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# ON EINSTEIN METRICS

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Let us consider a compact orientable $C^\infty$ manifold $M$. If a $C^\infty$ Riemannian metric $g$ is given on $M$, we get a Riemannian manifold $(M, g)$. Let us consider the set of Riemannian metrics $g$ on $M$ such that

$$\int_M dV = 1 ,$$

where $dV$ is the volume element of $(M, g)$. We denote this set by $\mathscr{M}(M)$.

Consider the integral

$$I(g) = \int_M K dV ,$$

where $K$ is the scalar curvature of $(M, g)$. It is well known that a critical point $\bar{g}$ of $I(g)$ in $\mathscr{M}(M)$ is an Einstein metric. A question then arises that whether a given Einstein metric gives a minimum of $I(g)$ or not. It has been established by M. Berger [1] that there exists some Einstein metric (on some $C^\infty$ manifold) for which both $I(g)$ and $-I(g)$ have nonfinite indices.

The purpose of the present paper is to prove the following main theorem.

**Theorem.** *Let $I(g)$ be the integral as defined above. Then the index of $I(g)$ and also the index of $-I(g)$ are positive at each critical point.*

In the last paragraph some suggestion is given about the index.

## 1. Infinitesimal deformations of a Riemannian metric from an Einstein metric

In the following we use local coordinates, and a tensor is expressed in its components with respect to the natural frame. Thus $K_{kji}{}^h$ means the curvature tensor

$$K_{kji}{}^h = \partial_k \begin{Bmatrix} h \\ ji \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ ki \end{Bmatrix} + \begin{Bmatrix} h \\ kl \end{Bmatrix} \begin{Bmatrix} l \\ ji \end{Bmatrix} - \begin{Bmatrix} h \\ jl \end{Bmatrix} \begin{Bmatrix} l \\ ki \end{Bmatrix} ,$$

where $\begin{Bmatrix} h \\ ji \end{Bmatrix}$ is the Christoffel symbol of the metric $g$. The Ricci tensor and the scalar curvature are respectively given by

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$$K_{ji} = K_{tji}{}^{t}\,, \qquad K = K_{ji}g^{ji} = K^{ji}g_{ji}\,.$$

When we take a $C^\infty$ curve $g(t)$ in $\mathscr{M}(M)$, we get several tensor fields defined by

$$D_{ji} = \frac{\partial}{\partial t}g_{ji}\,, \qquad D_i{}^h = D_{ik}g^{kh}\,, \qquad D^{ih} = D_{kj}g^{ki}g^{jh}\,,$$

$$D_{ji}{}^h = \tfrac{1}{2}(\nabla_j D_i{}^h + \nabla_i D_j{}^h - \nabla^h D_{ji})\,, \qquad D_{kji}{}^h = \nabla_k D_{ji}{}^h - \nabla_j D_{ki}{}^h\,,$$

where $\nabla$ means the covariant differentiation with respect to the metric $g(t)$. Then we get

$$\frac{\partial}{\partial t}K_{kji}{}^h = D_{kji}{}^h\,,$$

and consequently

$$\frac{d}{dt}I(g(t)) = \int_M \left[D_{kji}{}^k g^{ji} - K^{ji}D_{ji} + \frac{1}{2}Kg^{ji}D_{ji}\right]dV\,.$$

Since any divergence vanishes by integration, we immediately obtain

$$\text{(1.1)} \qquad \frac{d}{dt}I(g(t)) = \int_M \left(-K^{ji} + \frac{1}{2}Kg^{ji}\right)D_{ji}dV\,.$$

