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# THE STRUCTURE OF $\delta$-PINCHED MANIFOLDS WITH THE FUNDAMENTAL GROUP $\pi_1(M) = Z_3$

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The present paper is a continuation of the differentiable pinching theorems for the sphere (see [7]), and the real (see [3] and [4]) and the complex (see [5]) projective spaces. The diffeotopy theorem plays an essential role for obtaining the dimension independency in the proof of the sphere pinching theorem. In order to make a fibre preserving diffeotopy between the Hopf fibration $S^{2n+1} \to S^{2n+1}/S^1 = P(C)^n$ and the free $S^1$ action on $S^{2n+1}$ which is caused by a Riemannian manifold $N$ with certain conditions (see [5]), we made heavy use of the strong diffeotopy theorem to get the diffeomorphism between the complex projective space and such an $N$. In the real projective pinching theorem, a fibre preserving diffeotopy between the antipodal map on $S^m$ and the involutive diffeomorphism on $S^m$ obtained from a $\delta$-pinched $M$ with $\pi_1(M) = Z_2$ is constructed easily by the diffeotopy theorem, and in this case we again obtain the dimension independency. The reason why we need not use the strong diffeotopy theorem in the real projective pinching is based on the following two facts. First, for each point $p$ on a $\delta$-pinched $M$ with $\pi_1(M) = Z_2$, the cut locus $C(p)$ of $p$ is a compact hypersurface of $M$ without boundary. Second, the deck transformation on the universal covering Riemannian manifold $\tilde{M}$ of $M$ leaves the inverse image $\pi^{-1}(C(p))$ of $C(p)$ invariant, where $\pi : \tilde{M} \to M$ is the covering projection.

However, if the order of $\pi_1(M)$ is greater than 2, it will not be easy to investigate the structure of cut locus $C(x)$ of a point $x$ on $M$. This is because $C(x)$ has nonempty boundary, and furthermore each element of the deck transformation group does not leave $\pi^{-1}(C(x))$ invariant. For instance, let $L^{2n+1}(r_1, \cdots, r_n; k) = S^{2n+1}/G$ be a general lens space of constant curvature 1 of the type $(r_1, \cdots, r_n; k)$, i.e., $G$ has the generator $g$ such that it is expressed in terms of the orthonormal basis $(e_1, \cdots, e_{2n+2})$ of $R^{2n+2}$ as follows:

$$g = \begin{bmatrix} R(r_1/k) & & & \\ & R(r_2/k) & & \\ & & \ddots & \\ & & & R(r_n/k) \end{bmatrix}, \qquad R(\alpha) := \begin{bmatrix} \cos 2\pi\alpha & \sin 2\pi\alpha \\ -\sin 2\pi\alpha & \cos 2\pi\alpha \end{bmatrix}.$$

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Then for each point $p^* \in L^{2n+1}(r_1, \cdots, r_n; k)$ the cut locus $C(p^*)$ is a compact hypersurface with nonempty boundary, and $\pi^{-1}(\partial C(p^*))$ is isometiric to the great $(2n-1)$-sphere of $S^{2n+1}$, where $\pi: S^{2n+1} \to L^{2n+1}(r_1, \cdots, r_n; k)$ is the covering projection. Moreover each element of the deck transformation group leaves $\pi^{-1}(\partial C(p^*))$ invariant but not $\pi^{-1}(C(p^*))$.

In this paper we shall deal with the case where $\pi_1(M) = Z_3$ and dim $M$ must be odd. Our main result can be stated as follows.

**Main theorem.** *There exists a monotone increasing sequence $\{\delta_k\}$, $\delta_k \in (\frac{1}{4}, 1)$ in such a way that for any connected, complete and $\delta$-pinched Riemannian manifold $M$ with its fundamental group $\pi_1(M) = Z_3$, $\delta > \delta_n$ implies that $M$ is diffeomorphic to the lens space $L^{2n+1}(1, \cdots, 1; 3)$.*

We shall give in § 1 definitions, notation and the known results to be used in this paper, and shall investigate in § 2 the structure of the cut locus $C(p)$ of a suitably chosen point $p$ on $M$. Then we see that $C(p)$ is a compact hypersurface of $M$ with nonempty boundary. We shall study in § 3 the structure of the boundary $\partial C(p)$. On the universal covering Riemannian manifold $\tilde{M}$ of $M$, each element of the deck transformation group leaves the submanifold $\pi^{-1}(\partial C(p))$ invariant, and hence we get the $Z_3$ action on $\pi^{-1}(\partial C(p))$ via the deck transformation group. Clearly $\pi^{-1}(\partial C(p))/Z_3$ can be identified with $\partial C(p)$. In § 4, the strong diffeotopy theorem is employed to construct the fibre preserving diffeotopy between the $Z_3$ action on $\pi^{-1}(\partial C(p))$ and the *standard $Z_3$ action* on $S^{2n-1}$, where dim $M = 2n + 1$ and by definition the standard $Z_3$ action has the generator $g \in Z_3$ such that

$$g = \begin{bmatrix} R(1/3) & & & \\ & R(1/3) & & \\ & & \ddots & \\ & & & R(1/3) \end{bmatrix}$$

and its quotient space is $L^{2n-1}(1, \cdots, 1; 3) =: L^{2n-1}(1; 3)$. Finally we shall prove the main theorem as well as the homeomorphism theorem.

## 1. Preliminaries

Throughout this paper let $M$ be a complete and connected Riemannian manifold whose sectional curvature $K$ and fundamental group $\pi_1(M)$ satisfy, respectively,

$$\frac{1}{4} < \delta \leq K \leq 1 \qquad \text{for any plane section,} \tag{1.1}$$

$$\pi_1(M) = Z_3 . \tag{1.2}$$

As is well known, an even dimensional complete Riemannian manifold of positive sectional curvature either is simply connected or has its fundamental group $= Z_2$. Therefore from (1.1) and (1.2), dimension of $M$ must be odd and we set

$$\dim M = 2n + 1 \ . \tag{1.3}$$

Let $\tilde{M}$ be the universal Riemannian covering of $M$, and $\pi\colon \tilde{M} \to M$ the covering projection. For a point $x \in M$ ($\tilde{x} \in \tilde{M}$) we denote by $M_x$ or $T_xM$ ($\tilde{M}_{\tilde{x}}$ or $T_{\tilde{x}}\tilde{M}$) the tangent space of $M$ ($\tilde{M}$) at the point. Denote by $d\colon M \times M \to R$ the distance function on $M$ with respect to the Riemannian metric, and also by $d$ the distance function on $\tilde{M}$. $\pi_1(M)$ can be identified with the deck transformation group. Let $g \in \pi_1(M)$ be a generator. For each point $x \in M$ we denote by $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2 \in \tilde{M}$ all the elements of $\pi^{-1}(x)$ (depending on the choice of both $g$ and $\tilde{x}_0$) such that

$$g^i(\tilde{x}_0) = \tilde{x}_1 \ , \qquad i = 0, 1, 2 \ . \tag{1.4}$$

For a smooth curve $c\colon [0, 1] \to M$, $c'(t)$ is by definition its velocity vector at $c(t)$ and its length denoted by $L(c)$ is given by

$$L(c) = \int_0^1 \langle c'(t), c'(t)\rangle^{1/2} dt \ .$$

The cut locus at $x \in M$ is denoted by $C(x)$.

We shall now state the known results to be used in this paper. For each point on a complete and simply connected Riemannian manifold $N$ satisfying condition (1.1), the cut locus theorem due to Klingenberg states (see [2])

$$d(x, C(x)) \geq \pi \ . \tag{1.5}$$

In other words, let $U_\pi(x) \subset N_x$ be the open ball in $N_x$ with the radius $\pi$ and center at the origin, and let $B_\pi(x) \subset N$ be the open metric ball with the same radius and the center at $x$. Then $\exp_x | U_\pi(x) \to B_\pi(x)$ is a diffeomorphism.

