is of the form $\alpha v + \beta w$, where $\alpha, \beta > 0$. From this it follows that $\mathscr{P}$ is convex. Take $a \in$ int (cl $(\nu(M))$), and let $B_r(a)$ be a closed ball of center a and radius r contained in int (cl $(\nu(M))$). If $w \in \mathscr{P}$, then the length of the shortest of the two geodesic segments joining a and w does not exceed $\frac{1}{2}\pi - r$, and thus there exists $\delta > 0$ such that $\langle a, w\rangle \geq \delta$ for every $w \in \mathscr{P}$. Consider the cone $C = \{tw; t \geq 0, w \in \mathscr{P}\}$. Since $\mathscr{P}$ is closed and convex in Σ, C is closed and convex in H. If $a \in \mathscr{P}$, we have nothing to prove. Thus suppose that $a \notin \mathscr{P}$, and let $b \in C$ be such that $\|b - a\|$ is a minimum of the distance function $f(x) = \|a - x\|^2$ on C. It is not difficult to see that $0 < \|b\| < 1$. We shall now prove that the pole $v = b/\|b\|$ is in the interior of cl $(\nu(M))$. First we shall show that $\langle v, w\rangle \geq \delta$ for every pole w.

Since b is a minimum point for the function $f(x) = \|a - x\|^2$, $x \in C$, we have that $\langle a - b, b\rangle = 0$ and $\langle a - b, w\rangle \leq 0$ for every pole w. To prove this, note that the derivative of the function $g(t) = \|a - tb\|^2$ is zero at the point $t = 1$ since tb is in C for every number $t \geq 0$. This implies $\langle a - b, b\rangle = 0$, so that for every $t, 0 \leq t \leq 1, tb + (1 - t)w$ is in C if w is a pole. From this it follows that the derivative of the function $g(t) = \|a - (tb + (1 - t)w\|^2$ at the point $t = 1$ is nonpositive, and therefore that $\langle a - b, w - b\rangle \leq 0$, which combined with the first equality gives $\langle a - b, w\rangle \leq 0$. Since $\|b\| < 1$, it follows from the previous inequality that $\langle v, w\rangle \geq \langle b, w\rangle \geq \langle a, w\rangle \geq \delta$ for every pole w.

We now use this inequality to prove that v is in int (cl ($\nu(M)$)). Suppose that this is not true. Then take a sequence $\{y_n\}$ such that $\|y_n - v\| < 1/n$ and y_n is not in cl ($\nu(M)$). Consider the convex cone $C = \{tu;\ t \geq 0, u \in \text{cl}\,(\nu(M))\}$, and let $x_n \in C$ be a minimum point for the distance function $\|y_n - x\|^2, x \in C$. By the above argument, we have that $\langle x_n - y_n, y\rangle \geq 0$ for every y in $\nu(M)$. Thus we conclude that $w_n = x_n - y_n/\|x_n - y_n\|$ is a pole. Since $\|x_n\| < 1$, it follows that $\langle w_n, y_n\rangle \leq 0$ and therefore that $\langle w_n, v\rangle = \langle w_n, v - y_n\rangle + \langle w_n, y_n\rangle < 1/n$, which contradicts the inequality $\langle v, w\rangle \geq \delta$ for every pole w. Hence v is in int (cl ($\nu(M)$)).

Proof of Theorem B. Suppose that the convex hypersurface M is unbounded and that int ($\nu(M)$) is nonvoid. By Theorem A the set of poles is nonvoid. It follows from Lemma 3 that there exists a pole v in int (cl ($\nu(M)$)). We shall prove that v lies in int ($\nu(M)$) and that $M \cap L$ is bounded for every hyperplane L perpendicular to v. This will prove parts (a) and (d) in the definition of pseudograph.

