

## ON TWO NOTIONS OF STRUCTURAL STABILITY

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### Introduction

In the literature one can find two notions of structural stability. First the original one given by Andronov and Pontriagin (see [1], [2], [5]) stated for vector fields, that is, for the actions of the additive group of real number  $\mathbf{R}$  on a manifold  $M$ . This definition says roughly that an  $\mathbf{R}$ -action on  $M$  is structurally stable if, for any other  $\mathbf{R}$ -action close-to, in the sense that the vector fields generating these actions are close, there exists a homeomorphism of  $M$  onto itself mapping the orbits of the first action onto the orbits of the second. This definition can readily be extended (see below § 1), to actions on  $M$  of a given real Lie group  $G$  in particular  $G = \mathbf{Z} =$  additive group of all integers.

Another definition was proposed more recently by Smale (see [8] and [9]) for  $\mathbf{Z}$ -actions on  $M$ . Such an action is generated by an diffeomorphism  $\phi: M \rightarrow M$ . Smale's definition is roughly that  $\phi$  is structurally stable if any diffeomorphism  $\psi$  sufficiently close to  $\phi$  in the  $C^1$ -topology is topologically conjugate to  $\phi$ . Smale's definition, which can also be extended to action on  $M$  of any given real Lie group  $G$ , seems more restrictive than the one of Andronov and Pontrjagin.

The purpose of this note is to show that in the case  $G = \mathbf{Z}$ , the two definitions are equivalent if the dimension of  $M > 1$  and  $M$  is connected.

In § 1 we give precise statements of the two definitions, first in the case  $G = \mathbf{Z}$  (which is the one of interest to us) and then in the general case, for comparison sake.

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### 1. Definitions of structural stability

A  $C^1$ - $\mathbf{Z}$ -action on a compact  $C^\infty$  manifold  $M$  is generated by a  $C^1$ -diffeomorphism  $\phi: M \rightarrow M$ .

**Definition 1** (*Andronov-Pontrjagin*). A  $C^1$ -diffeomorphism  $\phi: M \rightarrow M$  is structurally stable if for any  $\varepsilon > 0$  there exists a neighborhood  $U$  of  $\phi$  in

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$\text{Diff}^1(M)$  for the  $C^1$ -topology ( $\text{Diff}^1(M) =$  group of all  $C^1$ -diffeomorphisms of  $M$  onto itself) such that for any  $\psi \in U$  there exists a homeomorphism  $h: M \rightarrow M$  with the following two properties: (a)  $h$  maps the orbits of  $\phi$  onto the orbits of  $\psi$ . (b)  $d(x, h(x)) \leq \varepsilon$  for all  $x \in M$  where  $d$  is some chosen metric on  $M$  compatible with its topology. A homeomorphism with property (b) is called an  $\varepsilon$ -homeomorphism.

**Definition 2 (Smale).** A  $C^1$ -diffeomorphism  $\phi: M \rightarrow M$  is said to be structurally stable if for any  $\varepsilon > 0$  there exists a neighborhood  $U$  of  $\phi$  in  $\text{Diff}^1(M)$  for the  $C^1$ -topology such that for any  $\psi \in U$  there exist an  $\varepsilon$ -homeomorphism  $h: M \rightarrow M$  and  $t$  equal to  $+1$  or  $-1$  such that  $\psi = h \circ \phi^t \circ h^{-1}$ .

In fact in Smale's definition  $t = 1$  always. We introduce this little complication in Smale's definition in order to square it off with the general definition. It will follow from what will be proved later that  $t = 1$  always (provided one chooses  $U$  small enough).