Let $S^m(k)$ be the standard $m$-sphere with the constant curvature $k$, and let $\gamma\colon [0, \beta] \to N$, $\gamma_\delta\colon [0, \beta] \to S^m(\delta)$, $\gamma_1\colon [0, \beta] \to S^m(1)$ be normal geodesics (i.e., parametrized to the arc length), where $N$ is complete, $\dim N = m$ and $N$ satisfies (1.1). Let $Y$, $Y_\delta$ and $Y_1$ be the Jacobi fields along $\gamma$, $\gamma_\delta$ and $\gamma_1$ respectively such that $Y(0) = Y_\delta(0) = Y_1(0) = 0$, $\|Y'(0)\| = \|Y_\delta'(0)\| = \|Y_1'(0)\|$. Then from Rauch's comparison theorem (see [2]) it follows that

$$\|Y_1(t)\| \leq \|Y(t)\| \leq \|Y_\delta(t)\| \qquad \text{for any } t \in [0, \pi] \ . \tag{1.6}$$

If these initial conditions are replaced by $\|Y(0)\| = \|Y_\delta(0)\| = \|Y_1(0)\|$ and $Y'(0) = Y_\delta'(0) = Y_1'(0) = 0$, then from Berger's comparison theorem we have (see [1])

$$(1.7)\qquad \|Y_1(t)\| \leq \|Y(t)\| \leq \|Y_\delta(t)\| \qquad \text{for any } t \in [0, \pi/2] .$$

A geodesic triangle $\varDelta$ on $N$ is by definition a triple of minimizing nontrivial geodesic segments, every one of whose extremals are on the others. Denote by $L(\varDelta)$ the circumference of $\varDelta$, and by $\alpha, \beta, \gamma \in [0, \pi]$ its angles. For a geodesic triangle $\varDelta$ on $N$, let $\varDelta_\delta$ and $\varDelta_1$ be the corresponding geodesic triangles on $S^2(\delta)$ and $S^2(1)$ respectively, where by definition $\varDelta_\delta$ and $\varDelta_1$ have the same side lengths as $\varDelta$. From Toponogov's theorem (see [2]), each angle of $\varDelta$ is not less than the corresponding angle of $\varDelta_\delta$, i.e.,

$$(1.8)\qquad \alpha_\delta \leq \alpha , \quad \beta_\delta \leq \beta , \quad \gamma_\delta \leq \gamma .$$

As a direct consequence of both theorems of Rauch and Klingenberg, we see that if a geodesic triangle $\varDelta$ on a simply connected $N$ satisfying (1.1) has circumference $L(\varDelta) \leq 2\pi$, then the corresponding $\varDelta_1$ with angles $\alpha_1$, $\beta_1$, $\gamma_1$ exists on $S^2(1)$ such that

$$(1.9)\qquad \alpha \leq \alpha_1 , \quad \beta \leq \beta_1 , \quad \gamma \leq \gamma_1 .$$

If $N$ satisfying (1.1) is *not simply connected*, its diameter $d(N)$ has an upper bound

$$(1.10)\qquad d(N) \leq \tfrac{1}{2}\pi/\sqrt{\delta} ,$$

where the diameter is defined by $d(N) := \operatorname{Max}\{d(x, y) ; x, y \in N\}$.

Finally we shall state the diffeotopy and the strong diffeotopy theorems.

**Diffeotory theorem** (*see* [7]). *Let $f$ be a diffeomorphism on $S^m(1) \subset R^{m+1}$, where $R^{m+1}$ is by definition a Euclidean $(m + 1)$-space, and assume that*

$$(1.11)\qquad \beta := \operatorname{Max}\{\sphericalangle (u, f(u)) ; u \in S^m(1)\} \leq \tfrac{1}{2}\pi ,$$

$$(1.12)\qquad \varepsilon := \operatorname{Max}\{\sphericalangle (A, dfA) ; A \in TS^m(1)\} < \cos^{-1}\{-\cos\beta \cdot \sqrt{\sin\beta/\beta}\} .$$

*Then $f$ is diffeotopic to the identity map via the following homotopy of diffeomorphisms: For each point $u \in S^m(1)$, let $\gamma_u : [0, 1] \to S^m(1)$ be the shortest great circular arc parametrized proportionally to the arc length, and let $F_t : S^m(1) \to S^m(1)$ be given by $F_t(u) = \gamma_u(t)$. Then $F_t$ is a diffeomorphism for any $t \in [0, 1]$.*

**Strong diffeotopy theorem** (*see* [5]). *Let $f$ satisfy* (1.1) *and* (1.2) *and let $L > 0$ be a constant such that*

$$(1.13)\qquad L^{-1} \leq \|dfA\| \leq L \qquad \text{for any } A \in TS^m(1), \quad \|A\| = 1 .$$

*Then for any $t \in [0, 1]$ and any unit vector $A \in TS^m(1)$ we have*

$$(1.14)\qquad \hat{L} \leq \|dF_tA\| \leq \hat{H} ,$$

$$(1.15) \qquad \measuredangle\, (A, dF_tA) \leq \hat{\varepsilon} + \beta \ ,$$

*where* $\hat{L}$, $\hat{H}$ *and* $\hat{\varepsilon}$ *are the constants defined by*

$$L^2 = L^{-2}\cos^2\varepsilon - (L^2 + 1 - L^{-2}\cos^2\varepsilon)\sin 2\alpha \ ,$$

$$H^2 = (L^2 + 1 - L^{-2}\cos^2\varepsilon)(1 + \sin 2\alpha) \ ,$$

$$\cos\varepsilon = (\cos\alpha - \sin\alpha)L^{-1}\cos\varepsilon/\sqrt{(L^2 + 1 - L^{-2}\cos^2\varepsilon)(1 + \sin 2\alpha)} \ ,$$

$$\cos\alpha = L^{-1}\cos\varepsilon/\sqrt{L^2 + 1 - L^{-2}\cos^2\varepsilon} \ .$$

In the above theorem we see $\lim_{\substack{\beta,\varepsilon\to 0\\ L\to 1}} \alpha = 0$, $\lim_{\substack{\beta,\varepsilon\to 0\\ L\to 1}} \varepsilon = 0$ and $\lim_{\substack{\beta,\varepsilon\to 0\\ L\to 1}} L = \lim_{\substack{\beta,\varepsilon\to 0\\ L\to 1}} H = 1$. We may actually assume that $\beta$, $\varepsilon$ and $L$ are taken so close to 0 and 1 respectively that these constants make sense. Indeed in this paper *we find upper bounds* $\beta(\delta)$, $\varepsilon(\delta)$ *and* $L(\delta)$ *of* $\beta$, $\varepsilon$ *and* $L$ *respectively so that* $\lim_{\delta\to 1}\beta(\delta) = \lim_{\delta\to 1}\varepsilon(\delta) = 0$ *and* $\lim_{\delta\to 1} L(\delta) = 1$ *hold.*

## 2. The structure of cut locus

Let $M$ satisfy conditions (1.1) and (1.2). Then for any point $x \in M$ we have (see [4])

$$(2.1) \qquad \tfrac{1}{3}\pi \leq d(x, C(x)) \leq \tfrac{1}{3}\pi/\sqrt{\delta} \ .$$

Since the function $x \to d(x, C(x))$ is continuous on $M$ and $M$ is compact, the function takes a minimum value. Let $p \in M$ be the point at which the minimum value, say $l$, is attained. From (2.1) there exists a simply closed geodesic $\gamma$ of length $2l$ such that

$$(2.2) \qquad \gamma\colon [0, 2l] \to M \ , \quad \gamma'(0) = \gamma'(2l) \ , \quad \tfrac{1}{3}\pi \leq l \leq \tfrac{1}{3}\pi\sqrt{\delta} \ .$$

The lifted geodesic is denoted by $\tilde{\gamma}$, and from (2.1) and (1.2) it is of length $6l$. Take a point $\tilde{p}_0 \in \pi^{-1}(p)$ and parametrize $\tilde{\gamma}\colon [0, 6l] \to \tilde{M}$ such that $\tilde{\gamma}(0) = \tilde{p}_0$. Then we can choose $g \in \pi_1(M)$ such that $\tilde{p}_i = g^i(\tilde{p}_0) = \tilde{\gamma}(2il)$. Setting

$$(2.3) \qquad \mathscr{F}_i := \{\tilde{x} \in \tilde{M}\,;\, d(\tilde{p}_{i+1}, \tilde{x}) = d(\tilde{p}_{i+2}, \tilde{x})\} \quad (\mathrm{mod}\ 3) \ ,$$

$\mathscr{F}_i$ is a compact subset of $\tilde{M}$, and $\tilde{p}_i \in \mathscr{F}_i$, $g^i(\mathscr{F}_0) = \mathscr{F}_i$ $(i = 0, 1, 2)$.

**Proposition 2.1.** *Assume that*

$$(2.4) \qquad \delta > 9/16 \ .$$

*Then* $\mathscr{F}_i$ *is a compact hypersurface of* $\tilde{M}$ *diffeomorphic to* $S^{2n}$.

