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CONDITIONS FOR CONSTANCY OF THE HOLOMORPHIC SECTIONAL CURVATURE

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In the present note we shall first prove an algebraic result (Theorem 1) on the curvature tensor of a Kaehlerian manifold. As applications we derive two results (Theorems 2 and 3) characterizing constancy of the holomorphic sectional curvature by the existence of sufficiently many complex or totally real submanifolds which are totally geodesic. A special case of Theorem 2 has been known as the axiom of holomorphic planes [3].

1. Curvature tensor

Let M be a Kaehlerian manifold. In the tangent space at a point we consider the curvature tensor R, the complex structure J, and the inner product $\langle\ ,\ \rangle$ arising from the Kaehlerian metric of M. We have $\langle Jx, Jy\rangle = \langle x, y\rangle$ for any two vectors x and y. In addition to the usual properties of the curvature tensor of a Riemannian manifold, R possesses the following properties:

$$R(x, y)J = JR(x, y) \ , \tag{1}$$

$$R(Jx, Jy) = R(x, y) \ . \tag{2}$$

A subspace S of the tangent space is holomorphic if $J(S) = S$. S is said to be *totally real* if it satisfies the following condition:

$$\langle Jx, y\rangle = 0 \qquad \text{for all } x, y \in S \ . \tag{$*$}$$

If P is a 2-dimensional subspace, with an orthonormal basis $\{x, y\}$, of the tangent space, then the sectional curvature $k(P)$ is given by $\langle R(x, y)y, x\rangle$. If P is holomorphic, then the holomorphic sectional curvature $k(P)$ is equal to $\langle R(x, Jx)Jx, x\rangle$, where x is an arbitrary unit vector in P. It is well known (for example, see [1, p. 167]) that $k(P)$ is equal to a constant c for all holomorphic planes P if and only if R is of the form

$$R_c(x, y) = \tfrac{1}{4}c(x \wedge y + Jx \wedge Jy + 2\langle x, Jy\rangle J) \ , \tag{3}$$

where, in general, $x \wedge y$ denotes the endomorphism which maps z into $\langle y, z\rangle x - \langle x, z\rangle y$.

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We now prove

Theorem 1. *The curvature tensor R at a point of a Kaehlerian manifold has constant holomorphic sectional curvature if and only if it has the following property*:

(A) *If* $\langle y, x\rangle = \langle y, Jx\rangle = 0$, *then* $\langle R(x, Jx)Jx, y\rangle = 0$.

Proof. The property is easily verified for the curvature tensor of the form (3). Before we prove the converse, we observe that Property (A) implies that if $\langle y, x\rangle = \langle y, Jx\rangle = 0$ (consequently, $\langle x, Jy\rangle = 0$), then the following terms vanish:

$$(4)\quad \begin{gathered}\langle R(x, Jy)Jx, x\rangle\,,\quad \langle R(x, Jy)Jy, y\rangle\,,\quad \langle R(y, Jx)Jx, x\rangle\,,\quad \langle R(y, Jx)Jy, y\rangle\,,\\ \langle R(y, Jy)Jx, y\rangle\,;\quad \langle R(y, Jy)Jy, x\rangle\,,\quad \langle R(x, Jx)Jy, x\rangle\,.\end{gathered}$$

For example,

$$\langle R(x, Jy)Jx, x\rangle = \langle R(Jx, x)x, Jy\rangle = \langle R(x, Jx)Jx, y\rangle = 0\,,$$

$$\langle R(y, Jx)Jy, y\rangle = 0 \text{ by simply interchanging } x \text{ and } y\,,$$

and so on.

Now let x and y be unit vectors such that $\langle y, x\rangle = \langle y, Jx\rangle = 0$. Setting

$$u = x\cos\theta + y\sin\theta\,,\qquad v = -x\sin\theta + y\cos\theta\,,$$

we find $\langle v, u\rangle = \langle v, Ju\rangle = 0$. Applying Property (A) to the pair (u, v), we have $\langle R(u, Ju)Ju, v\rangle = 0$. Expanding $\langle R(u, Ju)Ju, v\rangle$ we get 16 terms such as

$$\begin{gathered}-\sin\theta\cos^3\theta\langle R(x, Jx)Jx, x\rangle\,,\quad \cos^4\theta\langle R(x, Jx)Jx, y\rangle\,,\\ -\cos^2\theta\sin^2\theta\langle R(x, Jx)Jy, x\rangle\,,\cdots,\ \sin^3\theta\cos\theta\langle R(y, Jy)Jy, y\rangle\,.\end{gathered}$$

