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ALMOST COTANGENT MANIFOLDS

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1. The geometry of the cotangent manifold $T^*\mathcal{M}$ of a differentiable manifold $\mathcal{M}$ has been studied by K. Yano and E. M. Patterson [4], [5], [6]. Some of their results can be extended to a manifold M of dimension $2n$ carrying a G-structure whose group consists of all $2n \times 2n$ matrices of the form

$$\begin{bmatrix} A & 0 \\ B & (A^{-1})^t \end{bmatrix} \tag{1.1}$$

where $A \in GL(R^n)$ and $A^tB = B^tA$. Such a structure is an *almost cotangent structure,* and such a manifold M is an *almost cotangent manifold* (M. R. Bruckheimer [1]).

Example 1.1. Suppose that $\mathcal{M}$ is a manifold of dimension n, and that $\pi\colon T^*\mathcal{M} \to \mathcal{M}$ is the natural projection which takes a covector at $m \in \mathcal{M}$ to the point m. Any function f in $\mathcal{M}$ can be lifted to a function $f \circ \pi$ in $T^*\mathcal{M}$ but we shall denote it by the same symbol f. If x is a chart of $\mathcal{M}$ with domain V, we can define a standard chart (x, y) of $T^*\mathcal{M}$ with domain $\pi^{-1}V$. Two such charts (x, y), $(\bar{x}, \bar{y})$ with intersecting domains are related by a change of coordinates whose Jacobian matrix has the form (1.1) with

$$A = \left[\frac{\partial x^a}{\partial \bar{x}^b}\right], \qquad B = \left[\frac{\partial^2 \bar{x}^c}{\partial x^a \partial x^d} \frac{\partial x^d}{\partial \bar{x}^b} \bar{y}_c\right], \tag{1.2}$$

where $a, b, c, d = 1, \cdots, n$. The natural moving frames associated with these charts therefore define an almost cotangent structure on $T^*\mathcal{M}$.

Suppose that M is any almost cotangent manifold. We define a 2-form ω on M by specifying its components to be

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \tag{1.3}$$

relative to any adapted frame of M. ω determines an *almost symplectic structure* on M to which the given almost cotangent structure is subordinate. If $(\theta^1, \cdots, \theta^{2n})$ is any adapted moving coframe of M, then locally

$$\omega = \theta^a \wedge \theta^{a+n} \qquad (a = 1, \cdots, n) .$$

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The 1-forms $\theta^1, \cdots, \theta^n$ form a local cobasis for an n-dimensional *distribution* $\mathscr{D}$ on M. This determines a G-structure on M to which the given almost cotangent structure is subordinate. Its group consists of the $2n \times 2n$ matrices of the form

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \tag{1.4}$$

where $A, C \in GL(R^n)$.

Conversely, we have

Proposition 1.1. *If an n dimensional distribution and an almost symplectic structure on a 2n-dimensional manifold have a common subordinate structure, then this is an almost cotangent structure.*

Proof. The group of the G-structure defined by the distribution consists of $2n \times 2n$ matrices of the form (1.4). If such a matrix also belongs to the symplectic group, then

$$\begin{bmatrix} A^t & B^t \\ 0 & C^t \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix},$$

which implies that $A^tB = B^tA$ and $C = (A^{-1})^t$. Consequently such a matrix is of the form (1.1). q.e.d.

Let M be a differentiable manifold carrying an almost symplectic structure determined by a 2-form ω. Given any vector field X in M, we use ω to define a 1-form $Y \mapsto \omega(X, Y)$ in M with the same domain. Since ω is nonsingular, it maps independent vector fields to independent 1-forms.