On the other hand $D_{ji}$ must satisfy the only condition

$$\text{(1.2)} \qquad \int_M g^{ji}D_{ji}dV = 0\,.$$

Let $\bar{g}$ be a Riemannian metric such that, for every $C^\infty$ curve $g(t)$ of $\mathscr{M}(M)$ satisfying $g(0) = \bar{g}$, $I(g(t))$ has vanishing derivative at $t = 0$. Then from (1.1) and (1.2) we get

$$K^{ji} = \tfrac{1}{2}Kg^{ji} + cg^{ji}$$

for this metric $\bar{g}$. Here $c$ is a constant, and this leads to the well-known formula of an Einstein space $(M, \bar{g})$, namely,

$$\text{(1.3)} \qquad K_{ji} = \frac{K}{n}g_{ji}\,.$$

Now let us assume that $g(0)$ is an Einstein metric and study the behavior of $d^2I/dt^2$ at $t = 0$ for various curves $g(t)$.

Differentiating (1.1) again we get

$$
(1.4)\quad \begin{aligned}
\frac{d^2I(g(t))}{dt^2} = \int_M \Big[ & \Big(-g^{lj}g^{ki}\frac{\partial K_{lk}}{\partial t} + 2g^{lj}\frac{\partial g_{lk}}{\partial t}K^{ki}\Big)\frac{\partial g_{ji}}{\partial t} - K^{ji}\frac{\partial^2 g_{ji}}{\partial t^2} \\
& + \frac{1}{2}\frac{\partial K_{ji}}{\partial t}g^{ji}g^{lk}\frac{\partial g_{lk}}{\partial t} - \frac{1}{2}K^{ji}\frac{\partial g_{ji}}{\partial t}g^{lk}\frac{\partial g_{lk}}{\partial t} \\
& - \frac{1}{2}Kg^{lj}g^{ki}\frac{\partial g_{lk}}{\partial t}\frac{\partial g_{ji}}{\partial t} + \frac{1}{2}Kg^{ji}\frac{\partial^2 g_{ji}}{\partial t^2} \\
& + \Big(-K^{ji}\frac{\partial g_{ji}}{\partial t} + \frac{1}{2}Kg^{ji}\frac{\partial g_{ji}}{\partial t}\Big)\frac{1}{2}g^{lk}\frac{\partial g_{lk}}{\partial t}\Big]dV\,,
\end{aligned}
$$

in which we can put

$$
\frac{\partial K_{ji}}{\partial t} = \nabla_t D_{ji}{}^t - \nabla_j D_{ti}{}^t\,.
$$

Using Green's theorem we can write (1.4) in the form

$$
\begin{aligned}
\frac{d^2I}{dt^2} = \int_M \Big[ & D^{jih}\nabla_h D_{ji} - D_t{}^{it}\nabla^j D_{ji} + 2D_t{}^j K^{ti} D_{ji} \\
& - \frac{1}{2}D_s{}^{st}\nabla_t D_i{}^i + \frac{1}{2}D_{st}{}^t\nabla^s D_i{}^i - K^{ji}D_{ji}D_t{}^t \\
& - \frac{1}{2}KD_{ji}D^{ji} + \frac{1}{4}K(D_t{}^t)^2 - \Big(K^{ji} - \frac{1}{2}Kg^{ji}\Big)\frac{\partial^2 g_{ji}}{\partial t^2}\Big]dV\,.
\end{aligned}
$$

Since $g(0)$ is an Einstein point[1], we get

$$
\int_M \Big[\Big(K^{ji} - \frac{1}{2}Kg^{ji}\Big)\frac{\partial^2 g_{ji}}{\partial t^2}\Big]dV = -\Big(\frac{1}{2} - \frac{1}{n}\Big)K\int_M g^{ji}\frac{\partial^2 g_{ji}}{\partial t^2}dV
$$

at $t = 0$. On the other hand, differentiating $\int_M dV = 1$ we get

$$
\int_M g^{ji}\frac{\partial g_{ji}}{\partial t}dV = 0\,,
$$

$$
\int_M \Big[g^{ji}\frac{\partial^2 g_{ji}}{\partial t^2} - g^{lj}g^{ki}\frac{\partial g_{lk}}{\partial t}\frac{\partial g_{ji}}{\partial t} + \frac{1}{2}\Big(g^{ji}\frac{\partial g_{ji}}{\partial t}\Big)^2\Big]dV = 0\,.
$$