Since $\langle R(x, Jx)Jx, y\rangle$ and the 7 terms in (4) vanish, and since

$$\langle R(x, Jy)Jx, y\rangle = \langle R(y, Jx)Jy, x\rangle\,,\quad \langle R(x, Jx)Jy, y\rangle = \langle R(y, Jy)Jx, x\rangle\,,$$

the surviving terms in the expansion of $\langle R(u, Ju)Ju, v\rangle$ give rise to (for θ such that $\cos\theta \neq 0$, $\sin\theta \neq 0$)

$$(5)\quad \begin{gathered}-\cos^2\theta\langle R(x, Jx)Jx, x\rangle + \sin^2\theta\langle R(y, Jy)Jy, y\rangle\\ + (\cos^2\theta - \sin^2\theta)(2\langle R(x, Jy)Jy, x\rangle + \langle R(x, Jx)Jy, y\rangle) = 0\,.\end{gathered}$$

Choosing $\theta = \pi/4$, we obtain

$$(6)\qquad \langle R(x, Jx)Jx, x\rangle = \langle R(y, Jy)Jy, y\rangle\,.$$

Substituting (6) in (5) yields

$$(7) \qquad 2\langle R(x, Jy)Jy, x\rangle + \langle R(x, Jx)Jy, y\rangle = \langle R(x, Jx)Jx, x\rangle .$$

We are now in a position to prove that R has constant holomorphic sectional curvature under Property (A). First, the case where the complex dimension of M is at least 3 can be easily disposed of. Let x_1 and y_1 be any two unit vectors. Then there exists a unit vector z_1 such that

$$\langle z_1, x_1\rangle = \langle z_1, Jx_1\rangle = \langle z_1, y_1\rangle = \langle z_1, Jy_1\rangle = 0 .$$

By virtue of (6) we obtain

$$\langle R(x_1, Jx_1)Jx_1, x_1\rangle = \langle R(z_1, Jz_1)Jz_1, z_1\rangle$$

as well as

$$\langle R(y_1, Jy_1)Jy_1, y_1\rangle = \langle R(z_1, Jz_1)Jz_1, z_1\rangle .$$

Thus the holomorphic sectional curvature of the plane spanned by x_1 and Jx_1 is equal to that of the plane spanned by y_1 and Jy_1. Hence the holomorphic sectional curvature for R is constant.

Now assume that the complex dimension of M is equal to 2. We have an orthonormal basis of the form $\{x, Jx, y, Jy\}$, for which (6) and (7) are valid. Set

$$(8) \qquad c = \langle R(x, Jx)Jx, x\rangle = \langle R(y, Jy)Jy, y\rangle .$$

From

$$R(x, Jx)Jy + R(Jx, Jy)x + R(Jy, x)Jx = 0$$

we obtain

$$\begin{aligned}\langle R(x, Jx)Jy, y\rangle &= -\langle R(Jx, Jy)x, y\rangle - \langle R(Jy, x)Jx, y\rangle \\ &= \langle R(x, y)y, x\rangle + \langle R(x, Jy)Jx, y\rangle \\ &= \langle R(x, y)y, x\rangle + \langle R(x, Jy)Jy, x\rangle ,\end{aligned}$$

where we have used (1) and (2). This last identity and (7) imply

$$(9) \qquad 3\langle R(x, Jy)Jy, x\rangle + \langle R(x, y)y, x\rangle = c .$$

Since we may replace y in (9) by Jy, we get

$$(10) \qquad \langle R(x, Jy)Jy, x\rangle + 3\langle R(x, y)y, x\rangle = c .$$

From (9) and (10) we find

(11) $$\langle R(x, y)y, x\rangle = \langle R(x, Jy)Jy, x\rangle = c/4 ,$$

and thus

(12) $$\langle R(x, Jx)Jy, y\rangle = \langle R(y, Jy)Jx, x\rangle = c/2 .$$

Replacing x by Jx in (11) gives

(13) $$\langle R(Jx, y)y, Jx\rangle = \langle R(Jx, Jy)Jy, Jx\rangle = c/4 .$$

The curvature tensor R_c in (3) obviously satisfies the identities (8), (11), (12) and (13). Also, $\langle R_c(x, Jx)Jx, y\rangle$ and the terms in (4) for R_c are 0. It follows that

(14) $$\langle R(x_1, x_2)x_3, x_4\rangle = \langle R_c(x_1, x_2)x_3, x_4\rangle$$

if the vectors x_1, x_2, x_3 and x_4 are taken from the basis $\{x, Jx, y, Jy\}$. Thus (14) is valid for arbitrary vectors. Hence $R = R_c$.