Now we give the same definitions in the general case of any Lie group  $G$ . A  $C^1$ -action of  $G$  on a manifold  $M$  is a homomorphism  $\phi: G \rightarrow \text{Diff}^1(M)$  (also called representation) such that the mapping  $G \times M \rightarrow M: (g, x) \rightarrow \phi(g)[x]$  is of class  $C^1$ . Call  $A^1(G, M)$  the set of all these  $C^1$ -actions. It is a subset of  $C^0(G, \text{Diff}^1(M))$ , the set of all continuous mappings  $G \rightarrow \text{Diff}^1(M)$ . This set carries the compact open topology ( $\text{Diff}^1(M)$  being endowed with the  $C^1$ -topology). Hence  $A^1(G, M)$  as a subset of  $C^0(G, \text{Diff}^1(M))$  is endowed, by restriction, of the compact open topology. The orbit of  $x \in M$  under the  $G$ -action  $\phi$  is the set  $\{\phi(g)(x) \mid g \in G\}$ .

**Definition 1.** A  $C^1$ -action  $\phi_0: G \rightarrow \text{Diff}^1(M)$  of  $G$  on  $M$  is said to be structurally stable if for any  $\varepsilon > 0$  there exists a neighborhood  $U$  of  $\phi_0$  in  $A^1(G, M)$  with the compact open topology such that for any  $\phi \in U$  there exists an  $\varepsilon$ -homeomorphism  $h: M \rightarrow M$  mapping the orbits of  $\phi_0$  onto the orbits of  $\phi$ .

**Definition 2.** A  $C^1$ -action  $\phi_0: G \rightarrow \text{Diff}^1(M)$  of  $G$  on  $M$  is said to be structurally stable if for any  $\varepsilon > 0$  there exists a neighborhood  $U$  of  $\phi_0$  in  $A^1(G, M)$  with the compact open topology such that for any  $\phi \in U$  there exist an  $\varepsilon$ -homeomorphism  $h: M \rightarrow M$  and an algebraic automorphism  $t: G \rightarrow G$  such that  $h \circ \phi(g) = \phi_0(t(g)) \circ h$  for all  $g \in G$ .

It is obvious that Definitions 1 and 2 stated previously in the case  $G = Z$  are particular cases of the general Definitions 1 and 2 stated above.

**Remark (due to J. Palis).** For  $G \neq Z$  it is obvious that in general Definition 1 does not imply Definition 2. In fact in the case  $G = \mathbf{R}$  consider a Morse-Smale vector field with 2 or more closed orbits with different periods. It seems that it would be more appropriate to have the reparametrization  $t$  to depend on the points on the manifold too so that in Definition 2 one should replace  $t: G \rightarrow G$  by  $t: G \times M \rightarrow M$  and the following equation by:

$$h(\phi(g)x) = \phi_0(t(g, x))h(x) .$$

Now we state our main result.

**Theorem.** *In the case  $G = Z$  the two definitions are equivalent at least if  $\dim M \geq 1$  and  $M$  is connected.*

It is obvious that Definition 2 implies Definition 1. So we only have to show the converse. This will follow from some lemmas.

**Comment on the case  $\dim M = 1$ .** In that case  $M = S^1$ . Our proof below does not cover that case but the theorem is true in that case. The proof (due to J. Palis) is as follows: by Peixotos theorem [5] the structurally stable, in the sense of Definition 1,  $Z$ -actions are the Morse-Smale actions, and then it is an easy particular case of a theorem of J. Palis [3] that the Morse-Smale actions are stable in the sense of Definition 2.

## 2. Some auxiliary lemmas

**Lemma 1.** (a) *If  $\phi: M \rightarrow M$  and  $\psi: M \rightarrow M$  are two  $C^1$ -diffeomorphisms such that there exists a homeomorphism  $h: M \rightarrow M$  mapping the orbits of  $\phi$  onto the orbits of  $\psi$ , then  $h$  maps the set  $\text{Per}(\phi)$  of all periodic points of  $\phi$  onto the set  $\text{Per}(\psi)$  of all periodic points of  $\psi$ .*

(b) *If a  $C^1$ -diffeomorphism  $\phi: M \rightarrow M$  is structurally stable, then  $\text{Per}(\phi)$  is countable.*