Let L be a hyperplane which intersects the interior of the convex body K of M and is perpendicular to v. Note that L intersects M transversally, for otherwise if $p \in M \cap L$ and $TM_p + F \neq H$ (where $L = p + F$) then, since M is of codimension one, $L \subset TM_p$ which contradicts the fact that L intersects the interior of K. Note that $M \cap L$ is nonvoid, for otherwise M would be a hyperplane. Hence $S = M \cap L$ is a convex hypersurface of L.

To prove that S is bounded, we identify L with the subspace perpendicular to v and consider the unit sphere $\Sigma' = \{x \in \Sigma;\ \langle x, v\rangle = 0\}$ of L. Take $w \in \Sigma'$. Since v is in int (cl ($\nu(M)$)), there exists u in int (cl ($\nu(M)$)) such that $u = \alpha v + \beta w$, where $\alpha, \beta > 0$. Consider the height function $h_w(x) = \langle w, x\rangle, x \in S$. Since $\langle v, x\rangle = 0$ for every x in S, $h_u(x) = \beta h_w(x)$ for every x in S. By Lemma 2, h_u is bounded below on M. From this it follows that h_w is bounded below on S for every w in Σ'. By the argument used to prove part (3) of Theorem A, we obtain that S is bounded. Consider the cylinder $C = \{x + tv;\ x \in S, t \in \boldsymbol{R}\}$, and let π be the orthogonal projection on L. By Lemma 1, the part of K below L is contained in C, that is, the closed convex set $K_1 = \{x \in K;\ \langle \pi(x) - x, v\rangle \leq 0\}$ is contained in C. By Lemma 2, there exists a hyperplane L_1 below K, that is, there exists L_1 perpendicular to v such that $\langle \pi_1(x) - x, v\rangle \geq 0$ for every x in K, where π_1 is the orthogonal projection on L_1. It follows from this that K_1 is bounded. Thus the height function h_v assumes its minimum at a point p of K_1. We easily see that p is in M and conclude that $\nu(p) = v$.

We shall now prove that M is a pseudograph on the tangent hyperplane $F = TM_p$. Part (d) of the definition of pseudograph was proved above. It remains to prove part (b) and part (c). Let π be the orthogonal projection on F and set $A = \pi(M)$. It in not difficult to see that $\pi(M) = \pi(K)$ and that $\operatorname{int}(A) = \{x \in A\,;\ \pi^{-1}(x)$ is transversal to $M\} = \{x \in A\,;\ x$ is a regular value of $\pi\colon M \to F\} = \pi(\operatorname{int}(K))$. Take $x \in \operatorname{int}(A)$ and $q \in \pi^{-1}(x) \cap M$. By Lemma 1 and the above remark, the half-line $\{q + tv\,;\ t > 0\}$ is contained in the interior of K. It follows from this that $\pi^{-1}(x) \cap M$ is a unique point, which we shall denote by $s(x)$. Thus $\pi^{-1}(\operatorname{int}(A)) \cap M$ is the graph of the function $f\colon \operatorname{Int}(A) \to R = TM_p^{\perp}$ defined by $f(x) = \langle s(x), v\rangle v$ (by a translation; we may suppose that p is the origin). To verify that f is a C^∞ function we note that $\pi\colon \pi^{-1}(\operatorname{int}(A)) \cap M \to \operatorname{int}(A)$ is the inverse of the function s which takes $x \in \operatorname{int}(A)$ into $\pi^{-1}(x) \cap M$. Since every $x \in \operatorname{int}(A)$ is a regular value of $\pi\colon \pi^{-1}(\operatorname{int}(A)) \to \operatorname{int}(A)$, we conclude, by the inverse function theorem, that s is a C^∞ diffeomorphism.