Proposition 1.2. *An n-dimensional distribution $\mathscr{D}$ and an almost symplectic structure on a 2n-dimensional manifold M admit a common subordinate structure iff ω maps each basis of $\mathscr{D}$ to a cobasis of $\mathscr{D}$.*

Proof. Suppose that the two structures have a common subordinate structure. Choose any moving frame $(X_1, \cdots, X_{2n})$ adapted for this structure, and let $(\theta^1, \cdots, \theta^{2n})$ be the dual moving coframe. Then X_{a+n} $(a = 1, \cdots, n)$ is a local basis for $\mathscr{D}$, and θ^a $(a = 1, \cdots, n)$ is a local cobasis. ω maps the vector field X_{a+n} to the 1-form ψ^a defined by

$$\psi^a(X_i) = \omega(X_{a+n}, X_i) = \delta_{ai} \qquad (i = 1, \cdots, 2n),$$

and so $\psi^a = \theta^a$. More generally, ω maps any local basis Y_{a+n} $(a = 1, \cdots, n)$ for $\mathscr{D}$ to a cobasis, since we can choose the moving frame so that locally

$$Y_{b+n} = \alpha^a_b X_{a+n} \qquad \det \alpha \neq 0 .$$

This maps to $\alpha^a_b \theta^a$ which is a cobasis for $\mathscr{D}$.

Conversely, suppose that ω maps each basis for $\mathscr{D}$ to a cobasis. Choose any moving frame $(Y_1, \cdots, Y_{2n})$ which is adapted for $\mathscr{D}$. The vector fields Y_{a+n} $(a = 1, \cdots, n)$ form a basis for $\mathscr{D}$, and so the 1-forms

$$Y \to \omega(Y_{a+n}, Y) \qquad (a = 1, \cdots, n)$$

form a cobasis. Consequently $\omega(Y_{a+n}, Y_{b+n}) = 0$, and we may write the matrix

$$\omega(Y_i, Y_j) = \begin{bmatrix} P & -Q^t \\ Q & 0 \end{bmatrix},$$

where $P^t = -P$, and $\det Q = 0$. We now construct a new moving frame

$$[X_1, \cdots, X_{2n}] = [Y_1, \cdots, Y_{2n}] \begin{bmatrix} A & 0 \\ B & C \end{bmatrix},$$

where $A = Q^{-1}, B = \frac{1}{2}(Q^{-1})^t PQ^{-1}, C = I$. This too is adapted for the distribution, and also for the almost symplectic structure since

$$\omega(X_i, X_j) = \begin{bmatrix} A^t & B^t \\ 0 & C^t \end{bmatrix} \begin{bmatrix} P & -Q^t \\ Q & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

Since we can find such a moving frame at each point of M, the two structures have a common subordinate structure.

2. Suppose we are given two G-structures with a common subordinate structure on a manifold M. If the subordinate structure is integrable, then so are the given structures. The converse is not necessarily true, but in the case of an almost cotangent structure we have

Proposition 2.1. *An almost cotangent structure is integrable iff the underlying distribution and almost symplectic structure are both integrable.*

Proof. Suppose that the underlying structures are both integrable. Choose any point $m \in M$. There exists a chart x at m adapted for the distribution. Choose any moving coframe $\phi = (\phi^1, \cdots, \phi^{2n})$ at m adapted for the almost cotangent structure. Since it is adapted for the distribution,

$$\phi^a = A^a_b dx^b, \qquad \det A \neq 0 \quad (a, b = 1, \cdots, n).$$

The moving coframe θ at m defined by

$$\theta^a = dx^a, \qquad \theta^{a+n} = A^b_a \phi^{b+n}$$

is adapted for the almost cotangent structure. Suppose that

$$\theta^{a+n} = \alpha^a_b dx^b + \beta^a_b dx^{b+n}.$$

Since the almost symplectic structure is integrable, the canonical 2-form $\omega = \theta^a \wedge \theta^{a+n}$ is closed, and so

$$dx^a \wedge \left\{\left(\frac{\partial \alpha_b^a}{\partial x^c} dx^c + \frac{\partial \alpha_b^a}{\partial x^{c+n}} dx^{c+n}\right) \wedge dx^b + \left(\frac{\partial \beta_b^a}{\partial x^c} dx^c + \frac{\partial \beta_b^a}{\partial x^{c+n}} dx^{c+n}\right) \wedge dx^{b+n}\right\} = 0 .$$

One consequence of this is that

$$\frac{\partial \beta_b^a}{\partial x^{c+n}} = \frac{\partial \beta_c^a}{\partial x^{b+n}} .$$

It follows that the equations

$$\frac{\partial H^a}{\partial x^{b+n}} = \beta_b^a$$

admit differentiable solution $H^a(x^1, \cdots, x^{2n})$ on a neighborhood of m. We use them to construct a new chart y at m by defining

$$y^a = x^a , \qquad y^{a+n} = H^a(x^1, \cdots, x^{2n}) .$$

In terms of this chart

$$\theta^a = dy^a , \qquad \theta^{a+n} = \bar{\alpha}_b^a dy^b + dy^{a+n} ,$$

where $\bar{\alpha}_b^a = \alpha_b^a - \partial H^a / \partial x^b$.

