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THE SPHERICAL IMAGES OF CONVEX HYPERSURFACES

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1. Introduction

The primary object of study in this paper is the spherical image of a continuous convex hypersurface in euclidean space. The original motivation for this study comes from differential geometry. Therefore, for the benefit of differential geometers, we first present the principal result of interest in the C^∞ category before discussing the technical theorems in convexity theory. To this end, recall that a subset K of the unit n-sphere S^n is (geodescially) *convex* iff for any $p, q \in K$, at least one of the minimal arcs joining p, q lies in K.

Main theorem. *Let M be a complete noncompact orientable C^∞ hypersurface in $\boldsymbol{R}^{n+1}$ $(n > 1)$ with nonnegative but not identically zero sectional curvature. Let $\gamma: M \to S^n$ be the spherical (Gauss) map. Then the following statements are true:*

(α) *$\gamma(M)$ has a convex closure and a convex interior (relative to S^n). More precisely, there exist a unique totally geodesic k-sphere $S^k \subseteq S^n$ $(2 \leq k \leq n)$ and a unique open convex subset K of S^k such that $K \subseteq \gamma(M) \subseteq \operatorname{cl} K$ (=closure of K). In particular, $\gamma(M)$ lies in a closed hemisphere of S^n.*

(β) *The total curvature of M (cf. Chern-Lashof* [3]) *does not exceed one.*

(γ) *M has infinite volume.*

(δ) *If the sectional curvature of M is everywhere positive, then M is homeomorphic with $\boldsymbol{R}^n$, $\gamma: M \to S^n$ is a diffeomorphism onto an open convex subset of S^n, and coordinates in $\boldsymbol{R}^{n+1}$ may be so chosen that M is tangent to the hyperplane $\{x_{n+1} = 0\}$ at the origin and is the graph of a nonnegative strictly convex function (=Hessian is positive definite everywhere) defined in $\{x_{n+1} = 0\}$. Moreover, for any $c > 0$, $M \cap \{x_{n+1} = c\}$ is diffeomorphic to the $(n-1)$-sphere.*

Of the four assertions above, (α) is the crucial one from which (β)–(δ) follow. Now we would like to describe the complete generalizations of (α) and (δ) in the context of convex hypersurfaces. By definition, a *convex hypersurface M* in $\boldsymbol{R}^{n+1}$ is the full boundary of a closed convex C with interior. (We always assume $C \neq \boldsymbol{R}^{n+1}$.) We recall the definition of the (possibly multi-valued)

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spherical map $\gamma: M \to S^n$ due to A. D. Alexandrow [1]: $\gamma(p)$ = the set of all outer unit normals to C at p. Then we have the following two theorems.

Theorem 1. *Let M be a connected convex hypersurface in $\boldsymbol{R}^{n+1}$ $(n \geq 1)$, and $\gamma: M \to S^n$ the spherical map. Then there exist a unique totally geodesic k-sphere $S^k \subseteq S^n$ $(0 \leq k \leq n)$ and a unique open convex subset K of S^k such that $K \subseteq \gamma(M) \subseteq \operatorname{cl} K$.*

Theorem 2. *Let $M = \partial C$ be a convex hypersurface in $\boldsymbol{R}^{n+1}$ homeomorphic to $\boldsymbol{R}^n$. Then coordinates can be so chosen that $\{x_{n+1} = 0\}$ is a supporting hyperplane to C at the origin, and it has the following additional properties:*

(a) *Let $\pi: \boldsymbol{R}^{n+1} \to \{x_{n+1} = 0\}$ be the orthogonal projection, and A the convex set $\pi(C)$. Then over $\operatorname{ri} A$ (=the interior of A relative to $\{x_{n+1} = 0\}$), M is the graph of a nonnegative convex function $f: \operatorname{ri} A \to \boldsymbol{R}$. If M is C^∞, then f is a C^∞ function.*

(b) *For every $a \in A \setminus \operatorname{ri} A$, $M \cap \pi^{-1}(a)$ is a closed half-line (=semi-infinite line segment).*

(c) *If in addition $\gamma(M)$ has interior (relative to S^n), then for any $c > 0$, $M \cap \{x_{n+1} = c\}$ is homeomorphic to the $(n-1)$-sphere; this homeomorphism is a diffeomorphism if M is C^∞.*

It remains to briefly discuss the background of these theorems. In 1897, Hadamard [7] proved that an immersed compact orientable surface in $\boldsymbol{R}^3$ of positive Gaussian curvature is necessarily imbedded and is a convex surface. Forty years later, Stoker was able to extend Hadamard's theorem to the non-compact case, replacing compactness by completeness [12]. In addition, he proved the special case of our main theorem when $n = 2$ and the curvature of the surface is everywhere positive. In 1958, Chern and Lashof [3] proved that Hadamard's theorem remains valid if positive curvature is replaced by non-negative curvature. Further progress in this direction was made by van Heijenoort [8], but it remained for Sacksteder to prove in 1960 [11] the following comprehensive convexity theorem.