Now take $x \in A - \operatorname{int}(A)$ and let $q \in \pi^{-1}(x) \cap M$. By Lemma 1, for every point r close to q and in the interior of K the half-line $\{r + tv\,;\ t \geq 0\}$ is contained in the interior of K. It follows from this that the half-line $\{q + tv\,;\ t \geq 0\}$ is contained in K. This half-line does not intersect the interior of K, for otherwise x would be in the interior of A. Thus $\pi^{-1}(x) \cap M = \pi^{-1}(x) \cap K$. Since $\pi^{-1}(x) \cap K$ is a closed convex set which contains the half-line $\{q + tv\,;\ t \geq 0\}$ and is contained in the half-line $\{x + tv\,;\ t \geq 0\}$, we conclude that $\pi^{-1}(x) \cap M$ is a closed half-line, so that we have proved the first part of the theorem.

Conversely, suppose that the convex hypersurface M is a pseudograph over the tangent hyperplane $TM_p \neq M$. Set $v = \nu(p)$ and take a hyperplane L perpendicular to v and intersecting the interior of the convex body K of M. By hypothesis, $S = M \cap L \neq \emptyset$ is bounded. Since v is clearly a pole, we have already shown that the fact that S is bounded implies that the part of K bounded by L and TM_p is bounded. Denote this part of K by K_1. Then K_1 is a closed bounded convex set. Note that a point in the boundary of K_1 is in either M or $K \cap L = K_2$. Let π_1 denote the orthogonal projection on L and set $a = \pi_1(p)$. Let $m, \delta > 0$ be such that $\|p - x\| \leq m$ for every x in K_2 and $\langle v, a - p\rangle - \delta m > 0$. We claim that $V = \{w \in \Sigma;\ \|w - v\| < \delta\}$ is contained in $\nu(M)$. In fact, let $w \in V$ and $q \in K_1$ such that $h_w(q)$ is the minimum of the height function h_w on K_1. Clearly q is in the boundary of K_1. For every $x \in K_2 = K \cap L$ we have

$$\begin{aligned} h_w(x) - h_w(p) &= \langle v, x - p\rangle + \langle w - v, x - p\rangle \\ &= \langle v, a - p\rangle + \langle w - v, x - p\rangle \geq \langle v, a - p\rangle - \delta m > 0\,, \end{aligned}$$

which implies that the minimum of h_w is assumed not at a point of K_2 and therefore at a point q of M. This proves that the open ball V of Σ is contained in $\nu(M)$, and the proof of the theorem is complete.

5. Proof of Theorem C

The proof of Bonnet-Myers theorem, in the finite dimensional case, depends on the Hopf-Rinow theorem which is known to be not true in the infinite dimensional case [1]. In the case where M is a complete hypersurface of a Hilbert space, we shall prove the Bonnet-Myers theorem by reducing it to the finite dimensional case.

Lemma 4. *Let M be a hypersurface of a Hilbert space H, and L a linear submanifold. Suppose that L intersects M transversally and let $S = M \cap L$. Then for each two-plane σ tangent to S we have*

(1) $K_S(\sigma) \geq K_M(\sigma)$, *if* $K_M(\sigma) \geq 0$,

(2) $K_S(\sigma) \leq K_M(\sigma)$, *if* $K_M(\sigma) \leq 0$,

where K_S and K_M denote the sectional curvatures of S and M respectively.

Proof. Take $p \in S$ and let σ be a two-plane tangent to S at p. Take an orthonormal basis $\{x, y\}$ of σ, and extend x and y to vector fields X, Y defined on a neighborhood U of M containing p such that $\nabla_X X, \nabla_Y Y$ and $\nabla_Y X$ are tangent to L at p. Let N denote a unit normal vector of M at p. Then by the Gauss egregium theorem we have

$$K_M(\sigma) = \langle N, \nabla_X X\rangle\langle N, \nabla_Y Y\rangle - \langle N, \nabla_Y X\rangle^2 .$$

Since L intersects M transversally, $N = N_1 + N_2$, where $N_1 \neq 0$ is in L and N_2 is in the orthogonal complement of L. By the proper choice of X and Y and from the fact that N_1 is normal to S at p, we obtain

$$K_M(\sigma) = \langle N_1, \nabla_X X\rangle\langle N_1, \nabla_Y Y\rangle - \langle N_1, \nabla_Y X\rangle^2 = \|N_1\|^2 K_S(\sigma) .$$

Since $\|N_1\| \leq 1$, the lemma is proved.