Using these new coordinates, the condition $d\omega = 0$ implies that

$$\frac{\partial \bar{\alpha}_b^a}{\partial y^c} + \frac{\partial \bar{\alpha}_c^b}{\partial y^a} + \frac{\partial \bar{\alpha}_a^c}{\partial y^b} = 0 , \tag{2.1}$$

$$\frac{\partial}{\partial y^{c+n}}(\bar{\alpha}_b^a - \bar{\alpha}_a^b) = 0 . \tag{2.2}$$

Consider the equations

$$\frac{\partial F^a}{\partial y^b} - \frac{\partial F^b}{\partial y^a} = \bar{\alpha}_b^a - \bar{\alpha}_a^b .$$

Equations (2.2) show that the right-hand side depends only on $y^1, \cdots, y^n$, and equations (2.1) show that differentiable solutions $F^a(y^1, \cdots, y^n)$ exist at m. We define functions

$$z^a = y^a , \qquad z^{a+n} = y^{a+n} + F^a(y^1, \cdots, y^n) \quad (a = 1, \cdots, n) .$$

Since

$$dz^a = dy^a = \theta^a ,$$

$$dz^{a+n} = dy^{a+n} + \frac{\partial F^a}{\partial y^b} dy^b = \theta^{a+n} + \left(\frac{\partial F^a}{\partial y^b} - \bar{\alpha}^a_b\right)\theta^b ,$$

these functions $z^1, \cdots, z^{2n}$ form a chart z at m. This chart is adapted for the almost cotangent structure since $\partial F^a/\partial y^b - \bar{\alpha}^a_b$ is symmetric in a, b.

3. S. S. Chern [2] defined a structure tensor for any given G-structure on a manifold M. This is determined by specifying its components relative to any adapted moving coframe θ with domain U.

Let Z be the subspace of $V = \hom(R^n \wedge R^n, R^n)$ consisting of elements ρ such that

$$\rho(u, v) = (Su)v - (Sv)u$$

for all $u, v \in R^n$, where $L(G)$ is the Lie algebra of G and where $S \in \hom(R^n, L(G))$. If matrices W_A $(A = 1, \cdots, r)$ form a basis for $L(G)$, then the elements $\rho \in Z$ have components

$$\rho^i_{jk} = \xi^A_j (W_A)^i_k - \xi^A_k (W_A)^i_j ,$$

where $i, j, k = 1, \cdots, n$ and $\xi^A_j \in R$. We have to define a subspace of V complementary to Z. Given $\gamma \in V$ we impose sufficient linear conditions on $\gamma + \rho$, where $\rho \in Z$, so that ρ is determined uniquely. Then $\gamma + \rho$ lies in a subspace W of V complementary to Z and the canonical projection $\lambda\colon V \to W$ is given by $\gamma \to \gamma + \rho$.

Suppose that

$$d\theta^i = \tfrac{1}{2}\gamma^i_{jk}\theta^j \wedge \theta^k .$$

The coefficients γ^i_{jk} determine a function γ on U with values in V. The structure tensor has components $C = \lambda \circ \gamma$ relative to the moving coframe θ.