*Proof.* We first show that

$$l \leq d(\tilde{x}, \tilde{p}_{i+1}) \leq 2l \qquad \text{for any } \tilde{x} \in \mathscr{F}_i\,. \tag{2.5}$$

In fact, from the choice of $p$ the first inequality in (2.5) is trivial. Suppose there exists a point $\tilde{y} \in \mathscr{F}_i$ such that $\tilde{y} \neq \tilde{p}_i$ and $d(\tilde{y}, \tilde{p}_{i+1}) = d(\tilde{y}, \tilde{p}_{i+2}) \geq d(\tilde{p}_i, \tilde{p}_{i+1})$. Note that $d(\tilde{p}_i, \tilde{p}_{i+1}) = 2l \geq 2\pi/3 > \frac{1}{2}\pi/\sqrt{\delta}$ follows from (2.2) and (2.4). Apply Toponogov's theorem (1.8) to the geodesic triangle with vertices $\tilde{p}_i, \tilde{p}_{i+1}$ and $\tilde{y}$ (or $\tilde{p}_i, \tilde{p}_{i+2}$ and $\tilde{y}$) to derive a contradiction. Thus the function $\tilde{x} \to d(\tilde{x}, \tilde{p}_{i+1})$, $\tilde{x} \in \mathscr{F}_i$ attains its maximum value $2l$ exactly at the point $\tilde{p}_i$. Since $2l \leq \frac{2}{3}\pi/\sqrt{\delta} < \pi$ and (1.5) holds for each point on $\tilde{M}$, $\mathscr{F}_i \subset B_\pi(\tilde{p}_{i+1}) \cap B_\pi(\tilde{p}_{i+2})$. The function $\lambda_i : \tilde{M} \to R$ defined by

$$\lambda_i(\tilde{x}) = d(\tilde{p}_{i+1}, \tilde{x}) - d(\tilde{p}_{i+2}, \tilde{x})\,, \qquad \tilde{x} \in \tilde{M} \tag{2.6}$$

is continuous on $\tilde{M}$ and differentiable on $(B_\pi(\tilde{p}_{i+1}) - \{\tilde{p}_{i+1}\}) \cap (B_\pi(\tilde{p}_{i+2}) - \{\tilde{p}_{i+2}\})$. In particular, grad $\lambda_i|_{\tilde{x}} \neq 0$ holds for any $\tilde{x} \in \mathscr{F}_i$. Hence $\mathscr{F}_i$ is a compact hypersurface without boundary.

Finally we shall prove $\mathscr{F}_i$ to be diffeomorphic to $S^{2n}$. Let $\tilde{c} : [0, \pi] \to \tilde{M}$ be a normal geodesic such that $\tilde{c}(0) = \tilde{p}_{i+1}$. Clearly $\lambda_i(\tilde{c}(0)) = -2l < 0$ and $\lambda_i(\tilde{c}(\pi)) > 0$, so that $\tilde{c}$ intersects $\mathscr{F}_i$ at some point $\tilde{c}(t_0)$. Obviously the intersection is *unique*, and $\tilde{c}'(t_0)$ is never tangent to $T_{\tilde{c}(t_0)}\mathscr{F}_i$ in $T_{\tilde{c}(t_0)}\tilde{M}$. Thus the map $\tilde{c}'(0) \to \tilde{c}(t_0)$ from the unit hypersphere $S_{\tilde{p}_{i+1}}(1)$ in $T_{\tilde{p}_{i+1}}\tilde{M}$, centered at the origin, onto $\mathscr{F}_i$ is a diffeomorphism. Hence the proof is completed.

We note $\pi^{-1}(C(p)) \subset \mathscr{F}_0 \cup \mathscr{F}_1 \cup \mathscr{F}_2$, and each point $\tilde{x}$ on $\pi^{-1}(C(p))$ has one of the following properties:

$$d(\tilde{p}_i, \tilde{x}) = d(\tilde{p}_{i+1}, \tilde{x}) < d(\tilde{p}_{i+2}, \tilde{x}) \qquad \text{for some } i = 0, 1, 2\,, \tag{2.7}$$

$$d(\tilde{p}_i, \tilde{x}) = d(\tilde{p}_{i+1}, \tilde{x}) = d(\tilde{p}_{i+2}, \tilde{x})\,. \tag{2.8}$$

In order to investigate the structure of $\pi^{-1}(C(p))$, we shall define the function:

$$\mu_i : \tilde{M} \to R\,, \quad \mu_i(\tilde{x}) = d(\tilde{x}, \tilde{p}_{i+1}) + d(\tilde{x}, \tilde{p}_{i+2}) - 2d(\tilde{x}, \tilde{p}_i)\,, \quad \tilde{x} \in \tilde{M}\,. \tag{2.9}$$

$\mu_i$ is continuous on $\tilde{M}$ and differentiable on $(B_\pi(\tilde{p}_0) - \{\tilde{p}_0\}) \cap (B_\pi(\tilde{p}_1) - \{\tilde{p}_1\}) \cap (B_\pi(\tilde{p}_2) - \{\tilde{p}_2\})$. Let $\mathscr{E}$ be defined by

$$\mathscr{E} = \{\tilde{x} \in \tilde{M}\,;\, d(\tilde{x}, \tilde{p}_0) = d(\tilde{x}, \tilde{p}_1) = d(\tilde{x}, \tilde{p}_2)\}\,. \tag{2.10}$$

From definition follows

$$\mathscr{E} = \mathscr{F}_0 \cap \mathscr{F}_1 \cap \mathscr{F}_2 = \mathscr{F}_0 \cap \mathscr{F}_1 = \mathscr{F}_1 \cap \mathscr{F}_2 = \mathscr{F}_2 \cap \mathscr{F}_0 \tag{2.11}$$

(as point sets). Here we essentially use assumption (1.2). Clearly each element of the deck transformation group leaves $\mathscr{E}$ invariant. $\mu_i | \mathscr{F}_i : \mathscr{F}_i \to R$ is continuous on $\mathscr{F}_i$ and differentiable on $\mathscr{F}_i \cap B_\pi(\tilde{\gamma}((2i+3)l))$. From (2.11) and

$\mathscr{F}_i \subset B_\pi(\tilde{p}_{i+1}) \cap B_\pi(\tilde{p}_{i+2})$ we see

$$\mathscr{E} \subset B_\pi(\tilde{p}_0) \cap B_\pi(\tilde{p}_1) \cap B_\pi(\tilde{p}_2) \ . \tag{2.12}$$

**Proposition 2.2.** *Under assumption* (2.4), $\mathscr{E}$ *is a compact* $(2n-1)$*-dimensional submanifold without boundary and is diffeomorphic to* $S^{2n-1}$.

*Proof.* We may restrict our discussion to the case where $\tilde{\gamma}((3+2i)l)$ does not belong to the closure $\overline{B_\pi(\tilde{p}_i)}$ of the open ball. In fact, $\tilde{\gamma}((3+2i)l) \in \overline{B_\pi(\tilde{p}_i)}$ implies $l = \frac{1}{3}\pi$, and hence $\tilde{M}$ is isometric to $S^{2n+1}(1)$ (see [4]). Therefore $M$ is isometric to $L^{2n+1}(1,3)$. Consider the gradient field of $\mu_i$ restricted to $\mathscr{F}_i \cap B_\pi(\tilde{p}_i)$. We claim that on $\mathscr{F}_i \cap B_\pi(\tilde{p}_i)$, there exists no critical point of $\mu_i$ other than $\tilde{p}_i$. Recall that $(\tilde{x} \to d(\tilde{x}, \tilde{p}_{i+1}),\ \tilde{x} \in \mathscr{F}_i)$ takes its maximum value exactly of the point $\tilde{p}_i$, and hence $\tilde{p}_i$ is a critical point of $\mu_i$. For any point $\tilde{x}$ on $\mathscr{F}_i \cap B_\pi(\tilde{p}_i)$, $\tilde{x} \neq \tilde{p}_i$, let $\tilde{a}_{i+1}, \tilde{a}_{i+2}\colon [0,m] \to \tilde{M}$ and $\tilde{a}_i\colon [0,\tilde{m}] \to \tilde{M}$ be the minimizing geodesics such that $\tilde{a}_j(0) = \tilde{p}_j$, $j = 0,1,2$, $\tilde{a}_{i+1}(m) = \tilde{a}_{i+2}(m) = \tilde{a}_i(\tilde{m}) = \tilde{x}$. Then we have

$$\begin{aligned} \operatorname{grad}(\mu_i | \mathscr{F}_i \cap B_\pi(\tilde{p}))|_{\tilde{x}} = {} & \text{the tangential component of} \\ & \tilde{a}'_{i+1}(m) + \tilde{a}'_{i+2}(m) - 2\tilde{a}'_i(\tilde{m}) \quad \text{to } T_{\tilde{x}}\mathscr{F}_i \ . \end{aligned} \tag{2.13}$$

Since $\tilde{a}'_{i+1}(m)$ is symmetric to $\tilde{a}'_{i+2}(m)$ with respect to $T_{\tilde{x}}\mathscr{F}_i$, $\tilde{a}'_{i+1}(m) + \tilde{a}_{i+2}(m) \in T_{\tilde{x}}\mathscr{F}_i$, and clearly $m < 2l$. From Toponogov's theorem (1.8) follows

$$\begin{aligned} \cos \sphericalangle (\tilde{a}'_j(m), \tilde{a}'_i(m)) &\leq \frac{\cos 2l\sqrt{\delta} - \cos m\sqrt{\delta}\cos \tilde{m}\sqrt{\delta}}{\sin m\sqrt{\delta}\sin \tilde{m}\sqrt{\delta}} \\ &< \frac{\cos 2l\sqrt{\delta}\,(1 - \cos \tilde{m}\sqrt{\delta})}{\sin m\sqrt{\delta}\sin \tilde{m}\sqrt{\delta}} \ , \end{aligned}$$