Theorem of Sacksteder-van Heijenoort. *Let M be a C^∞ n-dimensional $(n > 1)$ complete orientable Riemannian manifold of nonnegative sectional curvature which is not identically zero, and let $x: M \to \boldsymbol{R}^{n+1}$ be an isometric immersion.*

(A) *Then x is an imbedding and $x(M)$ is a convex hypersurface.*

(B) *If r is the maximal rank of the second fundamental form of $x(M)$ (necessarily $2 \leq r \leq n$), then $\boldsymbol{R}^{n+1}$ can be decomposed into an orthogonal direct sum $\boldsymbol{R}^{n+1} = \boldsymbol{R}^{r+1} \oplus \boldsymbol{R}^{n-r}$ in such a way that the orthogonal projections of $\boldsymbol{R}^{n+1}$ into the two factors yield an isometry $x(M) \cong M_1 \oplus \boldsymbol{R}^{n-r}$, where M is a convex hypersurface in $\boldsymbol{R}^{r+1}$ containing no complete lines.*

We may observe, in view of this theorem, that Theorems 1 and 2 are honest generalizations of (α) and (δ) of the main theorem. It is also clear that the results of this paper together with the Sacksteder-van Heijenoort theorem form a complete extension of Stoker's theorem to n dimensions. Beyond this, Theorems 1 and 2 have applications in the theory of convex surfaces, cf. for instance Alexandrow's theory of spherical measures on an open convex surface (Busemann [2, p. 31]) and the rigidity and nonrigidity theorems of Pogorelov and Olovyanishnikov for open convex surfaces [2, pp. 167–168]. We should mention that the spherical image of a convex hypersurface was previously thought to be convex (cf. [2, p. 25, Theorem (4.4)]). In the appendix, we will present a counterexample to this assertion. Consequently, the more delicate statements concerning $\gamma(M)$ in Theorem 1 are actually optimal. It may also be of interest to point out that, coupled with Alexandrow's imbedding theorem for surfaces [2, p. 150, Theorem (20.1)], (γ) of the main theorem implies that a 2-dimensional C^∞ Riemannian manifold, which is complete noncompact and has nonnegative curvature, necessarily has infinite area. However, this result was already anticipated by Cohn-Vossen [4, p. 47, Theorem 4] (cf. also Huber, [9, p. 69, Theorem 14]).

Slightly weaker and less complete versions of the above theorems were first announced in [13]. Subsequently, do Carmo and Lawson [5] have found a different proof of one-half of (α) of the main theorem (the convexity of $\operatorname{cl}\gamma(M)$) within the framework of calculus, and using only this half of (α) do Carmo and Lima in [6] gave another proof of (β) and parts of (γ) and (δ) of the main theorem. The author would like to take this opportunity to acknowledge his indebtedness to Stoker's paper [12]. The theorems of this paper were originally inspired by [12], and were first proved along the lines of [12]; even now, the proof of (γ) of the main theorem is entirely based on Stoker's idea.

This paper is written for differential geometers. While certain concepts in convexity theory are used in a very essential way in the proofs, only a few elementary and plausible theorems from that discipline are drawn upon. The author would like to believe that this paper is completely intelligible to an average differential geometer.

2. Preparatory materials

We follow Rockafellar [10] as regards notation and terminology; definitions and elementary theorems from [10] will sometimes be used without comment.

As usual, C will denote a closed convex subset of $\boldsymbol{R}^{n+1}$ with interior, and $M = \partial C$ is a convex hypersurface (we always assume $C \neq \boldsymbol{R}^{n+1}$). The following general fact is easily proved (Busemann [2, p. 3]).

Lemma 1. *If C contains no lines, then M is homeomorphic to either S^n or $\boldsymbol{R}^n$.*

Given an arbitrary convex subset D of $\boldsymbol{R}^{n+1}$, we recall the definition of the *relative interior* $\operatorname{ri} D$ of D. The *affine hull* $\operatorname{aff} D$ of D is the minimal affine set containing D, and $\operatorname{ri} D$ is by definition the interior of D relative to $\operatorname{aff} D$. We have ri (cl D) $=$ ri D and cl (ri D) $=$ cl D [10, p. 46, Theorem 6.3]. A subset K of $\boldsymbol{R}^{n+1}$ is called a *cone* iff $x \in K$ implies $\lambda x \in K$ for all $\lambda > 0$. Given an arbitrary subset T, we denote by ray T the *cone generated by* T; in other words, ray $T = \{\lambda y : \lambda \geq 0, y \in T\}$ (see [10, p. 14]).