Suppose that M is an almost cotangent manifold, and let θ be an adapted moving coframe. We first calculate the structure tensor for the underlying almost symplectic structure. The Lie algebra of the symplectic group consists of $2n \times 2n$ matrices

$$\begin{bmatrix} A & C \\ B & -A^t \end{bmatrix}$$

where the $n \times n$ matrices B, C are symmetric. This admits a basis consisting of matrices

$$(W^a_b - W^{b+n}_{a+n}), (W^{a+n}_b + W^{b+n}_a), (W^a_{b+n} + W^b_{a+n}), \qquad (a, b = 1, \cdots, n),$$

where the matrix W^i_j $(i, j = 1, \cdots, 2n)$ has entry 1 in the (i, j)th position and zeros elsewhere. A straightforward calculation shows that we can define ρ so that $C = \gamma + \rho$ satisfies the linear conditions

$$C^a_{bc} = 0 \,, \qquad C^{a+n}_{b+n\ c+n} = 0 \,,$$
$$C^{a+n}_{b\ c+n} + C^{b+n}_{a\ c+n} = 0 \,, \qquad C^a_{b\ c+n} + C^c_{b\ a+n} = 0 \,,$$
$$C^{a+n}_{bc} = C^{b+n}_{ca} = C^{c+n}_{ab} \,, \qquad C^a_{b+n\ c+n} = C^b_{c+n\ a+n} = C^c_{a+n\ b+n} \,,$$

and that these conditions determine ρ uniquely. The components C of the structure tensor relative to the coframe θ are given by

$$C^a_{bc} = 0 \,, \qquad C^{a+n}_{b+n\ c+n} = 0 \,, \tag{3.1}$$

$$C^{a+n}_{b\ c+n} = \tfrac{1}{2}(\gamma^{a+n}_{b\ c+n} - \gamma^{b+n}_{a\ c+n} - \gamma^c_{ab}) \,, \tag{3.2}$$

$$C^a_{b\ c+n} = \tfrac{1}{2}(\gamma^a_{b\ c+n} - \gamma^c_{b\ a+n} + \gamma^{b+n}_{a+n\ c+n}) \,, \tag{3.3}$$

$$C^{a+n}_{bc} = \tfrac{1}{3}(\gamma^{a+n}_{bc} + \gamma^{b+n}_{ca} + \gamma^{c+n}_{ab}) \,, \tag{3.4}$$

$$C^a_{b+n\ c+n} = \tfrac{1}{3}(\gamma^a_{b+n\ c+n} + \gamma^b_{c+n\ a+n} + \gamma^c_{a+n\ b+n}) \,. \tag{3.5}$$

Proposition 3.1. *The underlying almost symplectic structure on M is integrable iff its structure tensor is zero.*

Proof. The structure is integrable if $d\omega = 0$, and this condition is satisfied locally if

$$\tfrac{1}{2}\gamma^a_{jk}\theta^j \wedge \theta^k \wedge \theta^{a+n} - \tfrac{1}{2}\gamma^{a+n}_{jk}\theta^a \wedge \theta^j \wedge \theta^k = 0 \,.$$

Equations (3.2), ..., (3.5) show that this is true if $C = 0$. q.e.d.

We next calculate the structure tensor for the underlying distribution on the almost cotangent manifold M. The Lie algebra for the distribution group consists of the $2n \times 2n$ matrices

$$\begin{bmatrix} A & 0 \\ B & D \end{bmatrix} ,$$

and it admits a basis

$$W^a_b, W^{a+n}_b, W^{a+n}_{b+n} \,.$$

In this case we can define ρ in just one way so that $C = \gamma + \rho$ satisfies the linear conditions

$$C^i_{bk} = 0 \,, \qquad C^{a+n}_{b+n\ c+n} = 0 \,.$$

The components C of the structure tensor relative to the coframe θ are then all zero except

$$C^a_{b+n\ c+n} = \gamma^a_{b+n\ c+n} \,. \tag{3.6}$$

Proposition 3.2. *The underlying distribution on M is integrable iff its structure tensor is zero.*

Proof. $\theta^1, \cdots, \theta^n$ is a local cobasis for the distribution. If $C = 0$, it follows from equation (3.6) that

$$d\theta^a = \tfrac{1}{2}(\gamma^a_{bc}\theta^b + 2\gamma^a_{b+n\ c}\theta^{b+n}) \wedge \theta^c ,$$

and Frobenius Theorem shows that the distribution is integrable. q.e.d.