$j \neq i$. But from (2.4) we have $2l\sqrt{\delta} > \frac{1}{2}\pi$, since $\sphericalangle (\tilde{a}'_j(m), \tilde{a}'_i(\tilde{m})) > \frac{1}{2}\pi$, $j \neq i$. Therefore the angle between $\tilde{a}'_{i+1}(m) + \tilde{a}'_{i+2}(m)$ and the tangential component of $-2\tilde{a}'_i(\tilde{m})$ to $T_{\tilde{x}}\mathscr{F}_i$ is less than $\frac{1}{2}\pi$. Thus we have proved

$$\operatorname{grad}(\mu_i | \mathscr{F}_i \cap B_\pi(\tilde{p}_i))|_{\tilde{x}} \neq 0 \qquad \text{for } \tilde{x} \neq \tilde{p}_i \ .$$

Hence we observe that $\mathscr{F}_i \cap B_\pi(\tilde{p}_i)$ is diffeomorphic to a $2n$-disk, and each level surface $\mu_i^{-1}(\{\text{constant}\})$ especially $\mu_i^{-1}(\{0\}) = \mathscr{E}$ is diffeomorphic to $S^{2n-1}$.

We shall define the sets

$$\begin{aligned} \mathscr{F}_i^+ &= \{\tilde{x} \in \mathscr{F}_i;\ \mu_i(\tilde{x}) > 0\} \ , \\ \mathscr{F}_i^- &= \{\tilde{x} \in \mathscr{F}_i;\ \mu_i(\tilde{x}) < 0\} \ , \end{aligned} \qquad i = 0,1,2 \ .$$

Then $\mathscr{F}_i^+$ has clearly no intersection with $\pi^{-1}(C(p))$, so that

$$\pi^{-1}(C(p)) = \mathscr{F}_0^- \cup \mathscr{F}_1^- \cup \mathscr{F}_2^- \cup \mathscr{E} \qquad \text{(disjoint union)}, \tag{2.14}$$

$$g^i(\mathscr{F}_0^-) = \mathscr{F}_i^- \ , \quad g^i(\mathscr{E}) = \mathscr{E} \ , \qquad i = 0, 1, 2. \tag{2.15}$$

(2.15) allows us to consider the free $Z_3$ action $(\mathscr{E}, \varphi^*, Z_3)$ on $\mathscr{E}$ by the deck transformation group. In fact, for any point $\tilde{x} \in \mathscr{E}$, let

$$\varphi^*(g^i, \tilde{x}) := g^i(\tilde{x}) \ , \qquad g^i \in Z_3 = \pi_1(M) \ . \tag{2.16}$$

Clearly the quotient space $\mathscr{E}/Z_3$ is diffeomorphic to the boundary $\partial C(p) = \pi(\mathscr{E})$.

For any $\tilde{x} \in \mathscr{E}$ let $X \in T_{\tilde{x}}\mathscr{E}$ and $\tilde{a}_i : [0, m] \to \tilde{M}$ be the unique shortest connection joining $\tilde{p}_i$ to $\tilde{x}$. From (2.8), we see $\measuredangle\, (X, \tilde{a}_0'(m)) = \measuredangle\, (X, \tilde{a}_1'(m)) = \measuredangle\, (X, \tilde{a}_2'(m)) \neq 0$. Hence with the aid of (2.12), the projection $p_r : \mathscr{E} \to \tilde{S}_0(1) \subset \tilde{M}_{\tilde{p}_0}$ defined by

$$p_r(\tilde{x}) := \tilde{a}_0'(0) \tag{2.17}$$

is a diffeomorphism, where $\tilde{S}_0(1)$ is the unit hypersphere in $\tilde{M}_{\tilde{p}_0}$ centered at the origin. We shall denote by $E$ the image

$$E = p_r(\mathscr{E}) \subset \tilde{S}_0(1) \ . \tag{2.18}$$

Obviously $E$ is a hypersurface of $\tilde{S}_0(1)$ and diffeomorphic to the standard sphere. Therefore we have the $Z_3$ action $(E, \varphi, Z_3)$ such that

$$\varphi(g^i, u) = p_r \cdot \varphi^*(g^i, p_r^{-1}(u)) \ , \qquad u \in E \ , \quad g^i \in Z_3 \ . \tag{2.19}$$

Clearly we have the following

**Lemma 2.3.** *The quotient space $E/Z_3$ is diffeomorphic to $\partial C(p) = \pi(\mathscr{E})$.*

*Proof.* Since $p_r$ is the fibre-preserving diffeomorphism, $(\mathscr{E}, \varphi^*, Z_3)$ is equivalent to $(E, \varphi, Z_3)$. The conclusion is now trivial from $\mathscr{E}/Z_3$ being diffeomorphic to $\pi(\mathscr{E})$.

## 3. The $Z_3$ action on $\mathscr{E}$

**Lemma 3.1.** *Assume that*

$$\delta > 25/36 \ . \tag{3.1}$$

*Then for any $\tilde{x} \in \mathscr{E}$ and any $i = 0, 1, 2$, we have*

$$\tfrac{1}{2}\pi \leq d(\tilde{x}, \tilde{p}_i) \leq \tfrac{1}{2}\pi/\sqrt{\delta} \ . \tag{3.2}$$

*Proof.* Let $\tilde{a}_i : [0, m] \to \tilde{M}$ be the minimizing geodesic such that $\tilde{a}_i(0) = \tilde{p}_i$, $\tilde{a}_i(m) = \tilde{x}$, $m = d(\tilde{p}_i, \tilde{x})$. From $\pi(\tilde{x}) \in C(p)$ and (1.10), follows $d(\tilde{x}, \tilde{p}_i) = d(x, p) \leq d(M) \leq \frac{1}{2}\pi/\sqrt{\delta}$. (3.1) ensures that the circumference of the geodesic triangle with the sides $\tilde{a}_i$, $\tilde{\gamma}\,|\,[2il, 2(i+1)l]$ and $\tilde{a}_{i+1}$ is less than $2\pi$ since $m \leq \frac{1}{2}\pi/\sqrt{\delta}$ and $2l \leq 2\pi/(3\sqrt{\delta})$. Thus $\tilde{a}_{i+1}(t) \in B_\pi(\tilde{p}_i)$ holds for any

$t \in [0, m]$, and without loss of generality we may assume that $\measuredangle\, (\tilde{a}_i'(0), \tilde{\gamma}'(2il)) \geq \frac{1}{2}\pi$. From (1.9) we get $m \geq \frac{1}{2}\pi$.

Now by the triangle arguments stated in (1.8) and (1.9) together with the cosine rule in spherical trigonometry, we get

**Lemme 3.2.** *Under the assumption* (3.1) *we have, for any* $\tilde{x} \in \mathscr{E}$ *and* $i = 0, 1, 2$,

$$\text{(3.3)} \qquad \begin{gathered} \operatorname{Max}\{\pi - \omega_H(\delta), \omega_L(\delta)\} \leq \measuredangle\, (\tilde{a}_i'(0)\,, \\ \tilde{\gamma}'(2il)) \leq \operatorname{Min}\{\omega_H(\delta), \pi - \omega_L(\delta)\}\,, \end{gathered}$$

$$\text{(3.4)} \qquad \theta_L(\delta) \leq \measuredangle\, (\tilde{a}_{i+1}'(m), \tilde{a}_{i+2}'(m)) \leq \theta_H(\delta)\,,$$

*where we set*

$$\text{(3.5)} \qquad \theta_L(\delta) = \frac{2\pi\sqrt{\delta}}{3}\,, \qquad \theta_H(\delta) = \cos^{-1}\left\{\frac{\cos\dfrac{2\pi}{3\sqrt{\delta}} - \cos^2\dfrac{\pi}{2\sqrt{\delta}}}{\sin^2\dfrac{\pi}{2\sqrt{\delta}}}\right\}\,,$$

$$\text{(3.6)} \qquad \begin{aligned} \omega_L(\delta) &= \cos^{-1}\left(\tan\frac{\pi}{3}\cot\frac{\pi\sqrt{\delta}}{3}\right)\,, \\ \omega_H(\delta) &= \cos^{-1}\left(\tan\frac{\pi}{3\sqrt{\delta}}\cot\frac{\pi}{2\sqrt{\delta}}\right)\,. \end{aligned}$$

The proof is left to the reader.

