J. DIFFERENTIAL GEOMETRY
9 (1974) 275-277

QUASI-SYMMETRIC IS LOCALLY SYMMETRIC

J. E. D'ATRI

1. Let M be the space of left cosets G/K, where G is a connected Lie group with Lie algebra $\mathfrak{g}$ and K is a closed Lie subgroup with Lie subalgebra $\mathfrak{k}$. For simplicity, assume G acts effectively on M on the left. The canonical projection $\pi: G \to M$ takes $e \in G$ to $o = \pi(e) \in M$, and induces the map $\pi_*: T_eG \to T_0M$. For any $X \in \mathfrak{g}$, let X^* be the global vector field on M generated by the 1-parameter group $\{\exp tX\}$. Then $X_0^* = \pi_*X$, and $X \to X^*$ is an injective Lie algebra anti-homomorphism. Assume M is reductive homogeneous, i.e., there is a given Ad (K)-invariant vector space decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. This defines the canonical connection $\tilde{\nabla}$ on M by $(\tilde{\nabla}_{X^*}Y^*)_0 = -[X, Y]_\mathfrak{p}$ for $X, Y \in \mathfrak{p}$. Here as usual we identify $\mathfrak{p}$ and T_0M by the restriction of π_*, and the subscript $\mathfrak{p}$ indicates $\mathfrak{p}$ component.

Now suppose M has a naturally reductive pseudo-Riemannian metric $\langle\ ,\ \rangle$, i.e., the metric is G-invariant and the associated Levi-Civita connection ∇ agrees with the canonical torsionless connection for the reductive decomposition $\mathfrak{g} = \mathfrak{k} \otimes \mathfrak{p}$ (see [4, Chapter 10]). Define tensors T and B on $\mathfrak{p}$ by

$$T(X, Y) = [X, Y]_\mathfrak{p}\ , \qquad B(X, Y)Z = [[X, Y]_\mathfrak{k}, Z]\ .$$

With our conventions of sign, these are just the negatives of the torsion and curvature for the canonical connection at o. Next define endomorphisms T_X and B_X on $\mathfrak{p}$ by

$$T_X(Y) = T(X, Y)\ , \qquad B_X(Y) = B(X, Y)X\ .$$

Then Chavel [2] has given the following definition (in a slightly more special setting).

Definition. The pseudo-Riemannian manifold $(M, \langle\ ,\ \rangle)$ is said to be quasi-symmetric if T_X and B_X commute for all $X \in \mathfrak{p}$.

There are a number of reasons why it seems plausible to study this class of spaces (see [2, p. 20]) but unfortunately, as we will show, all quasi-symmetric spaces are locally symmetric. On the other hand, one can consider the condition as an easy way to distinguish locally symmetric spaces amongst all naturally reductives, so it still may have some interest.

Remark. In [2], Chavel claims that certain spaces M_α^n are quasi-symmetric.

Received November 17, 1972.

These spaces are not locally symmetric and the author believes the claim is in error.

2. With notation as before, let M be naturally reductive. Define a tensor field E on M by $E = \nabla - \tilde{\nabla}$. If R is the curvature tensor of ∇, then it was shown in [3] (where the letter T was used in place of E) that, at the point o,

$$(1) \qquad (\nabla_Z R)(Y, X) = -R(E_Z Y, X) + R(E_Z X, Y) + [E_Z, R(Y, X)]$$

for all $X, Y, Z \in \mathfrak{p}$. Equivalent formulas occur in [1] and [5]. Note that at the point o we have

$$(2) \qquad E_X Y = \tfrac{1}{2}[X, Y]_{\mathfrak{p}} = \tfrac{1}{2}T_X(Y) \qquad \text{for } X, Y \in \mathfrak{p}\ ,$$

and also the curvature given by

$$(3) \qquad \begin{aligned} R(X, Y)Z = &-[[X, Y]_{\mathfrak{k}}, Z] - \tfrac{1}{2}[[X, Y]_{\mathfrak{p}}, Z]_{\mathfrak{p}} \\ &+ \tfrac{1}{4}[X, [Y, Z]_{\mathfrak{p}}]_{\mathfrak{p}} - \tfrac{1}{4}[Y, [X, Z]_{\mathfrak{p}}]_{\mathfrak{p}} \end{aligned}$$

for all $X, Y, Z \in \mathfrak{p}$ (see [4, p. 202]), so that

$$(4) \qquad R(X, Y)X = -B_X(Y) + \tfrac{1}{4}T_X{}^2(Y)\ .$$

Then (1) and (4) give

$$(5) \qquad (\nabla_X R)(Y, X)X = \tfrac{1}{2}T_X \circ B_X(Y) - \tfrac{1}{2}B_X \circ T_X(Y)$$

for all $X, Y \in \mathfrak{p}$. This immediately shows that if M is locally symmetric, it is quasi-symmetric.