Finally we calculate the structure tensor for the almost cotangent structure on M. The Lie algebra for the almost cotangent group consists of the $2n \times 2n$ matrices

$$\begin{bmatrix} A & 0 \\ B & -A^t \end{bmatrix} \tag{3.7}$$

where the $n \times n$ matrix B is symmetric. It admits a basis consisting of the matrices

$$(W^a_b - W^{b+n}_{a+n}), (W^{a+n}_b + W^{b+n}_a) .$$

We can define ρ in just one way so that $C = \gamma + \rho$ satisfies the linear conditions

$$C^a_{bc} = 0 , \qquad C^{a+n}_{b+n\ c+n} = 0 ,$$
$$C^{a+n}_{b\ c+n} + C^{b+n}_{a\ c+n} = 0 , \qquad C^a_{b\ c+n} + C^c_{b\ a+n} = 0 ,$$
$$C^{a+n}_{bc} = C^{b+n}_{ca} = C^{c+n}_{ab} .$$

The components C of the structure tensor relative to the coframe θ are then given by equations (3.1), (3.2), (3.3), (3.4), (3.6). From this we deduce

Proposition 3.3. *The structure tensor of an almost cotangent structure is zero iff the structure tensors of the underlying distribution and almost symplectic structure are both zero.*

Propositions 2.1, 3.1, 3.2, 3.3 now lead to

Proposition 3.4. *An almost cotangent structure is integrable iff its structure tensor is zero.*

Any G-structure is said to be *almost transitive* if its structure tensor is constant. If the group G includes an element αI, where the real number α is not 1, such a structure tensor is necessarily zero. Since the almost cotangent group includes the element $-I$, we have

Proposition 3.5. *An almost cotangent structure is almost transitive iff it is integrable.*

4. A nondegenerate Riemannian metric S on a manifold M defines a class of conjugate structures on M. S is said to be *related* to a given G-structure on M if one of these conjugate structures has a common subordinate structure with the given G-structure.

Among the conjugate structures is included one $O_s(R^n)$ structure, the components of S relative to any adapted frame of this structure being

$$\begin{bmatrix} I_s & 0 \\ 0 & -I_{n-s} \end{bmatrix} .$$

If this $O_s(R^n)$ structure has a common subordinate structure with the given G-structure, then the metric S is called a *G-metric*.

A positive-definite G-metric on an almost cotangent manifold will be called an *almost cotangent metric*. Such metrics are studied in this section.

Lemma 4.1. *If S is a positive-definite Riemannian metric on an almost cotangent manifold M, then there exists an adapted moving frame ρ at any given point $m \in M$ relative to which S has components of the form*

$$\begin{bmatrix} a & b \\ -b & I \end{bmatrix} . \tag{4.1}$$

Proof. Choose any adapted moving frame σ at m, and suppose that, relative to σ, S has components

$$\begin{bmatrix} P & Q \\ Q^t & R \end{bmatrix} . \tag{4.2}$$

Because this matrix is positive-definite, we can choose a differentiable function A at m with values in $GL(R^n)$ such that $AA^t = R$. We then define

$$B = -\tfrac{1}{2}[Q(A^{-1})^t + R^{-1}Q^tA] .$$

The moving frame

$$\rho = \sigma\begin{bmatrix} A & 0 \\ B & (A^{-1})^t \end{bmatrix}$$

satisfies our requirements, since it is adapted for the almost cotangent structure on M and the components of S relative to ρ

$$\begin{bmatrix} A^t & B^t \\ 0 & A^{-1} \end{bmatrix}\begin{bmatrix} P & Q \\ Q^t & R \end{bmatrix}\begin{bmatrix} A & 0 \\ B & (A^{-1})^t \end{bmatrix}$$

reduce to the form (4.1). q.e.d.

A Riemannian metric on a manifold determines a scalar product on each tangent space and each cotangent space. We denote both of these by the same symbol (.).

Proposition 4.2. *A positive-definite Riemannian metric S on an almost cotangent manifold M is an almost cotangent metric iff*

$$(\omega X \cdot \omega Y) = (X \cdot Y) \tag{4.3}$$

for all vector fields X and Y in M, where ω is the canonical 2-form on M.