For any $\tilde{x}_0 \in \mathscr{E}$, let $\tilde{x}_1, \tilde{x}_2 \in \mathscr{E}$ be such that $g^i(\tilde{x}_0) = \tilde{x}_i$, and $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i : [0, m] \to \tilde{M}$ be the shortest geodesics such that

$$\text{(3.7)} \qquad \begin{gathered} \tilde{a}_i(0) = \tilde{b}_i(0) = \tilde{c}_i(0) = \tilde{p}_i\,, \quad \tilde{a}_i(m) = \tilde{x}_0\,, \quad \tilde{b}_i(m) = \tilde{x}_1\,, \\ \tilde{c}_i(m) = \tilde{x}_2\,, \qquad i = 0, 1, 2, \quad m = d(\tilde{p}_i, \tilde{x}_j)\,. \end{gathered}$$

Clearly we see

$$\text{(3.8)} \qquad \begin{aligned} g\circ\tilde{a}_i &= \tilde{b}_{i+1}\,, \qquad & g^2\circ\tilde{a}_i &= g\circ\tilde{b}_{i+1} = \tilde{c}_{i+2}\,, \\ g\circ\tilde{b}_i &= \tilde{c}_{i+1}\,, \qquad & g^2\circ\tilde{b}_i &= g\circ\tilde{c}_{i+1} = \tilde{a}_{i+2}\,, \\ g\circ\tilde{c}_i &= \tilde{a}_{i+1}\,, \qquad & g^2\circ\tilde{c}_i &= g\circ\tilde{a}_{i+1} = \tilde{b}_{i+2}\,. \end{aligned}$$

Define the projection map $\bar{p}_r : E \to \tilde{S}_0^\perp(1) \subset \tilde{S}_0(1)$ with respect to the point $\tilde{\gamma}'(0)$ by

$$\text{(3.9)} \qquad \bar{p}_r(u) = u - \langle u, \tilde{\gamma}'(0)\rangle\tilde{\gamma}'(0)\,, \qquad u \in E\,,$$

where $\tilde{S}_0^\perp(1)$ is by definition the equator hypersphere with the north pole $\tilde{\gamma}'(0)$.

**Proposition 3.3.** *There exists $\delta' \in [25/36, 1)$ such that*

$$\delta > \delta' \tag{3.10}$$

*implies that $\bar{p}_r$ is a diffeomorphism.*

*Proof.* For any $u \in E$ and any $A \in T_uE$, $\|A\| = 1$, let $\sigma: I \to E$ be a smooth curve fitting $A$ so that $\sigma'(0) = A$ and $0 \in I$ is an interval. Let $\tilde{a}_0: [0, m] \to \tilde{M}$ be the minimizing geodesic such that $\tilde{a}_0(0) = \tilde{p}_0$, $\tilde{a}_0(m) \in \mathscr{E}$, and $\tilde{a}_0'(0) = u$. Define a 1-parameter geodesic variation $V: [0, m] \times I \to \tilde{M}$ along $\tilde{a}_0$ by

$$V(t, s) = \exp_{\tilde{p}_0} \frac{p_r^{-1}(\sigma s))}{m} t\ , \qquad t \in [0, m]\ , \quad s \in I\ . \tag{3.11}$$

We denote by $Y(t)$ the Jacobi field along $\tilde{a}_0$ associated with $V$, and by $Y_\perp(t)$ its normal component of $\tilde{a}_0'(t)$. Obviously $Y_\perp(t)$ is again a Jacobi field. From construction

$$Y(0) = 0\ , \qquad Y_\perp'(0) = A \in \tilde{M}_{\tilde{p}_0}\ , \tag{3.12}$$

where $A$ is identified with the vector obtained by the parallel displacement of $A \in T_uE \subset T_u\tilde{S}_0(1)$ in $\tilde{M}_{\tilde{p}_0}$. Let $P$ be the unit parallel field along $\tilde{a}_0$ such that $P(0) = A = Y_\perp'(0)$. From an approximation theorem of Jacobi fields (see [7]) we have an upper bound $\Theta(\delta)$ for the angle

$$\measuredangle\ (Y_\perp(m), P(m)) \leq \Theta(\delta)\ , \qquad \lim_{\delta \to 1} \Theta(\delta) = 0\ . \tag{3.13}$$

Apply Berger's theorem (1.7) to the curve $c: [0, m] \to \tilde{M}$, $c(t) := \exp \frac{1}{2}\pi P(t)$ to get

$$L(c) \leq \frac{\pi}{2\sqrt{\delta}} \cos \frac{\pi\sqrt{\delta}}{2}\ . \tag{3.14}$$

From (2.8) and Lemma 3.2, follows

$$\begin{aligned} \sin^{-1}\Big(\frac{2}{\sqrt{3}} \sin \frac{\theta_L}{2}\Big) &\leq \measuredangle\ (Y(m), \tilde{a}_0'(m)) = \measuredangle\ (Y(m), \tilde{a}_1'(m)) \\ &= \measuredangle\ (Y(m), \tilde{a}_2'(m)) \leq \pi - \sin^{-1}\Big(\frac{2}{\sqrt{3}} \sin \frac{\theta_L}{2}\Big)\ . \end{aligned} \tag{3.15}$$

(3.15) implies $\measuredangle\ (Y(m), Y_\perp(m)) \leq \frac{\pi}{2} - \sin^{-1}\Big(\frac{2}{\sqrt{3}} \sin \frac{\theta_L}{2}\Big)$, and hence we get an upper bound $\hat{\Theta}(\delta)$ for the angle

$$\measuredangle\ (Y(m), P(m)) \leq \hat{\Theta}(\delta)\ , \qquad \lim_{\delta \to 1} \Theta(\delta) = 0\ . \tag{3.16}$$

Thus we have a bound $d(\delta)$ for the distance

$$(3.17)\quad d\left(\tilde{q}, \exp\frac{\pi}{2}A\right) \leq d(\delta)\ , \quad \lim_{\delta\to 1} d(\delta) = 0\ , \quad \tilde{q} := \exp\frac{\pi}{2}\frac{Y(m)}{\|Y(m)\|}\ .$$

On the other hand, (3.15) implies

$$d(\tilde{q}, \tilde{p}_1) \geq \sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\theta_L}{2}\right)$$

together with Rauch's theorem (1.6), and therefore we obtain a lower bound for the distance

$$d\left(\exp\frac{\pi}{2}A, \tilde{p}_1\right) \geq \sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\theta_L}{2}\right) - d(\delta)\ .$$

To the geodesic triangle with vertices $\tilde{p}_0, \tilde{p}_1$ and $\exp\frac{1}{2}\pi A$, we apply (1.8) to get a lower bound for the angle

$$\begin{aligned}
&\cos\sqrt{\delta}\ \sphericalangle(\tilde{\gamma}'(0), A) \\
(3.18)\qquad &\leq \frac{\cos\sqrt{\delta}\left(\sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\theta_L}{2}\right) - d(\delta)\right) - \cos\frac{\pi\sqrt{\delta}}{2}\cos\frac{2\pi\sqrt{\delta}}{3}}{\sin\frac{\pi\sqrt{\delta}}{2}\sin\frac{2\pi\sqrt{\delta}}{3}} \\
&=: \cos\sqrt{\delta}\ \Phi_L(\delta)\ .
\end{aligned}$$

Analogously we have an upper bound $\Phi_H(\delta)$ for the angle $\sphericalangle(\tilde{\gamma}'(0), A)$ such that

$$(3.19)\quad \Phi_L(\delta) \leq \sphericalangle(\tilde{\gamma}'(0), A) \leq \Phi_H(\delta)\ , \qquad \lim_{\delta\to 1}\Phi_L(\delta) = \lim_{\delta\to 1}\Phi_H(\delta) = \tfrac{1}{2}\pi\ .$$

If $\delta'$ is chosen so close to 1 that

$$(3.20)\qquad \Phi_L(\delta') > 0\ , \qquad \Phi_H(\delta') < \pi$$

are satisfied, then $d(\bar{p}_r)_u A \neq 0$ holds for any $u \in E$ and any $A \in T_u E$. Since $\bar{p}_r$ is $1-1$ and onto, the proof is complete.

Let $\omega: E \to R$ be the function defined by

$$(3.21)\qquad \omega(u) = \sphericalangle(u, \bar{p}_r(u))\ , \qquad u \in E\ .$$

Clearly $\omega$ is a differentiable function on $E$ as far as (3.10) is satisfied, and from (3.3) follows

$$(3.22)\qquad 0 \leq \omega(u) \leq \mathrm{Max}\,\{\tfrac{1}{2}\pi - \omega_L(\delta), \omega_H(\delta) - \tfrac{1}{2}\pi\}\ .$$

Each $u \in E$ can be expressed as $u = \cos \omega(u) \cdot \bar{p}_r(u) + \sin \omega(u) \cdot \tilde{\gamma}'(0)$. Put

$$\bar{p}_r^\tau(u) = \cos \{(1-\tau) \cdot \omega(u)\} \bar{p}_r(u) + \sin \{(1-\tau) \cdot \omega(u)\} \cdot \tilde{\gamma}'(0) , \quad u \in E,\ \tau \in [0,1] . \tag{3.23}$$

Then $\bar{p}_r^\tau$ is an imbedding of $E$ into $\tilde{S}_0(1)$ for each $\tau \in [0,1]$ as far as (3.10) is satisfied. Therefore we have a 1-parameter family of $Z_3$ action $(\bar{p}_r^\tau(E), \varphi_\tau, Z_3)$, $\tau \in [0,1]$, defined by

$$\varphi_\tau(g^i, v) = \bar{p}_r^\tau \cdot \varphi(g^i, (\bar{p}_r)^{-1}(v)) , \quad v \in \bar{p}_r(E) , \quad g^i \in Z_3 . \tag{3.24}$$

Obviously the quotient space $\bar{p}_r(E)/Z_3$ is diffeomorphic to $\mathscr{E}/Z_3 = \pi(E) = \partial(C(p)$ for each $\tau \in [0,1]$.