Thus

$$
(1.5)\quad \begin{aligned}
\Big(\frac{d^2I}{dt^2}\Big)_0 = \int_M \Big[ & (\nabla^j D^{ih})\nabla_h D_{ji} - \frac{1}{2}(\nabla^h D^{ji})\nabla_h D_{ji} - (\nabla^i D_j{}^j)\nabla^h D_{hi} \\
& + \frac{1}{2}(\nabla^i D_j{}^j)\nabla_i D_h{}^h + \frac{K}{n}\Big\{D_{ji}D^{ji} - \frac{1}{2}(D_i{}^i)^2\Big\}\Big]dV
\end{aligned}
$$

[1] We call an Einstein metric an Einstein point in $\mathscr{M}(M)$.

for any symmetric tensor field $D_{ji}$ satisfying (1.2) if $g(0)$ is an Einstein point.

## 2. Infinitesimal conformal deformations of a Riemannian metric

If we put $D_{ji} = fg_{ji}$ where $f$ is a $C^\infty$ function such that $\int_M f dV = 0$, then

$$\left(\frac{d^2 I}{dt^2}\right)_0 = \frac{n-2}{2}\left[(n-1)\int_M (\nabla^i f)\nabla_i f dV - K\int_M f^2 dV\right].$$

There exist functions $f$ which make this quantity positive. This proves that the index of $-I(g)$ is positive.

## 3. Infinitesimal deformations of a Riemannian metric in a small neighborhood

Let $U$ be a coordinate neighborhood of $M$, and let $N \subset U$ be a neighborhood of a point $P_0 \in U$, where the local coordinates are such that

$$g_{ji} = \delta_{ji}, \qquad \left\{ {h \atop ji} \right\} = 0$$

at $P_0$. We assume that $N$ is sufficiently small so that there exists a positive number $\varepsilon$ such that $g$ satisfies in $N$

$$|g_{ji} - \delta_{ji}| < \varepsilon, \quad |g^{ji} - \delta_{ji}| < \varepsilon, \quad \left|\left\{ {h \atop ji} \right\}\right| < \varepsilon.$$

Now we want to take a suitable $C^\infty$ tensor field $D_{ji}$. We know that for any given tensor field $D_{ji}$ there exists $g(t)$ such that

$$\left(\frac{\partial g_{ji}}{\partial t}\right)_0 = D_{ji}.$$

First, we assume $D_i{}^i = 0$ on $M$. Then

$$\left(\frac{d^2 I}{dt^2}\right)_0 = \int_M \left[(\nabla^j D^{ih})\nabla_h D_{ji} - \frac{1}{2}(\nabla^h D^{ji})\nabla_h D_{ji} + \frac{K}{n} D_{ji} D^{ji}\right] dV.$$

As we have

$$|(\nabla^j D^{ih} + \nabla^i D^{jh})\nabla_h D_{ji}| = |(\nabla_c D_{ba} + \nabla_b D_{ca})(\nabla_h D_{ji}) g^{cj} g^{bi} g^{ah}|,$$

where

$$\nabla_j D_{ih} = \partial_j D_{ih} - \begin{Bmatrix} k \\ ji \end{Bmatrix} D_{kh} - \begin{Bmatrix} k \\ jh \end{Bmatrix} D_{ik} ,$$

we get

$$\begin{aligned} &|(\nabla^j D^{ih} + \nabla^i D^{jh}) \nabla_h D_{ji}| \\ &\quad = \left| \left( \partial_c D_{ba} + \partial_b D_{ca} - 2\begin{Bmatrix} t \\ cb \end{Bmatrix} D_{ta} - \begin{Bmatrix} t \\ ca \end{Bmatrix} D_{tb} - \begin{Bmatrix} t \\ ba \end{Bmatrix} D_{tc} \right) \right. \\ &\qquad \left. \cdot \left( \partial_h D_{ji} - \begin{Bmatrix} s \\ hj \end{Bmatrix} D_{si} - \begin{Bmatrix} s \\ hi \end{Bmatrix} D_{sj} \right) g^{cj} g^{bi} g^{ah} \right| . \end{aligned}$$

Let us denote this quantity by $F_1$.

