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HYPERSURFACES OF $S^{2n+1}(k)$ IN E^{2n+2}

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Introduction

The study of hypersurfaces of an odd-dimensional sphere regarded as a Sasakian manifold was started by the present author [3] and then done by Yamaguchi [5]. In these papers, we considered the case where the induced structure f commutes with the linear transformation H defined by the second fundamental tensor. Recently Blair, Ludden and Yano [1] jointly studied the case where f and H anti-commute with each other, and obtained some results under the condition that the scalar curvature of the hypersurface is constant.

In this paper the author considers the same case as that of Blair, Ludden and Yano but under different conditions.

1. A structure on S^{2n+1} in E^{2n+2}

Let E^{2n+2} be a $(2n+2)$-dimensional Euclidean space, and denote by $\bar{X}$ the position vector representing a point of E^{2n+2}. Since E^{2n+2} is even-dimensional, E^{2n+2} can be regarded as a flat Kaehlerian manifold with respect to a natural Kaehlerian structure, and hence in E^{2n+2} there exists a tensor field $\bar{J}$ of type (1.1) with constant components such that

$$\bar{J}^2 = -1 , \tag{1.1}$$

$$(\bar{J}\bar{X})\cdot(\bar{J}\bar{Y}) = \bar{X}\cdot\bar{Y} \tag{1.2}$$

for any vectors $\bar{X}$ and $\bar{Y}$, where 1 denotes the identity transformation of the tangent space of E^{2n+2}, and a dot the inner product in the Euclidean space E^{2n+2}. Since the structure is Kaehlerian, $\bar{J}$ satisfies

$$\bar{\nabla}_X\bar{J} = 0 , \tag{1.3}$$

where $\bar{\nabla}_{\bar{X}}$ denotes the covariant differentiation with respect to the inner product of E^{2n+2}.

Let $S^{2n+1}(k)$ be a sphere of radius k defined by

$$\bar{X}\cdot\bar{X} = k^2 , \qquad (k > 0) , \tag{1.4}$$

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and X', Y' be tangent vectors to $S^{2n+1}(k)$. Then

$$N' \equiv \bar{X}/k \tag{1.5}$$

is a unit vector normal to $S^{2n+1}(k)$. Since the vectors X', N' are linearly independent, their transforms $\bar{J}X', \bar{J}N'$ by $\bar{J}$ can be expressed respectively as

$$\bar{J}X' = FX' + \eta(X')N' , \tag{1.6}$$

$$\bar{J}N' = -\xi . \tag{1.7}$$

Then F, η, ξ define respectively a skew symmetric linear transformation on $TS^{2n+1}(k)$, a 1-form and a vector field. As is well known they satisfy the following properties:

$$F^2X' = -X' + \eta(X')\xi , \tag{1.8}$$

$$\eta(FX') = 0 , \qquad F\xi = 0 , \tag{1.9}$$

$$\eta(\xi) = 1 , \tag{1.10}$$

$$G(X', \xi) = \eta(X') , \tag{1.11}$$

$$G(FX', FY') = G(X', Y') - \eta(X')\eta(Y') , \tag{1.12}$$

where G denotes the induced Riemannian metric of $S^{2n+1}(k)$.

Differentiating (1.6) and (1.7) covariantly with respect to the induced metric and comparing the tangential parts and normal parts, we have easily

$$(\nabla'_{Y'}F)(X') = \eta(X')H'Y' - h'(X', Y')\xi , \tag{1.13}$$

$$\nabla'_{Y'}\xi = FH'Y' , \tag{1.14}$$

where h' and H' denote the second fundamental tensor of $S^{2n+1}(k)$ and the linear transformation defined by h' respectively.