We can now deal adequately with convex subsets of S^n. As mentioned above, $T \subseteq S^n$ is convex iff for any $p, q \in T$, either the shorter arc of the great circle through p, q (in case p and q are not antipodal) lies in T or at least one semi-great circle through p and q (in case p and q are antipodal) lies in T. It is easy to see that $T \subseteq S^n$ is convex iff ray T is a convex cone $\boldsymbol{R}^{n+1}$. Equivalently, a cone $K \subseteq \boldsymbol{R}^{n+1}$ is convex iff $K \cap S^n$ is convex in S^n. Since a convex cone either is $\boldsymbol{R}^{n+1}$ itself or possesses a supporting hyperplane at the origin 0, we have the following well-known fact:

Lemma 2. *A convex subset of S^n is either S^n itself or a subset of a closed hemisphere.*

Let $T \subseteq S^n$ be convex. Then ray T is convex so that we have the affine set $A \equiv$ aff (ray T). Thus $S^r \equiv A \cap S^n$ is the unique minimal totally geodesic r-sphere containing T. We now define the *relative interior* of T, also denoted by ri T, to be the interior of T relative to this S^r. Clearly, ri $T = \{$ri (ray T)$\} \cap S^n$. Therefore every fact about the relative interior of a convex set in $\boldsymbol{R}^{n+1}$ has an analogue pertaining to the relative interior of a convex set in S^n. In particular:

Lemma 3. *Let $T \subseteq S^n$ be convex. Then* ri T *and T are both convex in S^n, and* ri (cl T) $=$ ri T, cl (ri T) $=$ cl T.

Returning for a moment to a nonempty closed convex subset C of $\boldsymbol{R}^{n+1}$, we recall the concepts of the barrier cone and the recession cone of C. (C may have empty interior in this discussion.) C is said to be *bounded in the direction of a vector* $\xi \neq 0$ iff $\sup_{x \in C} \langle x, \xi \rangle < \infty$, that is, iff C is contained in some half-space with outer normal ξ. By definition, the *barrier cone* $B(C)$ of C is the set of all vectors ξ with this property together with the origin 0 [10, p. 15]; $B(C)$ is a (not necessarily closed) convex cone. Since C is clearly bounded in the direction of an outer normal of C, every outer normal of C belongs to $B(C)$. Now let C have interior, and let $M = \partial C$. Since by definition the spherical image $\gamma(M)$ consists of all outer unit normals of C, we have

$$\gamma(M) \subseteq B(C) \cap S^n .$$

The *recession cone* 0^+C of C is by definition the set of all vectors y such that $y + C \subseteq C$ (vector sum); 0^+C is a closed convex cone [10, pp. 61–63]. Equivalently, $y \neq 0$ is in 0^+C iff C contains a translate of the half-line $\{\lambda y : \lambda \geq 0\}$ [10, p. 63, Theorem 8.3]. Using this alternate description of 0^+C, we prove

Lemma 4. *Let C be a closed convex subset of $\boldsymbol{R}^{n+1}$, and let an n-dimensional subspace H be such that $H \cap 0^+C = \{0\}$. Then $C \cap H'$ is bounded for all translates H' of H.*

Proof. Suppose $C \cap H'$ is unbounded for one H'; then it must contain a half-line $\{x + \lambda y : \lambda \leq 0\}$ for some $x \in H'$ and some $y \neq 0$. Thus $y \in 0^+C$. Of course, $y \in H$ also, so that $0 \neq y \in 0^+C \cap H$, a contradiction. q.e.d.

0^+C and $B(C)$ are related. To state this relationship, we introduce the notion of the *polar cone* K^0 of a nonempty convex cone K [10, p. 121]. By definition, $K^0 = \{x^* : \langle x, x^* \rangle \leq 0$ for all $x \in K\}$. K^0 is a closed convex cone, and $K^{00} = \operatorname{cl} K$. Then according to [10, p. 123, Corollary 14.21] we have $B(C)^0 = 0^+C$.

3. Proof of Theorems 1 and 2

In this sectin, C is a closed convex proper subset of $\boldsymbol{R}^{n+1}$ with interior, and $M = \partial C$ is a connected convex hypersurface. We first prove Theorem 1.

Let H^r be the affine hull of $B(C)$, and assume $\dim H^r = r$. Note that $r \geq 1$. Let $J = B(C) \cap S^n$. Then J is convex in S^n. We will prove

$$(*) \qquad \operatorname{ri} J \subseteq \gamma(M) \subseteq \operatorname{cl} J .$$

Let us assume (*) for the moment, and also observe, using Lemma 3, that $\operatorname{cl}(\operatorname{ri} J) = \operatorname{cl} J$ and that $\operatorname{ri} J$ is a convex open subset of $S^{r-1} \equiv H^r \cap S^n$. Then (*) implies Theorem 1 (with $K = \operatorname{ri} J$) except for the uniqueness statements. Since we have already pointed out in § 2 that $\gamma(M) \subseteq J \subseteq \operatorname{cl} J$, to prove (*) it suffices to prove $\operatorname{ri} J \subseteq \gamma(M)$.