Define $S_{ji}$ by

$$g^{ji} = \delta_{ji} + \varepsilon S_{ji} .$$

Then $S_{ji}$ satisfy $|S_{ji}| < 1$.

Assume that $D_{ji}$ vanishes everywhere except in the interior of $N$, and define $M_1$ and $M_2$ by

$$\begin{aligned} M_1 &= \max \{|D_{ji}(P)|; P \in N; i, j = 1, \cdots, n\} , \\ M_2 &= \max \{|\partial_j D_{ih}(P)|; P \in N; h, i, j = 1, \cdots, n\} . \end{aligned}$$

Then

$$\begin{aligned} F_1 = &\left| \sum_{a,b,c} \sum_{h,i,j} \left( \partial_c D_{ba} + \partial_b D_{ca} - 2\begin{Bmatrix} t \\ cb \end{Bmatrix} D_{ta} - \begin{Bmatrix} t \\ ca \end{Bmatrix} D_{tb} - \begin{Bmatrix} t \\ ba \end{Bmatrix} D_{tc} \right) \right. \\ &\qquad \cdot \left( \partial_h D_{ji} - \begin{Bmatrix} s \\ hj \end{Bmatrix} D_{si} - \begin{Bmatrix} s \\ hi \end{Bmatrix} D_{sj} \right) \\ &\qquad \left. \cdot (\delta_{cj} + \varepsilon S_{cj})(\delta_{bi} + \varepsilon S_{bi})(\delta_{ah} + \varepsilon S_{ah}) \right| \\ \leq &\left| \sum_{h,i,j} (\partial_j D_{ih} + \partial_i D_{jh}) \partial_h D_{ji} \right| + 4n^4 M_1 M_2 \varepsilon + 4n^4 M_1 M_2 \varepsilon + 6n^4 (M_2)^2 \varepsilon \\ &+ (M_1 + M_2)^2 \varepsilon , \end{aligned}$$

where the last term $(M_1 + M_2)^2\varepsilon$ is a substitute for the terms containing $\varepsilon^2, \varepsilon^3, \cdots$.

Next consider

$$F_2 = (\nabla^h D^{ji}) \nabla_h D_{ji} .$$

For this quantity we get

$$F_2 = \Bigg| \sum_{a,b,c} \sum_{h,i,j} \left( \partial_c D_{ba} - \begin{Bmatrix} t \\ cb \end{Bmatrix} D_{ta} - \begin{Bmatrix} t \\ ca \end{Bmatrix} D_{tb} \right)$$

$$\cdot \left( \partial_j D_{ih} - \begin{Bmatrix} s \\ ji \end{Bmatrix} D_{sh} - \begin{Bmatrix} s \\ jh \end{Bmatrix} D_{si} \right)$$

$$\cdot (\delta_{cj} + \varepsilon S_{cj})(\delta_{bi} + \varepsilon S_{bi})(\delta_{ah} + \varepsilon S_{ah}) \Bigg|$$

$$\geq \sum_{h,i,j} (\partial_j D_{ih})^2 - 4n^4 M_1 M_2 \varepsilon - 3n^4 (M_2)^2 \varepsilon - (M_1 + M_2)^2 \varepsilon ,$$

and therefore

$$\int_M \left[ (\nabla^j D^{ih}) \nabla_h D_{ji} - \frac{1}{2} (\nabla^h D^{ji}) \nabla_h D_{ji} \right] dV$$