*Proof.* All this is well known. We only give the proofs for the sake of completeness. (a) follows immediately from the fact that a point  $X \in M$  is periodic for  $\phi$  if and only if its orbit is compact, and compactness is preserved by a homeomorphism. To prove (b) we use a general approximation theorem (see [4] or [7] or [8]) which implies that the set  $E$  of all  $C^1$ -diffeomorphisms  $\phi$ , such that  $\text{Per}(\psi)$  is countable, is a Baire subset of  $\text{Diff}^1(M)$ . Hence choose a  $\psi \in E$  so close to  $\phi$  that there exists a homeomorphism  $h: M \rightarrow M$  mapping the orbits of  $\phi$  onto the orbits of  $\psi$ . Then by (a)  $h$  maps  $\text{Per}(\phi)$  onto  $\text{Per}(\psi)$ , and hence  $\text{Per}(\phi)$  is countable.

**Lemma 2.** *Assume that  $\dim M > 1$  and  $M$  is connected, and that  $D$  is a countable subset of  $M$ . Then  $M - D$  is arcwise connected.*

*Proof.* Let  $U_1 \cup U_2 \cup \dots \cup U_n$  be a finite covering of  $M$ ,  $U_j$ ,  $j = 1, \dots, n$ , being domains of charts  $(U_i, \alpha_i)$ , and  $\alpha_i$  mapping  $U_i$  onto the unit open ball  $B^d(0, 1)$  in the euclidean space  $\mathbf{R}^d$  ( $d = \text{dimension of } M$ ). Let  $D_j = D \cap U_j$  and  $C_j = \alpha_j(D_j)$ . It is obviously sufficient to prove that for  $x$  and  $y$  in  $U_j - D_j$  there exists a continuous curve in  $U_j - D_j$  joining  $x$  to  $y$  or what is the same that there exists a continuous curve in  $B^d(0, 1) - C_j$  joining  $\alpha_j(x)$  to  $\alpha_j(y)$ . Let  $E$  be the 2-plane in  $\mathbf{R}^d$  spanned by  $0, \alpha_j(x), \alpha_j(y)$  (if  $\alpha_j(x) = 0$  or  $\alpha_j(y) = 0$ , take any 2-plane containing  $0$  and  $\alpha_i(y)$  or  $\alpha_j(x)$ ). Let  $\Delta(x)$  be the set of all lines in  $E$  joining  $\alpha_j(x)$  to the points of  $C_j \cap E$ , and  $\Delta(y)$  the corresponding set for  $\alpha_j(y)$ .  $\Delta(x)$  and  $\Delta(y)$  are countable. Hence they exist, as close as we want to the line  $(\alpha_j(x), \alpha_j(y))$ , a line  $\delta_x \notin \Delta(x)$  passing through  $\alpha_j(x)$  and a line  $\delta_y \notin \Delta(y)$  passing through  $\alpha_j(y)$ . These lines meet at a point  $\zeta$  close to the

segment  $\overline{\alpha_j(x)\alpha_j(y)}$ . As this segment is contained in  $B^d(0, 1)$  so will be  $\zeta$ ; as  $B^d(0, 1)$  is convex, the 2 segments  $\overline{\alpha_j(x)\zeta}$  and  $\overline{\zeta\alpha_j(y)}$  will be in  $B^d(0, 1)$  and, in fact, in  $B^d(0, 1) - C_j$  by construction since  $\overline{\alpha_j(x)\zeta}$  lies on  $\delta_x$  and  $\overline{\zeta\alpha_j(y)}$  on  $\delta_y$ . Hence the polygonal curve  $\overline{\alpha_j(x)\zeta} \cup \overline{\zeta\alpha_j(y)}$  joins  $\alpha_j(x)$  to  $\alpha_j(y)$  in  $B^d(0, 1) - C_j$ , and Lemma 2 is proved.

### 3. Proof of the theorem

Assume  $\phi$  and  $\psi$  are two  $C^1$ -diffeomorphisms  $M \rightarrow M$  such that there exists a homeomorphism  $h: M \rightarrow M$  mapping the orbits of  $\phi$  onto the orbits of  $\psi$  and that  $\text{Per}(\phi)$  is countable (hence by Lemma 1 (a)  $\text{Per}(\psi)$  is). We are going to show that there exists an integer  $q \in \mathbb{Z}$  such that  $\psi^q \circ h = h \circ \phi$ .