Let us first assume $r = n + 1$, i.e., $B(C)$ has interior. In this case $\operatorname{ri} J = \{\operatorname{int} B(C)\} \cap S^n$ (int = interior). Hence $\operatorname{ri} J \subseteq \gamma(M)$ is equivalent to $\operatorname{int} B(C) \subseteq \operatorname{ray} \gamma(M)$. Since $\operatorname{ray} \gamma(M)$ consists of all the outer normals to C, what we are trying to prove is that a nonzero vector $\xi \in \operatorname{int} B(C)$ is necessarily an outer normal to C. Suppose not, then we shall show that ξ is approached by vectors not in $B(C)$, thereby arriving at a contradiction. To this end, recall that ξ is an outer normal of C iff there exists $x^0 \in C$ such that $\sup_{x \in C} \langle x, \xi \rangle = \langle x^0, \xi \rangle$, [10, p. 100]. Therefore the assumptions that $\xi \in B(C)$ and ξ is not an outer normal imply the existence of a sequence of points $x^i \in C$ such that $\|x^i\| \to \infty$ and $\langle x^i, \xi \rangle \to \sup_{x \in C} \langle x, \xi \rangle < \infty$. We may assume that the sequence $\{x^i\}$ has been so chosen that the unit vectors $x^i/\|x^i\|$ converge to a vector e. For every $\lambda > 0$, we then have

$$\langle x^i, \xi + \lambda e \rangle = \langle x^i, \xi \rangle + \lambda \|x^i\| \langle x^i/\|x^i\|, e \rangle \to \infty ,$$

so that $(\xi + \lambda e) \notin B(C)$ for every $\lambda > 0$, and hence ξ cannot be an interior point of $B(C)$.

Before treating the general case, we pause to note that the preceding argument depends on $n \geq 1$ (which is part of the assumption of Theorem 1). Thus we have proved Theorem 1 in a euclidean space of dimension ≥ 2, provided the barrier cone of C has interior.

Now suppose $r < n + 1$, and we will reduce this case to the preceding case. We first note that we may assume $r \geq 2$. Indeed, if $r = 1$, then $\gamma(M) \subseteq B(C)$

$\cap S^n \subseteq H^1 \cap S^n = \{-1, 1\}$ so that $\gamma(M)$ would consist of one or two points. If $\gamma(M)$ consists of one point, then there is nothing to prove. If $\gamma(M)$ consists of two points, then M would consist of two parallel hyperplanes and therefore would be disconnected, contrary to assumption. Hence we may assume $r \geq 2$. Now recall that C always admit a direct sum decomposition:

$$C = L \oplus (C \cap L^\perp) ,$$

where L, the so-called *lineality space* of C, is the maximal vector subspace contained in the recession cone 0^+C [10, p. 65]. We *claim*: $H^r = L^\perp$. First we show $H^r \subseteq L^\perp$. Since $H^r = \operatorname{span} B(C)$, it suffices to show $B(C) \subseteq L^\perp$. Let $x \in B(C)$; then $x \in (0^+C)^0$, implying $\langle x, L\rangle \leq 0$ and $\langle x, -L\rangle \leq 0$ (because $-L = L$). Consequently, $\langle x, L\rangle = 0$ and $x \in L^\perp$. Conversely, we show $L^\perp \subseteq H^r$. This is equivalent to $H^{r\perp} \subseteq L$. Let $x \in H^{r\perp}$; then $\langle x, B(C)\rangle = 0$, implying $x \in 0^+C$ because $0^+C = B(C)^0$. Since also $-x \in H^{r\perp}$, $-x \in 0^+C$. Thus span $x \subseteq 0^+C$, and by definition of L we have $x \in L$, thereby proving the claim.

As a result of this claim, we have a decomposition:

$$(**) \qquad C = H^{r\perp} \oplus (C \cap H^r) .$$

Clearly, $\gamma(M) = \gamma(\partial(C \cap H^r))$ and $B(C) = B(C \cap H^r)$. In particular, since $B(C)$ is a convex subset of H^r with interior (relative to H^r), the barrier cone of the convex subset $C \cap H^r$ of H^r also has interior (relative to H^r). As mentioned above, we may assume $r \geq 2$. Therefore (*) follows by considering the space H^r instead of $\boldsymbol{R}^{n+1}$ and its closed convex subset $C \cap H^r$ instead of C.

To finish the proof of Theorem 1, it remains to prove the uniqueness assertions. Let us use the notation in the statement of Theorem 1. To show that K is unique, suppose another open convex subset K' of some totally geodesic k'-sphere in S^n also satisfies $K' \subseteq \gamma(M) \subseteq \operatorname{cl} K'$. Then $\operatorname{cl} K' = \operatorname{cl} \gamma(M) = \operatorname{cl} K$. By Lemma 3, $K' = \operatorname{ri} K' = \operatorname{ri}(\operatorname{cl} K') = \operatorname{ri}(\operatorname{cl} K) = \operatorname{ri} K = K$. The uniqueness of the k-sphere S^h is now trivial. q.e.d.