$$\leq \frac{1}{2} \int_N (F_1 - F_2) dV$$

$$\leq \frac{1}{2} \int_N \Bigg[ \Bigg| \sum_{h,i,j} (\partial_j D_{ih} + \partial_i D_{jh}) \partial_h D_{ji} \Bigg| - \sum_{h,i,j} (\partial_j D_{ih})^2$$

$$+ 12 n^4 M_1 M_2 \varepsilon + 9 n^4 (M_2)^2 \varepsilon + 2(M_1 + M_2)^2 \varepsilon \Bigg] dV .$$

Now let us consider a tensor field $T_{ji}$, which vanishes everywhere except in the interior of $N$, such that all components are identically zero except

$$T_{12} = T_{21} = f ,$$

where $f$ is a $C^\infty$ function. By putting

$$D_{ji} = T_{ji} - \frac{1}{n} T_{lk} g^{lk} g_{ji} = T_{ji} - \frac{2}{n} f g^{12} g_{ji} ,$$

we get $D_i{}^i = 0$ and

$$|\partial_j D_{ih} - \partial_j T_{ih}| \leq \left( \frac{4}{n} |f| + \frac{2}{n} |\partial_j f| \right) \delta_{ih} \varepsilon + 0(\varepsilon^2) .$$

Hence

$$M_1 = (\max |f|)(1 + 0(\varepsilon)) , \qquad M_2 \leq \left( \max |\partial_j f| + \frac{4}{n} |f| \varepsilon \right)(1 + 0(\varepsilon)) ,$$

and we can neglect all minor terms in $F_1$ and $F_2$. Moreover we can replace $D_{ih}$ and $\partial_j D_{ih}$ by $T_{ih}$ and $\partial_j T_{ih}$ respectively to obtain

$$\left(\frac{d^2I}{dt^2}\right)_0 = \int_N \left[\sum_{h,i,j} (\partial_j T_{ih})\partial_h T_{ji} - \frac{1}{2}\sum_{h,i,j} (\partial_j T_{ih})^2 + \frac{K}{n}\sum_{i,j} (T_{ji})^2\right] dV$$
$$= \int_N \left[-(\partial_3 f)^2 - \cdots - (\partial_n f)^2 + \frac{2K}{n} f^2\right] dV .$$

As there exist functions $f$ for which the last integral is negative, the index of $I$ is positive.

Thus we have proved the main theorem.

## 4. Index of an Einstein metric

Let $g$ be an Einstein metric. Then the right hand side of (1.5) is a quadratic functional of the tensor field $D_{ji}$. For convenience this integral will be denoted by $J(D)$. Let $\mathscr{D}$ be the set of tensor fields $D$ (with components $D_{ji}$) which satisfy

$$\int_M g^{ji} D_{ji} dV = 0 , \tag{4.1}$$

$$\int_M D^{ji} D_{ji} dV = 1 . \tag{4.2}$$

We now study critical points of $J(D)$ when $D$ moves in $\mathscr{D}$.

If $E_{ji}$ denotes an infinitesimal change of $D_{ji}$, we have at a critical point

$$\begin{aligned}\int_M \Big[&(\nabla^j E^{ih})\nabla_h D_{ji} + (\nabla^j D^{ih})\nabla_h E_{ji} - (\nabla^h E^{ji})\nabla_h D_{ji} \\ &- (\nabla^i E_j{}^j)\nabla^h D_{hi} - (\nabla^i D_j{}^j)\nabla^h E_{hi} + (\nabla^i E_j{}^j)\nabla_i D_h{}^h \\ &+ \frac{K}{n}(2D^{ji}E_{ji} - D_j{}^j E_i{}^i)\Big] dV = 0 ,\end{aligned} \tag{4.3}$$