Choose a point  $x_0$  in  $M - \text{Per}(\phi)$ . Then there exists a unique integer  $q \in \mathbb{Z}$  such that  $\psi^q(h(x_0)) = h(\phi(x_0))$ . So we are going to show that if  $y$  is any other point in  $M - \text{Per}(\phi)$ , then  $\psi^q(h(y)) = h(\phi(y))$ . By Lemma 2 there exists a continuous arc  $\gamma \subset M - \text{Per}(\phi)$ , joining  $x_0$  to  $y$  parametrized by the continuous map  $t \in I = [0, 1] \rightarrow x(t) \in M - \text{Per}(\phi)$ ,  $x(0) = x_0$ ,  $x(1) = y$ . Let  $T_m = \{t \mid t \in I, \psi^m(h(x(t))) = h(\phi(x(t)))\}$ .

**Lemma 3.** (a) *The  $T_m$  are closed.* (b)  *$T_m \cap T_k = \emptyset$  if  $m \neq k$ .*  
 (c)  *$\bigcup_{m \in \mathbb{Z}} T_m = I$ .*

*Proof.* (a) follows from the fact that the 2 functions  $t \rightarrow \psi^m(h(x(t)))$  and  $t \rightarrow h(\phi(x(t)))$  are continuous.

(b) If  $t_0 \in T_m \cap T_k$ , then  $\psi^m(h(x(t_0))) = h(\phi(x(t_0))) = \psi^k(h(x(t_0)))$ . Hence  $\psi^{m-k}(h(x(t_0))) = h(x(t_0))$ . So  $h(x(t_0)) \in \text{Per}(\psi)$ . By Lemma 1 (a)  $x(t_0) \in \text{Per}(\phi)$ . But  $x(t_0) \in \gamma \subset M - \text{Per}(\phi)$ , a contradiction.

(c) is obvious for given any  $x \in M$ ,  $h(\phi(x)) \in \psi$ -orbit of  $h(x)$  and hence there exists an  $n$  such that  $\psi^n(h(x)) = h(\phi(x))$ . If  $x \in M - \text{Per}(\phi)$ , this  $n$  is unique; otherwise *not*.

Let  $J_m$  be the interior of  $T_m$ , and let  $\omega = \bigcup_{m \in \mathbb{Z}} J_m$ .

**Lemma 4.** (a)  *$\omega$  is open and dense in  $I$ .*

(b) *Every connected component of  $\omega$  is contained in one and only one  $T_m$ .*

*Proof.* (a) Since  $I$  is the union of the countable class of closed sets  $T_m$ , (a) follows from the Baire property of  $I$ .

(b) Since  $\omega$  is the union of the open *disjoint* sets  $J_m$ , the components of  $\omega$  are those of the  $J_m$ . Hence (b) follows from this and Lemma 3 (b).

Let  $K = I - \omega$ . If  $K$  is empty,  $\omega = I$ .  $I$  is the sole connected component of  $\omega$  and hence contained in a unique  $T_m$  by Lemma 4 (b). As  $0 \in T_q$  ( $x(0) = x_0$  and  $\psi^q(h(x_0)) = h(\phi(x_0))$  by assumption),  $m = q$  so  $1 \in T_q$  and  $\psi^q(h(y)) = \psi^q(h(x(1))) = h(\phi(y))$ . We will show that  $K \neq \emptyset$  leads to a contradiction. Let  $K_m = K \cap T_m$ . Then the  $K_m$  are closed and  $K = \bigcup_{m \in \mathbb{Z}} K_m$ . As a closed subset of  $I$ ,  $K$  has the Baire property, so one of the set  $K_m$ , say  $K_p$ , has a non-empty interior in the space  $K$ . This means that there exists an open interval  $\delta \subset I$  such that  $\delta \cap K \neq \emptyset$  and  $\delta \cap K \subset K_p$ . By Lemma 4 (a),  $\delta \cap \omega \neq \emptyset$ . Let