Remark. Property (A) can be compared with E. Cartan's condition (see the lemma in [2]) for constancy of the sectional curvature of the curvature tensor of a Riemannian manifold.

2. Criteria for constancy of the holomorphic sectional curvature

Let M be a Kaehlerian manifold of dimension $2n$. If M has constant holomorphic sectional curvature, then for every $2k$-dimensional holomorphic subspace S of the tangent space $T_p(M), p \in M$, there exists a totally geodesic complex submanifold V containing p such that $T_p(V) = S$ (for example, see [1, pp. 277, 285]. On the other hand, suppose S is a k-dimensional totally real subspace of $T_p(M)$, where $k \leq n$ as is easily seen. Then there exists a k-dimensional totally geodesic submanifold V containing p such that $T_p(V) = S$. Indeed, for every point q of V, $T_q(V)$ is a totally real subspace of $T_q(M)$.

This assertion on the existence of totally real submanifolds which are totally geodesic can be proved most easily by the following observation. A Kaehlerian manifold of constant holomorphic sectional curvature c is locally either C^n (for $c = 0$) or CP^n with Fubini-Study metric (for $c > 0$) or the unit disk D^n in C^n with Bergman metric (for $c < 0$). For C^n, the submanifolds in question are simply R^k naturally imbedded in C^n as well as its images by holomorphic motions of C^n. For CP^n, they are the real projective space RP^k naturally imbedded in CP^n or its images by the holomorphic isometries of CP^n. Finally, for D^n, the submanifolds in question are the real disc: $\{(x^1, \cdots, x^k) \in R^k;\ (x^1)^2 + \cdots + (x^k)^2 < 1\}$ which is naturally imbedded in D^n or its images by the holomorphic transformations of D^n.

We are now concerned with the converse of these existence theorems. We formulate:

(B) **Axiom of holomorphic $2k$-planes.** *For any $2k$-dimensional holomorphic subspace S of $T_p(M)$, $p \in M$, there exists a $2k$-dimensional totally geodesic submanifold V of M containing p such that $T_p(V) = S$.*

(C) **Axiom of totally real k-planes.** *For any k-dimensional totally real subspace S of $T_p(M)$, $p \in M$, there exists a k-dimensional totally geodesic submanifold V of M containing p such that $T_p(V) = S$.*

We shall prove

Theorem 2. *If a Kaehleriam manifold M of dimension $2n$ satisfies the axiom of holomorphic $2k$-planes for some k, $1 \leq k \leq n-1$, then M has constant holomorphic sectional curvature.*

Theorem 3. *If a Kaehlerian manifold M of dimension $2n$ satisfies the axiom of totally real k-planes for some k, $2 \leq k \leq n$, then M has constant holomorphic sectional curvature.*

Proof of Theorem 2. Let $p \in M$, and let x, y be two vectors in $T_p(M)$ such that $\langle y, x\rangle = \langle y, Jx\rangle = 0$. We can find a holomorphic $2k$-plane S in $T_p(M)$ such that $x, Jx \in S$ and y is perpendicular to S. Since a totally geodesic submanifold V with $T_p(V) = S$ exists, $R(x, Jx)Jx \in S$ and hence $\langle R(x, Jx)Jx, y\rangle = 0$. By Theorem 1, M has constant holomorphic sectional curvature.

Proof of Theorem 3. Let $p \in M$, and let x, y be as above. We can find a k-dimensional totally real subspace S of $T_p(M)$ such that $Jx, y \in S$ and x is perpendicular to S. (For this, consider $T_p(M)$ as $C^n = R^{2n}$ and take a basis $e_1 = Jx, e_2 = y, e_3, \cdots, e_n$ of C^n as a vector space over C. Then let S be the real span of $\{e_1, e_2, \cdots, e_k\}$.) By the existence of a totally geodesic submanifold V with $T_p(V) = S$, we see that $R(Jx, y)Jx \in S$ so that $\langle R(Jx, y)Jx, x\rangle = 0$. But then $\langle R(x, Jx)Jx, y\rangle = 0$. Theorem 1 again applies.

Remark. The special case $k = 1$ of Theorem 2 has been known (see [3, p. 241]). Theorem 3 can be considered as a complex analogue of a result proved in [2] in a certain sense.

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