THE FIRST BETTI NUMBER OF A COMPACT ALMOST TACHIBANA SPACE

SEIICHI YAMAGUCHI

0. Introduction

It is well known that the $p$-th Betti number of a compact Kählerian space is zero or even if $p$ is odd [2]. A similar result is known for a compact Sasakian space [1], [6], [7]. In particular, the first Betti number is zero or even in a compact Sasakian space.

The purpose of this paper is to give the analogy for the first Betti number of a compact Tachibana space (= nearly Kähler space [3], = $K$-space [4]).

1. Preliminaries

Let $M$ be an $n$-dimensional almost Hermitian space with positive definite metric $g = (g_{ij})$ and almost complex structure $J = (J_i^j)$, $(i, j, \ldots = 1, \ldots, n)$.

A 1-form $\vartheta$ in $M$ is called a covariant almost analytic form [4] if it satisfies the equation

$$ F_j(J_i^r u_r) = \vartheta_r F_i J_j^r - J_j^r F_i u_r , $$

or equivalently

$$ F_j(J_i^r u_r) - F_i(J_j^r u_r) = J_j^r (F_i u_r - F_r u_r) , $$

where $F$ denotes the operator of covariant derivative with respect to the Riemannian connection.

An almost Hermitian space is called an almost Tachibana space (resp. a Kählerian space) if the associated 2-form $J = \frac{1}{2} J_{ji} dx^i \wedge dx^j$ is a Killing 2-form (resp. parallel), where we put $J_{ji} = g_{ij} J_i^j$ and $\{x^i\}$ is a local coordinate system of $M$.

Then the following theorems are known:

**Theorem A** [9]. A necessary and sufficient condition for a 1-form $\vartheta$ in a compact Kählerian space to be covariant analytic is that the 1-form $\vartheta$ be harmonic.

**Theorem B** [4]. In a compact almost Tachibana space, a necessary and sufficient condition for a 1-form $\vartheta = (u_i)$ to be covariant almost analytic is that $\vartheta$ and $\bar{\vartheta} = (\bar{u}_i)$ both be harmonic.
Throughout this paper, we shall deal with an almost Tachibana space $M$, that is, an almost Hermitian space satisfying

\[(1.1) \quad \nabla_j J_{\ell h} + \nabla_j J_{h \ell} = 0.\]

We shall recall the identities in $M$, which are necessary for later use. The following relations are well known [4], [8], [9]:

\[(1.2) \quad J^\ell R_{\ell \ell} + J_{\ell r} R_{\ell r} = 0,\]
\[(1.3) \quad \nabla^r \nabla_{\ell} J_{rf} = R_{\ell r} J_{\ell r} - \frac{1}{2} J_{\ell r} R_{\ell r} J_{\ell r}.\]

Next, let $u$ be any 1-form. Then by virtue of the Ricci’s identity we can obtain

\[(1.4) \quad J_{\ell r} \nabla_\ell u_\ell + (J_{\ell r} J_{\ell r} + J_{\ell r} J_{\ell r}) = 0,\]
\[(1.5) \quad (J_{\ell r} J_{\ell r} + J_{\ell r} J_{\ell r}) = 0,\]

which are valid in any Riemannian space.

\section{Theorems}

Let us prove the following theorem.

\textbf{Theorem 2.1.} In a compact almost Tachibana space $M$, if $u$ is a harmonic 1-form, then $\bar{u} = (J_{\ell r} u_\ell)$ is also so.

\textbf{Proof.} Since $u$ is a harmonic 1-form, we have

\[(u, \nabla_{\ell} J_{\ell r}) (J_{\ell r} u_\ell) + (J_{\ell r} J_{\ell r}) u_\ell (J_{\ell r} u_\ell) = 0,\]

and therefore

\[
(u, \nabla_{\ell} J_{\ell r}) u_\ell \nabla_{\ell} J_{\ell r} u_\ell + \frac{1}{2} (J_{\ell r} J_{\ell r} + J_{\ell r} J_{\ell r}) u_\ell (J_{\ell r} u_\ell) + 3(u, \nabla_{\ell} J_{\ell r} u_\ell) (J_{\ell r} u_\ell) + 3(u, \nabla_{\ell} J_{\ell r} u_\ell) (J_{\ell r} u_\ell)
\]

On the other hand, making use of (1.1), \ldots, (1.5) we easily see that
\[ V^r(J_r^s u_r) = V^s V_r(J_r^s u_r) , \quad (u_r V_r J_r^s) J_r^s u_j = 0 . \]

Hence, by Green's theorem and the obvious fact that \( V^r(J_r^s u_r) = 0 \), the theorem is proved.

As a corollary of this theorem, we obtain

**Theorem 2.2.** The first Betti number of a compact almost Tachibana space is zero or even.

By virtue of Theorem B and Theorem 2.1, we get

**Theorem 2.3.** In a compact almost Tachibana space, a necessary and sufficient condition for a 1-form \( u \) to be covariant almost analytic is that \( u \) be harmonic.

The author would like to express his hearty thanks to Professors S. Tachibana and Y. Ogawa for their criticisms and advices, and also to the referee for his suggestions regarding the revision of this paper.

**References**