Keeping the same notation, we proceed to prove Theorem 2. Thus let M be homeomorphic to $\boldsymbol{R}^n$. In the orthogonal direct sum decomposition (**) above, recall that $r \geq 1$ and that H^r is the affine hull of $B(C)$ as well as being identical with $L^\perp$, where L is the maximal vector subspace contained in 0^+C. We claim: $0^+(C \cap H^r) \neq \{0\}$. For if $0^+(C \cap H) = \{0\}$, then $C \cap H^r$ is bounded [10, p. 64, Theorem 8.4], and hence $\partial(C \cap H^r)$ would be homeomorphic with the sphere S^{r-1}. This means $M = \partial C$ would be homeomorphic with $H^r \times S^{r-1}$ $(r \geq 1)$, which is never homeomorphic with $\boldsymbol{R}^n$. Thus there is a nonzero $y \in 0^+ (C \cap H^r)$. In view of (**), $0^+C = H^r \oplus 0^+ (C \cap H^r)$, so that $0^+(C \cap H^r) = 0^+C \cap H^r$. Thus $0 \neq y \in 0^+C \cap H^r$. Now, no complete line parallel to y can be contained in C, for otherwise, span $y \subseteq 0^+C \cap H^r$, contradicting the maximality of $L = H^{r\perp}$. Hence every line parallel to y and intersecting C intersects $M = \partial C$ at one point or a closed half-line.

Now we further claim that y can be so chosen that $-y \in \operatorname{ri} B(C)$; in view of (*), this means among other things that $-y$ is an outer normal of C. We shall prove this claim by contradiction. So suppose no such y exists. Then, in H^r, the interiors of $-B(C)$ and $0^+C \cap H^r$ are disjoint. By the separation theorem [10, p. 97, Theorem 11.3], there exists an $(r-1)$-dimensional subspace of H^r separating $-B(C)$ and $0^+C \cap H^r$. This means that there exists a nonzero $\xi \in H^r$ such that $\langle x, \xi \rangle \geq 0$ for every $x \in -B(C)$, and $\langle y, \xi \rangle \leq 0$ for every $y \in 0^+C \cap H^r$. The first inequality may be rephrased as $\langle x, \xi \rangle \leq 0$ for every $x \in B(C)$; since $0^+C = B(C)^0$, this implies $\xi \in 0^+C$. Hence substituting ξ for y in the second inequality, we get $\langle \xi, \xi \rangle \leq 0$, and thus $\xi = 0$, a contradiction.

In summary, we have located a nonzero $y \in 0^+C \cap H^r$ such that $-y \in \operatorname{ri} B(C)$, and any line parallel to y and intersecting C intersects M at one point or a closed half-line. The proof of Theorem 2 now follows rapidly. Indeed, we choose an orthogonal coordinate system $\{x_1, \cdots, x_{n+1}\}$ so that y points in the direction of the positive x_{n+1}-axis. (*) implies that $-y$ is the outer normal vector of some supporting hyperplane to C; we may therefore assume that $\{x_{n+1} = 0\}$ is a supporting hyperplane to C at the origin 0 and that $C \subseteq \{x_{n+1} \geq 0\}$. Let $\pi: \boldsymbol{R}^{n+1} \to \{x_{n+1} = 0\}$ be the orthogonal projection, and $A = \pi(C)$ as in the theorem. Since $\pi(\operatorname{int} C) = \operatorname{ri} A$, clearly a line parallel to the x_{n+1}-axis through $p \in A$ intersects $M = \partial C$ at a point iff $p \in \operatorname{ri} A$. Thus every line through a point of $A \setminus \operatorname{ri} A$ parallel to the x_{n+1}-axis must intersect M at a closed half-line, and the portion of M over $\operatorname{ri} A$ is the graph of a function $f: \operatorname{ri} A \to \boldsymbol{R}$. That f is convex follows from the fact that M is a convex hypersurface. This proves (a) and (b) except for the statement that f is C^∞ whenever M is. Proving f is C^∞ is a local question, so if $p \in \operatorname{ri} A$, we only need to show that f is C^∞ in a neighborhood of p. We first observe that the normal to M at $(p, f(p))$ is never orthogonal to the x_{n+1}-axis; for if it were, then the supporting hyperplane to C at $(p, f(p))$, which in this case is the tangent plane to M at $(p, f(p))$, would contain the line l through p parallel to the x_{n+1}-axis. Since $p \in \operatorname{ri} A$, we know from the preceding analysis that $l \cap M = \{p\}$. Thus the half-line of l lying above $(p, f(p))$ is in $\operatorname{int} C$. This would mean that interior points of C also lie in a supporting hyperplane, a contradiction. Therefore the normals to M is a neighborhood of $(p, f(p))$ are never orthogonal to the x_{n+1}-axis. This is equivalent to saying that each tangent plane of M in this neighborhood is mapped isomorphically by $d\pi$ onto the corresponding tangent plane of $\{x_{n+1} = 0\}$. Hence $\pi | M$ (the restriction of π to M) is a diffeomorphism in this neighborhood. Since the map φ given by $p \to (p, f(p))$ is the inverse of $\pi | M$, φ is C^∞ in this neighborhood and hence so is f.