In Theorem A we have that if the convex hypersurface M is unbounded and does not bound any half-line, then the spherical image of M has void interior. This fact reflects on the sectional curvatures of M. It follows from the following lemma that in this case the sectional curvatures of M are not bounded away from zero at any point of M.

Lemma 5. *Let M be a hypersurface of a Hilbert space H. Suppose that the Gauss map is defined on M. If the sectional curvatures of M at a point p are all bounded away from zero, then the Gauss map is a local diffeomorphism on a neighborhood of p.*

Proof. By the Gauss egregium theorem, the sectional curvatures of M at p are given by

$$K(\sigma) = \langle A(x), x\rangle\langle A(y), y\rangle - \langle A(y), y\rangle^2 ,$$

where $\{x, y\}$ is an orthonormal basis of σ, and A is the derivative of the Gauss map at p. By assumption, there exists $\delta > 0$ such that $K(\sigma) \geq \delta$ for every two-plane $\sigma \subset TM_p$. This implies that there exists $\alpha > 0$ such that $\|A(x)\| \geq \alpha \|x\|$

for every x in TM_p. Thus A is invertible. It is clear from the above inequality that A is one to one. To prove that A is onto, set $F = TM_p$ and let us first show that $A(F)$ is a closed subspace of the Hilbert space F. Let $\{y_n\}$ be a Cauchy sequence on $A(F)$. Then $y_n = A(x_n)$ and $\|y_n - y_m\| = \|A(x_n - x_m\|$ $\geq \alpha\|x_n - x_m\|$. This shows that $\{x_n\}$ is a Cauchy sequence on F and therefore that $A(F)$ is closed. Denote by L the orthogonal complement of $A(F)$. Since $A(F)$ is closed, $F = A(F) \oplus L$. From the fact that A is self-adjoint, we have that $A(L) = L$. This implies that $L = \{0\}$, and we conclude that A is onto. Now the lemma follows from the inverse function theorem.

Proof of Theorem C. Suppose that M is a complete connected hypersurface of a Hilbert space H whose sectional curvatures satisfy $K(\sigma) \geq \delta > 0$. By the result proved in [1] we see that M is convex. Let p and q be two arbitrary points on M, and take a finite dimensional linear submanifold L containing p, q and intersecting the interior of the convex body of M. Then L intersects M transversally, and $S = M \cap L$ is a (finite dimensional) convex hypersurface of L. By Lemma 4 and the Bonnet-Myers theorem, the connected components of S are bounded, and thus S is connected. By Lemma 4, the sectional curvatures of S satisfy $K_S(\sigma) \geq \delta$. Then the Bonnet-Myers theorem shows that the distance $d_S(p,q)$ relative to S satisfies $d_S(p,q) \leq \pi/\sqrt{\delta}$. Since the distance $d_M(p,q)$ relative to M is less then or equal to $d_S(p,q)$, the diameter ρ of M satisfies $\rho \leq \pi/\sqrt{\delta}$.

We shall now prove that the Gauss map $\nu: M \to \Sigma$ is a diffeomorphism. By part (1) of Theorem A and Lemma 5, ν is a local diffeomorphism onto Σ. It remains to prove that ν is one to one. Let $p, q \in M$, and suppose that $p \neq q$ and $\nu(p) = \nu(q) = v$. Since p and q are minimum points of the height function $h_v: K \to \Sigma$, where K is the convex body of M, we have $h_v(p) = h_v(q)$ and hence $h_v(tp + (1-t)q) = h_v(p)$. From this it follows that the points on the segment $\{tp + (1-t)q;\ 0 \leq t \leq 1\}$ are minimum points of h_v. Since such points cannot occur in the interior of K, this segment is contained in M. This contradicts the fact that M has positive curvature, and we conclude that ν is one to one.

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