Since $S^{2n+1}(k)$ is a totally umbilical hypersurface of E^{2n+2} and the second fundamental tensor of $S^{2n+1}(k)$ of E^{2n+2} has the form

$$h'(X', Y') = G(X', Y')/k , \tag{1.15}$$

or

$$H'X' = X'/k , \tag{1.15'}$$

we can rewrite (1.13) and (1.14) as follows:

$$k(\nabla'_{Y'}F)(X') = \eta(X')Y' - G(X', Y')\xi , \tag{1.16}$$

$$k\nabla'_{Y'}\xi = FY' . \tag{1.17}$$

2. Hypersurfaces of $S^{2n+1}(k)$

Let M be a hypersurface of $S^{2n+1}(k)$, $X, Y, Z, \cdots$ tangent vectors to M, and N the unit normal vector to M. Since $S^{2n+1}(k)$ has a linear transformation F, we consider the transforms FX and FN of X and N by F. Put

$$FBX = BfX + u(X)N , \tag{2.1}$$

$$FN = -BU , \tag{2.2}$$

where B denotes the differential of the immersion $i\colon M \to S^{2n+1}(k)$. Then f, u, U respectively define a (1.1)-tensor, a 1-form and a vector field on M.

On the other hand, ξ can be decomposed into its tangential parts and normal parts to M, and so we put

$$\xi = BV + \lambda N . \tag{2.3}$$

Moreover, by defining a 1-form v on M by

$$v(X) = \eta(BX) , \tag{2.4}$$

we obtain

$$v(X) = G(\xi, BX) = G(BV, BX) = g(V, X) , \tag{2.5}$$

where g denotes the induced Riemannian metric from G.

Since F is skew symmetric, we have also

$$u(X) = G(FBX, N) = -G(BX, FN) = G(BX, BU) = g(X, U) . \tag{2.6}$$

Applying the operator F to (2.1) and taking account of (2.1), (2.2), (2.3), (2.5) and (1.8), we obtain

$$F^2BX = FBfX + u(X)FN = Bf^2X + u(fX)N - u(X)BU ,$$

from which follows

$$-BX + v(X)BV + \lambda v(X)N = Bf^2X - u(X)BU + u(fX)N .$$

Thus

$$f^2X = -X + u(X)U + v(X)V , \tag{2.7}$$

$$u(fX) = \lambda v(X) . \tag{2.8}$$

Similarly, application of the operator F to (2.2) gives

$$fU = -\lambda V , \tag{2.9}$$

$$u(U) = g(U, U) = 1 - \lambda^2 . \tag{2.10}$$

On the other hand, (1.9) and (1.11) imply that

$$fV = \lambda U , \tag{2.11}$$

$$u(V) = g(U, V) = 0 , \tag{2.12}$$

$$v(V) = g(V, V) = 1 - \lambda^2 . \tag{2.13}$$

Since F is skew-symmetric, we can find that f is also skew-symmetric, that is,

$$g(fX, Y) = -g(X, fY) . \tag{2.14}$$

Denote by ∇_X the operator of covariant differentiation with respect to the Riemannian connection. Then the equations of Gauss and Weingarten are given respectively by

$$'\nabla_{BX}BY = B\nabla_X Y + h(X, Y)N , \tag{2.16}$$

$$'\nabla_X N = -BHX , \tag{2.17}$$

where $h(X, Y)$ is the second fundamental tensor of M in $S^{2n+1}(k)$, and H is the linear transformation induced from $h(X, Y)$ in such a way that

$$g(HX, Y) = h(X, Y) . \tag{2.18}$$

It is easily seen that H is a symmetric transformation. The eigenvalues $h_1, \cdots, h_{2n}$ of H are called the principal curvatures of M. The i-th mean curvature H_i, $i = 1, \cdots, 2n$, are defined in terms of the elementary symmetric functions as follows:

$$\binom{2n}{i} H_i = \sum h_1 \cdots h_i , \qquad i = 1, \cdots, 2n , \tag{2.19}$$

where $\binom{2n}{i} = (2n)!/[i!\,(2n-i)!]$. If the first mean curvature vanishes everywhere, then the hypersurface is said to be *minimal*.