It remains to prove (c). If $\gamma(M)$ has interior, then $B(C)$ has interior and $r = n + 1$ in the preceding analysis. In this case, the negative x_{n+1}-axis, which is the direction of $-y$, would be in the interior of $B(C)$, and the positive x_{n+1}-axis would be in 0^+C. The former implies that every nonzero vector in $\{x_{n+1} = 0\}$ will have a positive inner product with at least one $x \in B(C)$. Since

$0^+C = B(C)^0$, only the zero vector of $\{x_{n+1} = 0\}$ can be in 0^+C. Equivalently, $0^+C \cap \{x_{n+1} = 0\} = \{0\}$. By Lemma 4, $C \cap \{x_{n+1} = c\}$ must be bounded for every $c \in \boldsymbol{R}$. If $c > 0$, this intersection must be nonempty since the whole positive x_{n+1}-axis lies in C. Thus $M \cap \{x_{n+1} = c\}$ is a bounded convex hypersurface for every positive c; consequently it must be homeomorphic with S^{n-1}. To show that this homeomorphism is a diffeomorphism when M is C^∞, we define a height function $h\colon M \to \boldsymbol{R}$ by $h(q) = \langle q, (0, \cdots, 0, 1)\rangle$. An elementary computation shows that q is a critical point of h iff $\gamma(q)$ is either $(0, \cdots, 0, -1)$ or $(0, \cdots, 0, 1)$. Since the positive x_{n+1}-axis is in C, the second possibility never arises. Furthermore, if $x_{n+1}(q) = \beta > 0$, then $\gamma(q) = (0, \cdots, 0, -1)$ would imply that M lies above the hyperplane $\{x_{n+1} = \beta\}$, which is absurd since M contains the origin. Thus no $q \in M$ with $x_{n+1}(q) > 0$ can be a critical point of h. Since $M \cap \{x_{n+1} = c\} = h^{-1}(c)$, and $h^{-1}(c)$ for $c > 0$ contains no critical point of h, we see that $M \cap \{x_{n+1} = c\}$ is a C^∞ submanifold of M for $c > 0$. Since it is the boundary of a bounded convex set with interior in $\{x_{n+1} = c\}$, it is diffeomorphic to S^{n-1}.

4. Proof of the main theorem

By part (A) of the theorem of Sacksteder-van Heijenoort quoted in § 1, M is a C^∞ convex hypersurface. Let us say $M = \partial C$, where C is a closed convex set in $\boldsymbol{R}^{n+1}$.

(α) is an immediate consequence of Theorem 1 and Lemma 2; the fact that $2 \leq k$ follows from the assumption that the sectional curvature of M is not identically zero.

We now approach (β). Let us first recall the definition of *total curvature* $\tau(M)$ of M (cf. Chern-Lashof [3]). Let B denote the unit sphere bundle of M in $\boldsymbol{R}^{n+1}$, and for each $v \in B$ let $S_v\colon M_p \to M_p$ be the second fundamental form of M determined by v. Then by definition,

$$\tau(M) = \frac{1}{\sigma^n} \int_B |\det S_v|\, dv_B \ ,$$

where det denotes determinant, dv_B is the volume form on B induced by the metric of M, and σ^n is the volume of S^n. Now in our case, B is just two copies of M corresponding to the outer and inner unit normals of M. So if we let S be the function which assigns to each $p \in M$ the second fundamental form of M corresponding to the outer unit normal at p, then clearly $\det S \geq 0$ since M has nonnegative curvature. Consequently,

$$\tau(M) = \frac{2}{\sigma^n} \int_M (\det S)\omega \ ,$$

where ω is the volume element of M. To handle this improper integral, we make a digression. Now as later, we shall need the following

Observation. If $(\det S)(r) > 0$, then for any $p \in M$, $\gamma(p) = \gamma(r)$ implies that $p = r$.

Indeed, we know that M bounds C, so every tangent plane of M is the supporting hyperplane to C. Thus the tangent planes M_r and M_p are both supporting hyperplanes to C. Since $\gamma(p) = \gamma(r)$, M_r and M_p are parallel supporting hyperplanes with outer normals pointing in the same direction. This is possible only if M_r and M_p, when considered as hyperplanes of $\boldsymbol{R}^{n+1}$, are identical. This being the case, the line segment $\overline{pr}$ is contained in M_r. By the convexity of C, $\overline{pr} \subseteq C$. Now suppose $p \neq r$. Then $M_r \cap C \supseteq \overline{pr} \supsetneqq \{r\}$. However, since $(\det S)(r) > 0$, the standard interpretation of the second fundamental form as the Hessian of a function shows that, locally near r, C can intersect the tangent plane M_r only at r. Since C is convex, this implies that $C \cap M_r = \{r\}$, a contradiction.

We now return to the consideration of $\tau(M)$. To prove $\tau(M) \leq 1$, it suffices to prove that $\int_M (\det S)\omega \leq \frac{1}{2}\sigma^n$. For this purpose, we partition M into two sets A and B, where $A = \{p \in M\colon (\det S)(p) > 0\}$ and $B = \{p \in M\colon (\det S)(p) = 0\}$. A is a submanifold of M, and obviously

$$\int_M (\det S)\omega = \int_A (\det S)\omega \,.$$

By the above observation, γ is an injection on A. Furthermore, it is a classical fact that $(\det S)\omega = \gamma^*\Omega$, where Ω is the canonical volume element of S^n. Thus γ is nonsingular at every point of A, and hence $\gamma\colon A \to \gamma(A)$ is a diffeomorphism. This leads to

$$\int_M (\det S)\omega = \int_{\gamma(A)} \Omega \,.$$

Since $\gamma(A) \subseteq \gamma(M) \subseteq$ a closed hemisphere, clearly $\int_{\gamma(A)} \Omega \leq \frac{1}{2}\sigma^n$. Thus $\int_M (\det S)\Omega \leq \frac{1}{2}\sigma^n$, as desired.