Proof. The condition (4.3) can be expressed in tensor form as

$$\omega = -S\omega^{-1}S . \tag{4.4}$$

If S is an almost cotangent metric, then at any given point of M there is a frame relative to which S and ω have components

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

respectively. The tensor relation (4.4) is therefore satisfied on M.

Conversely, suppose that (4.4) is satisfied. Choose a special adapted moving frame ρ (as defined in Lemma 4.1) at a given point $m \in M$. Evaluating the relation (4.4) in terms of ρ shows that

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = -\begin{bmatrix} a & b \\ -b & I \end{bmatrix}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} a & b \\ -b & I \end{bmatrix} .$$

It follows that $b = 0$ and $a = I$. Consequently ρ is adapted for the $O(R^{2n})$ structure defined by S as well as for the almost cotangent structure. These two structures therefore have a common subordinate structure. q.e.d.

That almost cotangent metrics exist on any paracompact almost cotangent manifold follows from

Proposition 4.3. *Any given positive-definite Riemannian metric S on an almost cotangent manifold M determines an almost cotangent metric on M.*

Proof. Lemma 4.1 shows that there exists a set of special adapted moving frames for the almost cotangent structure whose domains cover M and for which S has components (4.1). Any two such moving frames $\rho, \bar{\rho}$ with intersecting domains $U, \bar{U}$ are related by

$$\bar{\rho} = \rho\begin{bmatrix} A & 0 \\ B & (A^{-1})^t \end{bmatrix}$$

where $A^tB = B^tA$. Since the components of S relative to $\bar{\rho}$ are given on $U \cap \bar{U}$ by

$$\begin{bmatrix} A^t & B^t \\ 0 & A^{-1} \end{bmatrix}\begin{bmatrix} a & b \\ -b & I \end{bmatrix}\begin{bmatrix} A & 0 \\ B & (A^{-1})^t \end{bmatrix}$$

it follows that $A \in O(R^n)$ and $B = 0$. Consequently the special adapted moving frames also define an $O(R^{2n})$-structure on M. The associated metric on M is an almost cotangent metric. q.e.d.

We continue with the problem of constructing an almost cotangent metric. An easy calculation using Proposition 4.2 leads to

Propositon 4.4. *A positive-definite Riemannian metric on an almost cotangent manifold M is an almost cotangent metric iff its components relative to any adapted frame of M are of the form*

$$\begin{bmatrix} R^{-1} + QR^{-1}Q^t & Q \\ Q^t & R \end{bmatrix} \tag{4.5}$$

where R is a positive-definite $n \times n$ matrix and RQ is symmetric.

This proposition shows that if σ is an adapted moving frame of M with domain U we can construct an almost cotangent metric on U when we are given differentiable $n \times n$ matrix-valued functions Q, R on U such that R is positive-definite and RQ is symmetric. If $\bar{\sigma}$ is an adapted moving frame on $\bar{U}$ such that

$$\bar{\sigma} = \sigma \begin{bmatrix} A & 0 \\ B & (A^{-1})^t \end{bmatrix}$$

with corresponding functions $\bar{Q}, \bar{R}$, and if

$$\bar{R}A^t = A^{-1}R\ , \qquad \bar{Q}A^t = A^tQ + B^tR\ , \tag{4.6}$$

then the two metrics agree on $U \cap \bar{U}$. We use this result in

Example 4.1. Starting from a positive-definite metric g on a manifold $\mathscr{M}$ we construct an almost cotangent metric on $T^*\mathscr{M}$. If x is a chart of $\mathscr{M}$, the moving frame σ associated with the standard chart (x, y) is adapted for the almost cotangent structure on $T^*\mathscr{M}$. Suppose that g^{ab} are the components of g^{-1} associated with the chart x, and that Γ^a_{bc} are the corresponding Christoffel symbols. We use these to define matrix-valued functions

$$Q = [-g^{bc}\Gamma^d_{ca}y_d]\ , \qquad R = [g^{ab}]$$

on the domain U of σ. Since R is positive-definite and RQ is symmetric, we have an almost cotangent metric on U with components (4.5) relative to σ. The corresponding functions $\bar{Q}, \bar{R}$ on $\bar{U}$ are related to Q, R by equations (4.6), where A and B are defined in (1.2).