where $E_{ji}$ satisfies

$$\int_M g^{ji} E_{ji} dV = 0 , \qquad \int_M D^{ji} E_{ji} dV = 0 . \tag{4.4}$$

From (4.3) we get

$$\begin{aligned}\int_M \Big[&-(\nabla_h \nabla^j D^{ih} + \nabla_h \nabla^i D^{jh}) + \nabla_h \nabla^h D^{ji} + (\nabla^k \nabla^h D_{hk}) g^{ji} + \nabla^j \nabla^i D_k{}^k \\ &- (\nabla_k \nabla^k D_h{}^h) g^{ji} + \frac{2K}{n} D^{ji} - \frac{K}{n} D_k{}^k g^{ji}\Big] E_{ji} dV = 0 ,\end{aligned}$$

where $E_{ji}$ is restricted only by (4.4). Hence we have

$$-\nabla_k\nabla^j D^{ik} - \nabla_k\nabla^i D^{jk} + \nabla_k\nabla^k D^{ji} + \nabla^j\nabla^i D_k{}^k + \left(\nabla^t\nabla^s D_{ts} - \nabla_t\nabla^t D_s{}^s - \frac{K}{n}D_k{}^k\right)g^{ji} = C_1 g^{ji} + C_2 D^{ji}, \tag{4.5}$$

where $C_1$ and $C_2$ are constants.

If (4.5) is transvected with $g_{ji}$ and then integrated over $M$, we get $C_1 = 0$ because of (4.1). If (4.5) is transvected with $D_{ji}$ and then integrated over $M$, we get

$$C_2 = \int_M \Big[2(\nabla^j D^{ik})\nabla_k D_{ji} - (\nabla^k D^{ji})\nabla_k D_{ji} - 2(\nabla_j D^{ji})\nabla_i D_k{}^k + (\nabla^t D_s{}^s)\nabla_t D_l{}^l - \frac{K}{n}D_l{}^l D_k{}^k\Big] dV$$

because of (4.2). Hence from (1.5) it follows that

$$\left(\frac{d^2 I}{dt^2}\right)_0 = \frac{1}{2}C_2 + \frac{K}{n}. \tag{4.6}$$

Let a symmetric tensor field $A_{ji}$ be a solution of the following system of partial differential equations with an unknown constant $C_2$,

$$-\nabla^k\nabla_j A_{ik} - \nabla^k\nabla_i A_{jk} + \nabla_k\nabla^k A_{ji} + \nabla_j\nabla_i A_k{}^k + \left(\nabla^t\nabla^s A_{ts} - \nabla_t\nabla^t A_s{}^s - \frac{K}{n}A_k{}^k\right)g_{ji} = C_2 A_{ji}, \tag{4.7}$$

when $C_2$ takes the eigenvalue $a$. If $A_{ji}$ satisfies $\int_M g^{ji}A_{ji}dV = 0$, and moreover $a < -2K/n$, then we have $J(A) < 0$ for this tensor field $A_{ji}$.

If (4.7) is transvected with $g^{ji}$ and integrated over $M$, we get

$$-K\int_M A_k{}^k dV = C_2 \int_M A_k{}^k dV .$$

Hence any solution $D_{ji}$ of (4.7) satisfies (4.1) if $K + C_2 \neq 0$. (4.2) is always satisfied if $D_{ji}$ is replaced by $D_{ji}$ multiplied by a suitable number.

Let us call each value of $C_2$, for which (4.7) has a solution $D_{ji}$ satisfying (4.1), an effective eigenvalue of the system (4.7).

Define a differential operator $L$ such that from any symmetric tensor field $A_{ji}$ a symmetric tensor field $L_{ji}(A)$ $(= (L(A))_{ji})$ is induced by

$$L_{ji}(A) = -\nabla^k\nabla_j A_{ik} - \nabla^k\nabla_i A_{jk} + \nabla_k\nabla^k A_{ji} + \nabla_j\nabla_i A_k{}^k + \left(\nabla^t\nabla^s A_{ts} - \nabla_t\nabla^t A_s{}^s - \frac{K}{n}A_k{}^k\right)g_{ji} . \tag{4.8}$$

Then we can write (4.7) in the form[2]: $L_{ji}(A) = C_2 A_{ji}$.