To tackle (γ), we need to distinguish between two cases. First, we assume that the maximal rank of the second fundamental form S of M is strictly smaller than n. By part (B) of the theorem of Sacksteder-van Heijenoort quoted in § 1, M is then isometric to a direct product $M_1 \times \boldsymbol{R}^s$, where $s \geq 1$. In this case, volume M = (volume M_1) $\times$ (volume $\boldsymbol{R}^s$) $= \infty$. The second case where the maximal rank of S is n is slightly more difficult. We want to apply Theorem 2, and to this end we must show that M is homeomorphic to $\boldsymbol{R}^n$. This we now do.

For the rest of this section, we assume that the maximal rank of S in n. Let $r \in M$ be a point at which rank S is n. Thus $(\det S)(r) > 0$. Notation as in the proof of (β), the identity $\gamma^*\Omega = (\det S)\omega$ implies that $(\gamma^*\Omega)(r) \neq 0$. In particular, γ is nonsingular at r, and therefore $\gamma(M)$ has interior (relative to S^n).

We *claim*: For a convex hypersurface M, $\gamma(M)$ having interior (relative to S^n) and M being noncompact imply that M is homeomorphic to $\boldsymbol{R}^n$. Indeed, this means $B(C)$ has interior since $\gamma(M) \subseteq B(C) \cap S^n$. 0^+C is therefore the polar cone of a cone with interior (because $0^+C = B(C)^0$) and consequently cannot contain any line. This implies that C contains no line. By Lemma 1, M is homeomorphic to $\boldsymbol{R}^n$.

We can now apply Theorem 2 to our C^∞ M. We may therefore assume that orthogonal coordinates $\{x_1, \cdots, x_{n+1}\}$ have been so chosen that M is tangent to $\{x_{n+1} = 0\}$ at 0 and that $M \subseteq \{x_{n+1} \geq 0\}$. Furthermore, if $\pi\colon \boldsymbol{R}^{n+1} \to \{x_{n+1} = 0\}$ is the orthogonal projection and A is the convex set $\pi(C)$, then M is the graph of a C^∞ function f above $\operatorname{ri} A$. If $c > 0$, $M \cap \{x_{n+1} = c\}$ is diffeomorphic to S^n. Also, we know from the proof of Theorem 2 that $(0, \cdots, 0, 1) \in 0^+C$.

We can now finish the proof of (γ). Take any $c > 0$, and let $Q = M \cap \{x_{n+1} = c\}$. Q is then diffeomorphic to S^n. We erect a cylinder $\mathscr{C}$ over Q, i.e., $\mathscr{C} = \{(x_1, \cdots, x_n, t)\colon (x_1, \cdots, x_n, c) \in Q \text{ and } t \geq c\}$. $\mathscr{C}$ thus consists of all closed half-lines in the direction of $(0, \cdots, 0, 1)$ issuing from Q, and hence $\mathscr{C} \subseteq C$ because $(0, \cdots, 0, 1) \in 0^+C$. Let M^c be that portion of M above $\{x_{n+1} = c\}$; then it is clear that the volume of M^c is not less than the volume of $\mathscr{C}$ (this is just Cavalieri's principle). Hence volume $M \geq$ volume $\mathscr{C}$. But it is elementary to show that $\mathscr{C}$ has infinite volume, so the proof of (γ) is complete.

Using the same notation, we now prove (δ). If the sectional curvature of M is everywhere positive, then the second fundamental form S has rank n everywhere. In particular, $\gamma(M)$ has interior (relative to S^n), and we have already seen that this implies that M is homeomorphic to $\boldsymbol{R}^n$. The fact that γ is everywhere nonsingular follows from $\gamma^*\Omega = (\det S)\omega$ and the fact that $\det S > 0$ everywhere. The injectivity of γ follows from the observation found in the proof of (β). Thus $\gamma(M)$ is an open subset of S^n. (α) now implies that $\gamma(M)$ is itself convex. Hence $\gamma\colon M \to S^n$ is a diffeomorphism onto an open convex subset of S^n. It remains to show that $A(=\pi(C))$ is open in $\boldsymbol{R}^n$ (so that M is the graph of the C^∞ function f defined on $\operatorname{ri} A = A$), and that f has everywhere positive Hessian. If $p \in A \setminus \operatorname{ri} A$, then $\pi^{-1}(p) \cap M$ is a closed half-line (Theorem 2(b)). By the classical lemma of Synge, M will have nonpositive sectional curvature along the half-line $\pi^{-1}(p) \cap M$, contradicting the positivity of the curvature of M. Thus $A = \operatorname{ri} A$. To prove $(\partial^2 f/\partial x_i \partial x_j)_{1 \leq i, j \leq n}$ is a positive definite matrix, let U be the outer unit normal vector field of M, and S be the second fundamental form of M corresponding to U. Let $X_i = F_*(\partial/\partial x_i)$, $i = 1, \cdots, n$, where $F\colon A \to \boldsymbol{R}^{n+1}$ is the map $F(x_1, \cdots, x_n) = (x_1, \cdots, x_n, f(x_1, \cdots, x_n))$. Then $X_1, \cdots, X_n$ form a basis of the tangent space at each point of M, and furthermore