5. An almost cotangent metric is an example of a related metric. We now describe another related metric on an almost cotangent manifold M.

A Riemannian metric on M such that

(i) $(\omega X \cdot \omega Y) = -(X \cdot Y)$ for all vector fields X, Y in M,

(ii) $(X, Y) = 0$ for all vector fields X, Y in M tangent to the distribution $\mathscr{D}$ will be said to be *skew invariant*. That such metrics always exist on a paracompact almost cotangent manifold follows from

Proposition 5.1. *Any given positive-definite Riemannian metric S on an* *almost cotangent manifold M determines a skew invarient metric $\bar{S}$ on M.*

Proof. Suppose that σ is any adapted moving frame of M, and that, relative to σ, S has components (4.2). We define a (2,0) tensor field locally by taking its components relative to σ to be

$$\begin{bmatrix} R^{-1}Q^t + QR^{-1} & I \\ I & 0 \end{bmatrix}.$$

It is easy to verify that such local fields agree on the intersection of their domains, and so they define a (2,0) tensor field $\bar{S}$ on M. $\bar{S}$ is a skew invariant metric.

Example 5.1. We use the above proposition to construct a skew invariant metric $\bar{S}$ on $T^*\mathscr{M}$ starting from the almost cotangent metric S described in Example 4.1. The components of $\bar{S}$ relative to the natural moving frame associated with the chart (x, y) reduce to

$$\begin{bmatrix} -2\Gamma^c_{ab}y_c & I \\ I & 0 \end{bmatrix}.$$

Consequently $\bar{S}$ is the *Riemann extension* of the Riemannian connection of the metric g on $\mathscr{M}$ as defined by E. M. Patterson and A. G. Walker [3]. The Riemannian connection may be replaced by any symmetric linear connection on $\mathscr{M}$.

Not every skew invariant metric arises in the way we have described in Proposition 5.1, and in general we have

Proposition 5.2. *A Riemannian metric on an almost cotangent manifold is skew invariant iff its components related to every adapted frame are of the form*

$$\begin{bmatrix} P & Q \\ Q^t & 0 \end{bmatrix} \tag{5.1}$$

where P is a symmetric $n \times n$ matrix, $Q^2 = I$ and $QPQ^t = P$.

Proof. A metric S has components (4.2) relative to an adapted frame. Suppose that S is skew invariant. Condition (ii) implies that $R = 0$, and then condition (i) implies that

$$\begin{bmatrix} P & Q \\ Q^t & 0 \end{bmatrix}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} P & Q \\ Q^t & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

This shows that $Q^2 = I$ and $QPQ^t = P$. The converse result is proved in a similar way. q.e.d.

Next we show that a skew invariant metric on a connected almost cotangent manifold is a related metric.

Lemma 5.3. *If S is a skew invariant metric on an almost cotangent manifold M, then there exists an adapted moving frame ρ at any given point $m \in M$ relative to which S has components*

$$\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$$

where K is some diagonal $n \times n$ matrix of the form

$$\operatorname{diag}\{1, 1, \cdots, 1, -1, -1, \cdots, -1\} .$$

Proof. Let σ be an adapted moving frame at m, and suppose that S has components (5.1) relative to σ. The differentiable matrix-valued function Q satisfies $Q^2 = I$, and so we can find a differentiable function A on some connected neighborhood U of m such that $AQA^{-1} = K$ where $K = \operatorname{diag}\{1, 1, \cdots, -1\}$. If we define B on U by $PA^t + 2QB = 0$, then, since $QPQ^t = P$,

$$\rho = \sigma \begin{bmatrix} A^t & 0 \\ B & A^{-1} \end{bmatrix}$$

is also an adapted moving frame at m. It has the property required. q.e.d.