## 4. Fibre-preserving diffeotopy between $\varphi_1$ and the standard $Z_3$ action

A differentiable deformation of the $Z_3$ action $\varphi_1$ on $\tilde{S}_0^\perp(1)$ is a 1-parameter family $\varphi_t$, $t \in [0,1]$, of $Z_3$ actions on $\tilde{S}_0^\perp(1)$ such that the map $(g, u, t) \to \varphi_t(g, u)$ is differentiable. For a diffeotopy $F_t$, $t \in [0,1]$, on $\tilde{S}_0^\perp(1)$ such that $F_0 =$ identity, we have a deformation of $\varphi_1$ in the following way: For each $t \in [0,1]$ let $\varphi_t$ be defined by

$$\varphi_t(g^i, u) = F_t \circ \varphi_1(g^i, F_t^{-1}(u)) , \qquad i = 0, 1, 2 . \tag{4.1}$$

In this section we shall construct a deformation $\Psi_t$ of $\varphi_1$ such that $\Psi_1 = \varphi_1$ and $\Psi_0 =$ the standard $Z_3$ actions on $\tilde{S}_0^\perp(1)$. If such a deformation exists, then the quotient space $\tilde{S}_0^\perp(1)/Z_3$ of $(\tilde{S}_0^\perp(1), \varphi_1, Z_3)$ is diffeomorphic to $L(1; 3)$ and so is $\pi(\mathscr{E}) = \partial C(p)$.

**Theorem 4.1.** *We have a monotone increasing sequence $\{\delta_k'\}$, $\delta_k' \in [25/36, 1)$, such that*

$$\delta > \delta_N' \tag{4.2}$$

*ensures the existence of a deformation $\Psi_t$ of $Z_3$ action such that $\Psi_1 = \varphi_1$, and $\Psi_0$ is the standard $Z_3$ action on $\tilde{S}_0^\perp(1)$, where $N$ depends on* $\dim M$.

To prove the theorem, we shall prepare Lemmas 4.2–4.4 and Propositions 4.5–4.6 below. The first step of constructing the deformation in the theorem is to choose an orthonormal basis $(e_1, \cdots, e_{2n+1})$ for $\tilde{M}_{\bar{p}_0}$ in such a way that $\tilde{\gamma}'(0) = e_{2n+1}$ and the standard action is roughly speaking near to $\varphi_1$ and is expressed in terms of $(e_1, \cdots, e_{2n})$ as follows:

$$\Psi_0(g, u) = \begin{pmatrix} R(\frac{1}{3}) & & & \\ & R(\frac{1}{3}) & & \\ & & \ddots & \\ & & & R(\frac{1}{3}) \end{pmatrix} \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{2n} \end{pmatrix} , \qquad u = \sum_{j=1}^{2n} u_j e_j , \quad \sum_{j=1}^{2n} u_j^2 = 1 . \tag{4.3}$$

To do this we define an isometry $h^* \tilde{M}_{\tilde{p}_0} \to \tilde{M}_{\tilde{p}_0}$ by

$$h^*(X) = s\cdot\tau\cdot dg_{\tilde{p}_0}X\ , \qquad X \in \tilde{M}_{\tilde{p}_0}\ , \tag{4.4}$$

where $g \in \pi_1(M)$ is the fixed element of the deck transformation group in § 2, $\tau: \tilde{M}_{\tilde{p}_1} \to \tilde{M}_{\tilde{p}_0}$ is the parallel displacement along $\tilde{\gamma}\,|\,[0, 2l]$, and $s: \tilde{M}_{\tilde{p}_0} \to \tilde{M}_{\tilde{p}_0}$ is the reflection with respect to the hyperplane normal to $\tilde{\gamma}'(0)$. Obviously $h^*$ is a linear isometry, and $h^*(\tilde{\gamma}'(0)) = -\tilde{\gamma}'(0)$.

**Lemma 4.2.** *There exist functions $\alpha_L(\delta)$ and $\alpha_H(\delta)$ such that for any $u \in \tilde{S}_0^{\perp}(1)$*

$$\alpha_L(\delta) \leq \sphericalangle\,(u, h^*(u)) \leq \alpha_H(\delta)\ , \tag{4.5}$$

$$\lim_{\delta\to1} \alpha_L(\delta) = \lim_{\delta\to1} \alpha_H(\delta) = \tfrac{2}{3}\pi\ . \tag{4.6}$$

*Proof.* For any $u \in \tilde{S}_0^{\perp}(1)$ let $\tilde{x}_0 = \exp m\bar{p}_r^{-1}(u) \in \mathscr{E}$, $m \in [\frac{1}{2}\pi, \frac{1}{2}\pi/\sqrt{\delta}\,)$. From (2.1), Lemma 3.2 and (1.7) we obtain the following inequalities by the same method as in the proof of Proposition 3.3:

$$\frac{2\pi}{3} - \bar{d}(\delta) \leq d\left(\exp\frac{\pi}{2}h^*(u), \exp\frac{\pi}{2}u\right) \leq \frac{2\pi}{3\sqrt{\delta}} + \bar{d}(\delta)\ . \tag{4.7}$$

where $\bar{d}(\delta)$ is an upper bound for the distance $d(\tilde{x}_1, \exp\frac{1}{2}\pi h^*(u))$ such that

$$\begin{aligned}\bar{d}(\delta) = \frac{2\pi}{3\sqrt{\delta}}\cos\frac{\pi\sqrt{\delta}}{2} + \frac{2}{\sqrt{\delta}}\cos^{-1}\Big\{&\cos^2\frac{\pi\sqrt{\delta}}{2} \\ &+ \sin^2\frac{\pi\sqrt{\delta}}{2}\cos\Omega(\delta)\Big\}\ ,\end{aligned} \tag{4.8}$$

$$\Omega(\delta) := \mathrm{Max}\,\{\tfrac{1}{2}\pi - \omega_L, \omega_H - \tfrac{1}{2}\pi\}\ .$$

Applying (1.8) and (1.9) to the geodesic triangle with vertices $\tilde{p}_0$, $\exp\frac{1}{2}\pi u$, and $\exp\frac{1}{2}\pi h^*(u)$, we obtain the conclusions.

Therefore, if $\delta$ is chosen so that $\alpha_L(\delta) > 0$ and $\alpha_H(\delta) < \pi$, then we can choose the orthonormal basis $(e_1, \cdots, e_{2n+1})$ for $\tilde{M}_{\tilde{p}_0}$ such that $h^*$ is expressed as follows:

$$h^*(v) = \begin{pmatrix} R(\alpha_1) & & & & \\ & R(\alpha_2) & & & \\ & & \ddots & & \\ & & & R(\alpha_n) & \\ & & & & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_{2n+1}\end{pmatrix}, \tag{4.9}$$

$$v = \sum_{i=1}^{2n+1} v_i e_i\ , \quad \sum_{i=1}^{2n+1} v_i^2 = 1\ , \quad e_{2n+1} = \tilde{\gamma}'(0)\ .$$

Lemma 4.2 implies $\alpha_L(\delta) \leq \alpha_k \leq \alpha_H(\delta)$ for any $k = 1, \cdots, n$. Thus we can restrict $h^*$ to $\tilde{S}_0^\perp(1)$, and denote it by

$$h = h^* \,|\, \tilde{S}_0^\perp(1) \ . \tag{4.10}$$

Below we fix the orthonormal basis which allows the expression (4.9). Clearly $h$ can be joined to $\Psi_0(g) \,|\, \tilde{S}_0^1(1)$ in the orthogonal group as follows: For each $t \in [0, 1]$, let $h_t$ be

$$h_t = \begin{bmatrix} R(\alpha_1 t + \frac{1}{3}(1-t)) & & \\ & \ddots & \\ & & R(\alpha_n + \frac{1}{3}(1-t)) \end{bmatrix} . \tag{4.11}$$

Then $h_0 = \Psi_0(g, \ ) \,|\, \tilde{S}_0^1(1)$ and $h_1 = h$.