Now suppose M is quasi-symmetric. For a given $Y \in \mathfrak{p}$, the tri-linear map $Q(X_1, X_2, X_3) = (\nabla_{X_1} R)(Y, X_2)X_3$ then satisfies $Q(X, X, X) = 0$ from (5). If we polarize, this implies that

$$(6) \qquad \sum_{\sigma} (\nabla_{X_{\sigma(1)}} R)(Y, X_{\sigma(2)})X_{\sigma(3)} = 0\ ,$$

where the sum is over all 6 permutations σ. Replace three of the terms in (6) using

$$(7) \qquad (\nabla_{X_1} R)(X_3, Y)X_2 = -(\nabla_{X_1} R)(Y, X_2)X_3 - (\nabla_{X_1} R)(X_2, X_3)Y$$

and the two equations derived from (7) by cyclically permuting $(1, 2, 3)$. Then using the second Bianchi identity (and relabeling) gives

$$(8) \qquad 0 = (\nabla_W R)(X, Y)Z + (\nabla_X R)(Z, Y)W + (\nabla_Z R)(W, Y)X$$

for all $W, X, Y, Z \in \mathfrak{p}$. Reversing the roles of X and Y in (8) yields

$$(8)' \qquad 0 = (\nabla_W R)(X, Y)Z + (\nabla_Y R)(X, Z)W + (\nabla_Z R)(X, W)Y\ .$$

Adding (8) and (8)$'$ and applying the analog of (7) and the second Bianchi identity, we find

$$(\nabla_W R)(X, Y)Z = -(\nabla_Z R)(X, Y)W . \tag{9}$$

If the analog of (9) is substituted for the second term of (8) and the second Bianchi identity is used again on the resulting last two terms, we have

$$(\nabla_W R)(X, Y)Z = (\nabla_Y R)(Z, W)X . \tag{10}$$

Now we use the analogs of (9) and (10) with standard identities to get

$$\begin{aligned}\langle(\nabla_W R)(X, Y)Z, V\rangle &= \langle(\nabla_W R)(V, Z)Y, X\rangle = \langle(\nabla_Z R)(Y, W)V, X\rangle \\ &= \langle(\nabla_Z R)(X, V)W, Y\rangle = -\langle(\nabla_W R)(X, V)Z, Y\rangle = \langle(\nabla_W R)(Y, Z)X, V\rangle .\end{aligned}$$

So using the nondegeneracy of $\langle\ ,\ \rangle$ we have

$$(\nabla_W R)(X, Y)Z = (\nabla_W R)(Y, Z)X . \tag{11}$$

Applying this twice in the analog of (7) shows that $\nabla R = 0$ at the point o. Since M is homogeneous, $\nabla R = 0$ everywhere. Hence we have

Theorem. *Suppose that M is naturally reductive pseudo-Riemannian. Then M is quasi-symmetric if and only if M is locally symmetric.*

References

[1] W. Ambrose & I. M. Singer, *On homogeneous Riemannian manifolds,* Duke Math. J. **25** (1958) 647–669.
[2] I. Chavel, *A class of Riemannian homogeneous spaces,* J. Differential Geometry **4** (1970) 13–20.
[3] J. E. D'Atri & H. K. Nickerson, *Geodesic symmetries in spaces with special curvature tensors,* J. Differential Geometry **9** (1974) 251–262.
[4] S. Kobayashi & K. Nomizu, *Foundations of differential geometry,* Vol. II, Interscience, New York, 1969.
[5] B. Kostant, *A characterization of invariant affine connections,* Nagoya Math. J. **16** (1960) 35–50.

RUTGERS UNIVERSITY