As a simple consequence of the above lemma we have

Proposition 5.4. *Every skew invariant metric on a 2n-dimensional almost cotangent manifold has signature (n, n).*

Proposition 5.5. *Any skew invariant metric on a connected almost cotangent manifold is related to the almost cotangent structure.*

Proof. Suppose that $\rho, \bar{\rho}$ are two moving frames as described in Lemma 5.3 and that the corresponding components of the metric S are

$$\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}, \begin{bmatrix} 0 & \bar{K} \\ \bar{K} & 0 \end{bmatrix} .$$

Suppose that the domains of these moving frames intersect, and that

$$\bar{\rho} = \rho \begin{bmatrix} A & 0 \\ B & (A^{-1})^t \end{bmatrix} .$$

Then $A^{-1}KA = \bar{K}$. Since the matrices $K, \bar{K}$ have the same trace, they are equal. Because M is connected, we can find a set of such adapted moving frames ρ whose domains cover M and with respect to which the components of S are the same. It follows that these moving frames are also adapted to one of the G-structures defined by S.

6. Suppose that a manifold M carries a G-structure. A connection on the adapted frame bundle $P(M, G)$ determines a linear connection on M called a G-connection. Any linear connection on M is a G-connection iff the local connection forms which correspond to adapted moving frames of M have values in the Lie algebra of G. It is sufficient if this connection is satisfied for a set of adapted moving frames whose domains cover M. When M is an almost cotangent manifold this leads to

Proposition 6.1. *A linear connection on an almost cotangent manifold is an almost cotangent connection iff it is a connection for both the underlying distribution and almost symplectic structure.*

Since the Lie algebra of the almost cotangent group consists of the $2n \times 2n$ matrices of the form (3.7), we deduce

Proposition 6.2. *A linear connection on an almost cotangent manifold is an almost cotangent connection iff its coefficients relative to each adapted moving coframe satisfy the conditions*

$$\Gamma^a_{j\,c+n} = 0\,, \quad \Gamma^a_{jc} = -\Gamma^{c+n}_{j\,a+n}\,, \quad \Gamma^{a+n}_{j\,c} = \Gamma^{c+n}_{j\,a}\,,$$

where $a, c = 1, \cdots, n\,; j = 1, \cdots, 2n$.

Example 6.1. Let ∇ be any symmetric linear connection on a manifold $\mathscr{M}$. The Riemann extension of ∇ (Example 5.1) is a metric on $T^*\mathscr{M}$. The Riemannian connection $\bar{\nabla}$ of this metric is called the *complete lift* of ∇. K. Yano and E. M. Patterson [5] show that its components relative to any standard chart (x, y) are given by

$$\bar{\Gamma}^a_{bc} = \Gamma^a_{bc}\,, \qquad \bar{\Gamma}^a_{b\,c+n} = \bar{\Gamma}^a_{b+n\,c} = \bar{\Gamma}^a_{b+n\,c+n} = 0\,,$$

$$\bar{\Gamma}^{a+n}_{bc} = y_d\left(\frac{\partial \Gamma^d_{bc}}{\partial x^a} - \frac{\partial \Gamma^d_{ca}}{\partial x^b} - \frac{\partial \Gamma^d_{ba}}{\partial x^c} + 2\Gamma^d_{ae}\Gamma^e_{bc}\right),$$

$$\bar{\Gamma}^{a+n}_{b\,c+n} = -\Gamma^c_{ba}\,, \quad \bar{\Gamma}^{a+n}_{b+n\,c} = -\Gamma^b_{ac}\,, \quad \bar{\Gamma}^{a+n}_{b+n\,c+n} = 0\,,$$

where $a, b, c, d, e = 1, \cdots, n$. It follows that if ∇ has zero curvature, then $\bar{\nabla}$ is an almost cotangent connection.

Example 6.2. Starting from a symmetric connection ∇ on $\mathscr{M}$, the same authors [6] have defined another connection $\tilde{\nabla}$ on $T^*\mathscr{M}$ called the *horizontal lift* of ∇. Its components relative to any standard chart (x, y) only differ from the corresponding components of the complete lift by

$$\tilde{\Gamma}^{a+n}_{bc} = y_d\left(-\frac{\partial \Gamma^d_{ca}}{\partial x^b} + \Gamma^d_{ae}\Gamma^e_{bc} + \Gamma^d_{ce}\Gamma^e_{ba}\right).$$

$\tilde{\nabla}$ is therefore always an almost cotangent connection.

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