In the following we want to construct a diffeotopy between $h$ and $\varphi_1(g, \ )$. For simplicity we write $f_t(u) = \varphi_t(g, u)$, $u \in \tilde{S}_0(1)$, $t \in [0, 1]$. As a direct consequence of Lemma 4.2 we obtain

**Lemma 4.3.** *For any $u \in \tilde{S}_0^\perp(1)$ we have*

$$\measuredangle\, (f_1(u), h(u)) \leq \bar{d}(\delta) + \Omega(\delta) = : \beta'(\delta) \ . \tag{4.12}$$

On the other hand, from (1.6) we obtain

**Lemma 4.4.** *For any $A \in T\tilde{S}_0^\perp(1)$, $\|A\| = 1$, we have*

$$L^{-1}(\delta) \leq \|df_1 A\| \leq L(\delta) \ , \tag{4.13}$$

*where*

$$L(\delta) := \frac{\mathrm{Max}\,\{\sin \Phi_L(\delta), \sin \Phi_H(\delta)\}}{\mathrm{Min}\,\{\sin \Phi_L(\delta), \sin \Phi_H(\delta)\}} \cdot \left(\sqrt{\delta}\, \sin \frac{\pi}{2\sqrt{\delta}}\right)^{-1} . \tag{4.14}$$

We note that $\lim_{\delta \to 1} \beta'(\delta) = 0$ and $\lim_{\varrho \to 1} L(\delta) = 1$.

**Proposition 4.5.** *There exists $\varepsilon'(\delta)$ such that for any $u \in \tilde{S}_0^\perp(1)$ and any $A \in T\tilde{S}_0(1)$, we have*

$$\measuredangle\, (df_1 A, dhA) \leq \varepsilon'(\delta) \ , \tag{4.15}$$

$$\lim_{\delta \to 1} \varepsilon'(\delta) = 0 \ . \tag{4.16}$$

*Proof.* For any $u \in \tilde{S}_0^\perp(1)$ and any $A \in T_u\tilde{S}_0^\perp(1)$, from Proposition 3.3 we have the nonzero vector $\bar{A} := d(\bar{p}_r)^{-1}A \in T_{\bar{p}_r{}^{-1}u}E$. Then

$$\measuredangle\, (df_1 A, dhA) \leq \measuredangle\, (df_1 A, df_0 \bar{A}) + \measuredangle\, (df_0 \bar{A}, dh^* \bar{A}) + \measuredangle\, (dh^* \bar{A}, dhA) \ .$$

Because of (3.19), (3.21) and (3.22), the first and the last terms on the right hand side of the above inequality tend to zero as $\delta \to 1$. Let $\tilde{x}_0 = \exp_{\bar{p}_0}(\bar{p}_r p_r)^{-1}(u)$, and let $Y, Z$ be the Jacobi fields along $\tilde{a}_0, \tilde{b}_1$ respectively such that $Y$ is associated with the geodesic variation (3.11) and $Z(t) = dg_{\tilde{a}_0(t)}Y(t)$. From the construction,

$$(4.17)\qquad \begin{aligned} Y(0) = Z(0) = 0 \ , &\qquad \|Y(m)\| = \|Z(m)\| \ , \\ \bar{A} = Y'_\perp(0) \ , &\qquad dg\bar{A} = Z'_\perp(0) \ , \end{aligned}$$

where $Y$ and $Z$ are the normal components of $Y$ and $Z$ respectively. Furthermore we get the 1-parameter geodesic variation $\tilde{V} : [0, m] \times I \to \tilde{M}$ along $\tilde{b}_0$ such that $\tilde{V}(m, s) = g(V(m, s))$, $V(0, s) = \tilde{p}_0$ for any $s \in I$. Let $\tilde{Y}$ be the Jacobi field associated with $\tilde{V}$, and $\tilde{Y}_\perp$ its normal component to $\tilde{b}_0$. Then we see

$$(4.18)\qquad \tilde{Y}'_\perp(0) = df_0\bar{A} \ , \qquad \tilde{Y}(m) = Z(m) \ .$$

Since

$$\begin{aligned} \sphericalangle\,(dh^*A, d(\tau\circ g)\bar{A}) &= 2\{\tfrac{1}{2}\pi - \sphericalangle\,(\bar{\gamma}'(0), \bar{A})\} \\ &\le 2\,\mathrm{Max}\,\{\tfrac{1}{2}\pi - \Phi_L(\delta), \Phi_H(\delta) - \tfrac{1}{2}\pi\} \end{aligned}$$

and $\lim_{\delta\to1} \sphericalangle\,(dh^*\bar{A}, d(\tau\circ g)\bar{A}) = 0$ from (3.19), we have only to verify $\lim_{\delta\to1} \sphericalangle\,(d(\tau\circ g)\bar{A}, df_0\bar{A}) = 0$. This is equivalent to show

$$(4.19)\qquad \lim_{\delta\to1} d(\exp_{\tilde{p}_0} \tfrac{1}{2}\pi d(\tau\circ g)\bar{A},\ \exp_{\tilde{p}_0} \tfrac{1}{2}\pi df_0\bar{A}/\|df_0\bar{A}\|) = 0 \ .$$

Combining the approximation theorem for Jacobi fields with (1.7), (4.18), (3.19) gives

$$\lim_{\delta\to1} d(\exp_{\tilde{p}_0} \tfrac{1}{2}\pi df_0\bar{A}/\|df_0\bar{A}\|,\ \exp_{\tilde{p}_1} \tfrac{1}{2}\pi dg\bar{A}) = 0 \ .$$

The approximation theorem for Jacobi fields implies

$$\lim_{\delta\to1} d(\exp_{\tilde{p}_0} \tfrac{1}{2}\pi d(\tau\circ g)\bar{A},\ \exp_{\tilde{p}_1} \tfrac{1}{2}\pi dg\bar{A}) = 0 \ .$$

From these relations we obtain (4.19), and thus the proof of the proposition is complete.

**Corollary.** *There exists $\delta''$ independent of* $\dim M$ *such that*

$$(4.20)\qquad \delta > \delta''$$

*implies that both $f_1$ and $f_1^2$ are diffeotopic to $h$ and $h^2$ respectively.*

*Proof.* By the same arguments as in Lemma 4.3 and Proposition 4.5, we can verify

$$(4.21)\qquad \mathrm{Max}\,\{\sphericalangle\,(f_1^2(u), h^2(u))\,;\, u \in \tilde{S}_0^\perp(1)\} \le 2\beta'(\delta) \ ,$$

$$\text{(4.22)} \qquad \operatorname{Max}\{\measuredangle\,(d(f_1^2)A, d(h^2)A)\,;\, A \in T\tilde{S}_0^{\perp}(1)\} \le 2\varepsilon'(\delta)\ .$$

Hence we can find $\delta''$ independent of dim $M$ such that (4.20) ensures the diffeotopy conditions (1.11) and (1.12) for both $f_1 \cdot h^{-1}$ and $f_1^2 \cdot h^{-2}$. Thus the proof is complete.

*Proof of Theorem 4.1.* Let $\delta$ satisfy (4.20), and $F_t, H_t$, $t \in [0, 1]$ be the diffeotopies such that

$$\text{(4.23)} \qquad \begin{aligned} F_0 = H_0 = \text{id.}\,|\,\tilde{S}_0^{\perp}(1)\ , \qquad F_1 &= f_1(h_0\,|\,\tilde{S}_0^{\perp}(1))^{-1}\ , \\ H_1 &= f_1^2\cdot(h_0\,|\,\tilde{S}_0^{\perp}(1))^{-2}\ , \end{aligned}$$

where we have employed (4.11).

We now fix a point $u_0 \in \tilde{S}_0^{\perp}(1)$, and set $U_0$ to be the domain of $\tilde{S}_0^{\perp}(1)$ such that $u_0 \in U_0$ and $U_0 \cap f_1(U_0) \cap f_1^2(U_0) = \emptyset$, $\overline{U}_0 \cap f_1(\overline{U}_0) \neq \emptyset$, where $\overline{U}_0$ is the closure of $U_0$. Let $V_0 \subset U_0$ be the open ball contered at $u_0$ with the radius $r_0$ in such a way that for any $v \in \partial U_1$ the distance between $v$ and $\overline{V}_0$ is greater than $2\beta'(\delta)$. If $\delta$ is chosen so close to 1, we can find nonempty $V_0$. So we may consider $V_0 \neq \emptyset$. Let $W_0$ be the open ball centered at $u_0$ with the radius $r_1$, where $r_1$ is fixed in $(r_0, r_0 + 2\beta'(\delta))$. We define the functions $r\colon U_0 \to R$ and $\eta\colon [0, r_1] \to [0, 1]$ as follows:

$$\text{(4.24)} \qquad \begin{gathered} r(v) = \measuredangle\,(u_0, v)\ , \quad v \in U_0\ , \\ \eta(t) = 0 \quad \text{for } t \in [0, r_0]\ , \quad \eta'(t) > 0 \quad \text{for } t \in (r_0, r_1)\ , \quad \eta(r_1) = 1\ , \\ \eta^{(k)}(r_0) = \eta^{(k)}(r_1) = 0 \qquad \text{for } k = 1, 2, \cdots\ . \end{gathered}$$