Let $A_{ji}$ and $B_{ji}$ be any $C^\infty$ symmetric tensor fields. Then

$$\int_M L_{ji}(A)B^{ji}dV = \int_M \Big[2(\nabla_j A_{ik})\nabla^k B^{ji} - (\nabla_k A_{ji})\nabla^k B^{ji} - (\nabla_i A_k{}^k)\nabla_j B^{ji} - \nabla^s A_{ts}\nabla^t B_l{}^l + \nabla^t A_s{}^s \nabla_t B_l{}^l - \frac{K}{n} A_k{}^k B_l{}^l\Big] dV ,$$

which will be denoted by $(A, B)$. If $A_{ji}$ and $B_{ji}$ are solutions of (4.7) when $C_2 = a$ and $C_2 = b$ respectively, then

$$(A, B) = a \int_M A_{ji}B^{ji}dV , \qquad (B, A) = b \int_M A_{ji}B^{ji}dV .$$

As $(A, B)$ is symmetric in $A$ and $B$, we get

$$\int_M A_{ji}B^{ji}dV = 0 , \qquad (A, B) = 0 ,$$

if $a \neq b$.

Now let $a_1 > \cdots > a_p$ be any discrete subset of effective eigenvalues of the system of equations

$$L_{ji}(A) = C_2 A_{ji}$$

such that $a_1 < -2K/n$. Let $\underset{k}{A}_{ji}$ be a solution of (4.7) corresponding to the eigenvalue $a_k$ and satisfying

$$\int_M g^{ji} \underset{k}{A}_{ji} dV = 0 .$$

As

$$\int_M \underset{l}{A}{}^{ji} \underset{k}{A}_{ji} dV = 0$$

for $l \neq k$, $\underset{1}{A}_{ji}, \cdots, \underset{p}{A}_{ji}$ are linearly independent.

**Definition.** Let us say that the index of the Einstein metric under consideration is $m$ if (4.7) has just $m$ linearly independent solutions with effective eigenvalues less than $-2K/n$.

Then we get the following theorem.

**Theorem 4.1.** *If* (4.7) *has $p$ effective eigenvalues $a_1 > \cdots > a_p$ such that $a_1 < -2K/n$, then the index of the Einstein metric under consideration is not less than $p$. The same holds with the index of $I(g)$ at this Einstein point.*

[2] $L$ is not an elliptic operator.

Evidently all eigenvalues of (4.7) are effective except the only possible one, namely, $C_2 = -K$. However, the author does not know whether $-K$ is an eigenvalue or not.

It has been pointed out by M. Berger and D. Ebin [2], [3] that in studying the variation of $I(g)$ we need to consider only tensor fields $D_{ji}$ which satisfy $\nabla^j D_{ji} = 0$, and that a tensor field $D_{ji} = \nabla_j X_i + \nabla_i X_j$ never changes $I(g)$. If we put $A_{ji} = \nabla_j X_i + \nabla_i X_j$, we immediately find that this is a solution of (4.7) with $C_2 = -2K/n$.

The author wishes to express his cordial thanks to Professor M. Berger for his kind information on relevant studies.

## References

[1] M. Berger, *Quelques formules de variation pour une structure riemannienne,* Ann. sci. École Norm. Sup. (4) **3** (1970) 285–294.
[2] M. Berger & D. Ebin, *Some decompositions of the space of symmetric tensors on a Riemannian manifold,* J. Differential Geometry **3** (1969) 379–392.
[3] D. G. Ebin, *The manifold of Riemannian metrics,* Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968) Amer. Math. Soc., 1970, 11–40.

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