$$X_i = (0, \cdots, 0, 1, 0, \cdots, 0, \partial f/\partial x_i) \qquad (i\text{-th spot}) ,$$

$$U = \frac{1}{W}(\partial f/\partial x_1, \cdots, \partial f/\partial x_n, -1) ,$$

where $W = (1 + \sum_i (\partial f/\partial x_i)^2)^{\frac{1}{2}}$. The following is then routine:

$$\langle S(X_i), X_j\rangle = \langle \partial U/\partial x_i, X_j\rangle = -\langle U, \partial X_j/\partial x_i\rangle = \frac{1}{W}\partial^2 f/\partial x_i\cdot\partial x_j .$$

Since S is positive definite, the matrix $(\langle S(X_i), X_j\rangle)$ is also positive definite, and this proves that $\{\partial^2 f/\partial x_i\cdot\partial x_j\}_{1\le i,j\le n}$ is positive definite.

Appendix

We shall produce a closed convex subset of $\boldsymbol{R}^3$ with a connected boundary whose spherical image fails to be convex.

Let $\boldsymbol{R}^3 = \{(x, y, z)\}$. The closed convex set in question will be a subset of the *first quadrant* $Q = \{(x, y, z)\colon x \ge 0, y \ge 0, z \ge 0\}$. We first describe the boundary of this set. Let α be the curve in the yz-plane defined by $z = (1+y)^{-1}$, and A be that portion of the first quadrant of yz-plane above α. Precisely,

$$A = \{(0, y, z)\colon y \ge 0, z \ge (1 + y)^{-1})\} .$$

Let β be the curve in the xy-plane defined by $x = (1 + y)^{-1}$, and B be that portion of the first quadrant of the xy-plane to the right of β. Precisely,

$$B = \{(x, y, 0)\colon y \ge 0, x \ge (1 + y)^{-1}\} .$$

Let δ be the straight line segment in the xz-plane joining $(0, 0, 1)$ to $(1, 0, 0)$, and D be that portion of the first quadrant of the xz-plane above δ. Precisely,

$$D = \{(x, 0, z)\colon x + z \ge 1, x \ge 0, y \ge 0\} .$$

Finally, we let E be the union of all the straight line segments joining points of $\alpha \cap Q$ and $\beta \cap Q$ with the same y-coordinate. Precisely, if

$$l_y = \{t(0, y, (1 + y)^{-1}) + (1 - t)((1 + y)^{-1}, y, 0)\colon y \ge 0, 0 \le t \le 1\} ,$$

then $E = \bigcup_{y\ge 0} l_y$.

We define C to be the closed subset of Q with A, B, D, E as boundary. It is easy to see that C is convex. Consider $\gamma\colon \partial C \to S^2$, and take an interior point p of A (relative to the yz-plane), then $\gamma(p) = (-1, 0, 0)$. Take an interior point q of B (relative to the xy-plane). Then $\gamma(q) = (0, 0, -1)$. The geodesic on S^2 joining $\gamma(p)$ and $\gamma(q)$ is $\xi\colon [0, 1] \to S^2$ such that $\xi(t) = (-\cos \frac{1}{2}\pi t, 0, -\sin \frac{1}{2}\pi t)$. In order that $\gamma(r) = \xi(t)$ for some $r \in \partial C$, it is necessary for r to have an outer normal lying in the xz-plane, or equivalently, it is necessary for r to have a supporting plane parallel to the y-axis. By the construction of C, no supporting plane to C can be parallel to the y-axis unless it is the xy-plane or the yz-plane. Thus $\gamma(\partial C) \cap \operatorname{Im} \xi = \{(-1, 0, 0), (0, 0, -1)\}$, and $\gamma(\partial C)$ is not covex.

In the above example, ∂C is not smooth. However, it is not difficult to see that if we smooth out ∂C along α, β, δ and the z- and x-axes, the essential feature of C is retained. Thus we have a C^∞ convex hypersurface whose spherical image contains $(0, 0, -1)$ and $(-1, 0, 0)$, but none of the other points on the minimal arc connecting them.

Added in proof. (γ) of the main theorem has been generalized by S. T. Yau (*Nonexistence of continuous convex functions on certain Riemannian manifolds*, to appear) and independently by R. E. Greene and H. Wu. The theorem of the latter implies that a noncompact complete Riemannian manifold whose sectional curvature is nonnegative outside a compact set has infinite volume; see their paper *Integrals of subharmonic functions on manifolds of nonnegative curvature*, to appear in Invent. Math.

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