We observe that both of the mappings $F_{\eta\cdot r}h_0\,|\,U_0\colon U_0 \to \tilde{S}_0^{\perp}(1)$ and $H_{\eta\cdot r}\circ h_0^2\,|\,U_0\colon U_0 \to \tilde{S}_0^{\perp}(1)$ defined by $v \to F_{\eta(r(v))}\circ h_0(v)$, $v \in U_0$ and $v \to H_{\eta(r(v))}\circ h_0^2(v)$, $v \in U_0$ respectively are imbeddings. In fact, $F_{\eta r}\circ h_0$ is locally regular and $F_{\eta r}\circ h_0\,|\,\overline{U}_0$ is $1-1$. Thus there exists $\hat{r}_0 \in (r_0, r_1]$ such that $F_{\eta r}\circ h_0\,|\,r^{-1}(\hat{r}_0)$ is imbedding. Suppose $F_{\eta r}\circ h_0\,|\,\overline{r^{-1}(\hat{r}_0)}$ is not $1-1$. Then we can find $v_1, v_2$ such that $r(v_1) = r(v_2) = \hat{r}_2$ and $F_{\eta(r(v_1))}h_0(v_1) = F_{\eta(r(v_2))}h_0(v_2)$. However this is a contradiction since $F_{\eta\hat{r}_0}\circ h_0$ is a diffeomorphism on $\tilde{S}_0^{\perp}(1)$. With these notations we can define a deformation $\Psi_t^1$ of $\Psi_1$ such that $\Psi_t^1$ coinsides with $\varphi_1$ on the open set $\tilde{S}_0^{\perp}(1) - \bigcup_{i=0}^{2} f_1^i(\overline{W}_0)$ and coinsides with the standard action on $\bigcup_{i=1}^{2} f_1^i(V_0)$. Indeed, for each $t \in [0, 1]$ let $\xi_t^1\colon \tilde{S}_0^{\perp}(1) \to \tilde{S}_0^{\perp}(1)$ be the diffeomorphism

$$\text{(4.25)} \qquad \xi_t^1(v) = \begin{cases} v, & \text{for } v \notin f_1(W_0) \cup f_1^2(W_0)\ , \\ F_{t\eta(r(h_0^{-1}(v)))}(v), & \text{for } v \in f_1(W_0)\ , \\ H_{t\eta(r(h_0^{-2}(v)))}(v), & \text{for } v \in f_1^2(W_0)\ . \end{cases}$$

Then $\xi_0^1$ is the identity, we see

(4.26) $$\Psi_t^1(g^i, u) := \xi_t^1 \cdot \varphi_1(g^i, (\xi_t^1)^{-1}(u))$$

is the desired deformation. We shall call $u_0$ the center of the deformation. From the strong diffeotopy theorem we see that the new action $\Psi_1^1$ is able to play the same role as $\varphi_1$ if $\delta$ is taken sufficiently close to 1 and independent of $\dim M$. Thus we can find the sequence $\delta_k$ of pinching numbers such that $\delta > \delta_k$ can be carried out $k$ times of deformations mentioned above, where the centers can be arbitrarily chosen. Let us take the finite open cover $U_0, U_1, \cdots, U_N$ of $\tilde{S}_0^{\perp}(1)$, where each $U_i$ is the ball with the radius $r_0$ and center $u_i$, and $W_0, W_1, \cdots, W_N$ are the open balls each of which has the radius $r_1$ with the same center $u_i$. If $\delta > \delta_N$, we can define $N$ deformation $\Psi_t^1, \Psi_t^2, \cdots, \Psi_t^N$ such that

$$\Psi_t^j(g^i, u) = \xi_t^j \circ \Psi_1^{j-1}(g^i, (\xi_t^{j-1})^{-1}(u)) , \qquad \Psi_t^0 := \varphi_1 .$$

Then clearly $\Psi_1^N = \Psi_1$. Thus the proof is completed.

It should be remarked that the number $N$ depends on $\dim M$ since the boundary $\partial U_0$ has so large diameter (indeed close to $\pi$) that $N$ increases rapidly with $\dim M$.

As a direct consequence of Theorem 4.1 we have the

**Corollary to Theorem 4.1.** *Under the same assumption as in Theorem* 4.1, *$M$ is homeomorphic to* $L^{2n+1}(1; 3)$.

*Proof of the Main Theorem.* Since $E \subset \tilde{S}_0(1)$ is diffeomorphic to $S^{2n}$, $\tilde{S}_0(1) - E$ consists of the components each bounded by $E$. Let $D_+ \ni \tilde{\gamma}'(0)$ and $D_- \ni -\tilde{\gamma}'(0)$ be the components. By means of the deck transformation $g$, we have the diffeomorphism $f^* : D_- \to D_+$ defined by

$$f^*(v) = \frac{1}{m}(\exp_{\tilde{p}_0} | U_\pi(\tilde{p}_0))^{-1} \circ g \circ (\exp_{\tilde{p}_0} mv) ,$$

$$\exp_{\tilde{p}_0} mv \in \mathscr{F}_1^- , \qquad l \le m \le \tfrac{1}{2}\pi\sqrt{\delta} .$$

Clearly we get $f^*(-\tilde{\gamma}'(0)) = \tilde{\gamma}'(0)$. From the construction for any $A \in TE$ and any $A_i \in TD_-$ such that $\lim A_i = A$ we see

(4.28) $$dfA_0 = \lim df^*A_i .$$

Making use of $\bar{p}_r : E \to \tilde{S}_0^{\perp}(1)$ (defined in (3.23)), we can construct a homotopy $\hat{p}_r : [0, 1] \times \tilde{S}_0(1) \to \tilde{S}_0(1)$ of diffeomorphism satisfying the following conditions: (1) If $\hat{p}_r^\tau(v) := p_r(\tau, v)$ for each $\tau \in [0, 1]$, then $\hat{p}_r^\tau$ is a diffeomorphism on $\tilde{S}_0(1)$ (and $\hat{p}_r^0 = \mathrm{id.} | \tilde{S}_0(1)$). (2) For each point $v \in \tilde{S}_0(1)$, $\hat{p}_r([0, 1], v)$ lies on the great circular arc joining $v$ to $\tilde{\gamma}'(0)$ when $v \in D_+$ (or joining $v$ to $-\tilde{\gamma}'(0)$ when $v \in D_-$). (3) For each $\tau \in [0, 1]$, $\hat{p}_r^\tau(\pm\tilde{\gamma}'(0)) = \pm\tilde{\gamma}'(0)$. (4) $\hat{p}_r^1 | E = \bar{p}_r$. Clearly $\hat{f} := \hat{p}_r^1 \cdot f^* \cdot (\hat{p}_r^1)^{-1}$ is a diffeomorphism from the southern hemisphere $\tilde{S}_-$ onto the northern hemisphere $\tilde{S}_+$, where the north pole is $\tilde{\gamma}'(0)$. Then

$\hat{f}(-\tilde{\gamma}'(0)) = \tilde{\gamma}'(0)$, and $\lim d\hat{f}A_i = df_1 A$ holds for any $A \in T\tilde{S}_0^{\perp}(1)$ and $A_i \in T\tilde{S}_-$ such that $\lim A_i = A$.

The final step of the proof is to verify that $\hat{f}$ is diffeotopic to $h_0 | \tilde{S}_-$. By means of Lemmas 3.2 and 4.4, there exists a constant $\hat{L}(\delta)$ such that

$$\hat{L}(\delta)^{-1} \leq \|d\hat{f}A\| \leq \hat{L}(\delta) \quad \text{for any } A \in T\tilde{S}_- \,, \quad \|A\| = 1 \,, \quad \lim_{\delta \to 1} \hat{L}(\delta) = 1 \,.$$

Therefore we can find $\beta''(\delta)$ such that $\operatorname{Max}\{\measuredangle\, (h_0(u), \hat{f}(u))\,;\, u \in \tilde{S}_-\} \leq \beta''(\delta)$ and $\lim_{\delta \to 1} \beta''(\delta) = 0$.

On the other hand, by the same method as in Proposition 3.3 there exists $\varepsilon''(\delta)$ such that

$$\operatorname{Max}\{\measuredangle\, (dh_0 A, dfA)\,;\, A \in T\tilde{S}_-\} \leq \varepsilon''(\delta) \,, \qquad \lim_{\delta \to 1} \varepsilon''(\delta) = 0 \,.$$

Thus we can find $\delta_n \geq \delta'_N$ such that $\delta > \delta_n$ ensures that $\hat{f}$ is diffeotopic to $h_0 | \tilde{S}_-$, and the proof of the main theorem is complete.

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