$]a, b[ = \{t \mid a < t < b\}$  be a connected component of  $\omega$  meeting  $\delta$ . Then one of the extremities  $a, b$  is contained in  $\delta$ , for otherwise  $]a, b[ \supset \delta$  and then  $\delta \cap K = \emptyset$ . Assume for example that  $a \in \delta$ . As  $a \in K$ ,  $a \in \delta \cap K \subset K_p$ . So  $a \in T_p$ . But by Lemma 4 (b),  $]a, b[$  is contained in a unique  $T_k$ . As  $a \in$  closure of  $]a, b[$  and this closure is also contained in  $T_k$  ( $T_k$  being closed),  $a \in T_k$ . As  $T_k \cap T_p = \emptyset$  if  $k \neq p$ , by Lemma 3 (b) it follows that  $k = p$ . So we have proved that if a connected component of  $\omega$  meets  $\delta$ , then it is contained in  $T_p$ . Thus  $\delta \cap \omega \subset T_p$ . As  $\delta \cap K \subset K_p \subset T_p$ , it follows that  $\delta \subset T_p$ ; hence  $\delta \subset J_p$  as  $\delta$  is open. But then  $\delta \subset \omega$  which contradicts the fact that  $\delta \cap K \neq \emptyset$ . So  $K = \emptyset$ , and we have proved the following:

**Lemma 5.**  $\psi^q \circ h = h \circ \phi$ .

**Lemma 6.**  $q = \pm 1$ .

*Proof.* If  $q \neq \pm 1$ , then  $h_0 \phi^k = \psi^{qk} \circ h$  for all  $k \in \mathbb{Z}$ , (easy to see by induction), and hence  $h(\phi$ -orbit of  $x) = \{\psi^{qk}(h(x)) \mid k \in \mathbb{Z}\}$ . This last set is not the whole orbit of  $x$ , unless  $q = \pm 1$  if  $x \notin \text{Per}(\phi)$ .

Finally we have proved the following:

**Proposition.** If  $\phi, \psi: M \rightarrow M$  are two  $C^1$ -diffeomorphisms such that there exists a homeomorphism  $h: M \rightarrow M$  mapping the orbits of  $\phi$  onto the orbits of  $\psi$ , then  $\psi^q \circ h = h \circ \phi$  where  $q = 1$  or  $-1$ .

Now assume  $\phi: M \rightarrow M$  is structurally stable in the sense of Definition 1. For any  $\varepsilon > 0$  there exists a neighborhood  $U$  of  $\phi$  in  $\text{Diff}^1(M)$  such that for any  $\psi \in U$  there exists an  $\varepsilon$ -homeomorphism  $h: M \rightarrow M$  mapping the orbits of  $\phi$  onto the orbits of  $\psi$ . By the proposition it follows that  $\psi^q \circ h = h \circ \phi$  where  $q = +1$  or  $-1$ . But there exists a possibly smaller neighborhood  $U' \subset U$  of  $\phi$  in  $\text{Diff}^1(M)$  such that if  $\psi \in U'$  then  $q = 1$ . For, if such a neighborhood did not exist, then we could find a sequence  $\{\psi_j \mid j = 1, 2, \dots\}$ ,  $\psi_j \in U$  and  $\psi_j \rightarrow \phi$  in  $\text{Diff}^1(M)$  as  $j \rightarrow +\infty$ , and a sequence  $\{h_j \mid j = 1, 2, \dots\}$  of homeomorphisms such that  $h_j$  maps the orbits of  $\phi$  onto the orbits of  $\psi_j$  and  $\sup_{x \in M} (x, h_j(x)) \rightarrow 0$  as  $j \rightarrow \infty$ , and  $\psi_j^q \circ h_j = h_j \circ \phi$  with  $q = -1$ . Thus taking the limits as  $j \rightarrow +\infty$  we get  $\phi^{-1} = \phi$ , and all points in  $M$  would be periodic of period 2 contradicting Lemma 1 (a). Hence the theorem is proved.

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