The structure equations of the hypersurface M of a sphere $S^{2n+1}(k)$ of radius k are given by

$$\begin{aligned} R(X, Y)Z = {} & [g(Y, Z)X - g(X, Z)Y]/k^2 \\ & + h(Y, Z)HX - h(X, Z)HY , \end{aligned} \tag{2.20}$$

$$(\nabla_X H)Y - (\nabla_Y H)X = 0 , \tag{2.21}$$

where $R(X, Y)Z$ denotes the curvature tensor of M.

Now applying the operator $'\nabla_{BY}$ of covariant differentiation to (2.1) we find

$$('\nabla_{BY}F)BX + FB\nabla_Y X + h(Y,X)FN$$
$$= B\nabla_Y(fX) + h(Y,fX)N + \nabla_Y(u(X))N - u(X)HY ,$$

from which taking account of (1.13) we obtain

(2.22) $(\nabla_Y f)X = [v(X)Y - g(X,Y)V]/k + u(X)HY - h(X,Y)U$,

(2.23) $(\nabla_Y u)(X) = g(X,fHY) - \lambda g(X,Y)/k$.

Similarly, application of $'\nabla_{BY}$ to (2.2) yields

(2.23)′ $\nabla_Y U = fHY - \lambda Y/k$,

(2.24) $h(Y,U) = u(HY)$.

Applying $'\nabla_{BY}$ to (2.3) and taking account of (1.14), we have

(2.25) $\nabla_Y V = \lambda HY + fY/k$,

(2.26) $\nabla_Y \lambda = h(Y,V) + u(Y)/k$.

3. The length of the second fundamental tensor

Suppose that in the hypersurface M the linear transformation f and H are anti-commutative with each other, that is,

(3.1) $fH + Hf = 0$,

or equivalently,

(3.1)′ $g(HX,fY) = g(fX,HY)$.

First of all we prove

Theorem 3.1. *Let M be a hypersurface of $S^{2n+1}(k)$ satisfying condition (3.1), and the function λ be almost everywhere nonzero. Then M is a minimal hypersurface.*

Proof. From (3.1) it follows that

$$fHf + Hf^2 = 0 ,$$

so that

$$\mathrm{tr}\,(Hf^2) = -\mathrm{tr}\,(fHf) = -\mathrm{tr}\,(Hf^2) ,$$

or

$$\operatorname{tr}(Hf^2) = 0 .$$

Thus for an orthonormal frame E_i,

$$\begin{aligned}\operatorname{tr}(Hf^2) &= \sum_i g(Hf^2E_i, E_i) = \sum_i g(f^2E_i, HE_i)\\ &= \sum_i g(-E_i + g(U, E_i)U + g(V, E_i)V, HE_i)\\ &= -\operatorname{tr} H + g(U, g(HU, E_i)E_i) + g(V, g(HV, E_i)E_i) ,\end{aligned}$$

and hence

$$\operatorname{tr} H = h(U, U) + h(V, V) . \tag{3.2}$$

On the other hand, from (3.1) we get

$$g((Hf + fH)U, V) = -\lambda g(HV, V) - \lambda g(HU, U) ,$$

and therefore

$$\lambda h(V, V) = -\lambda h(U, U) , \tag{3.3}$$

which, together with (3.2), gives

$$\operatorname{tr} H = 0 . \tag{3.4}$$

This completes the proof.

Lemma 3.2. *Under the conditions of Theorem* 3.1, *we have*

$$HU = \alpha U + \beta V , \tag{3.5}$$

$$HV = \beta U - \alpha V , \tag{3.6}$$

where α, β are suitable functions.

Proof. From (3.1) we get

$$fHfX + H(-X + u(X)U + v(X)V) = 0 ,$$

that is

$$g(fHfX, Y) - h(X, Y) + u(X)h(U, Y) + v(X)h(V, Y) = 0 ,$$

and therefore

$$g(fHfY, X) - h(Y, X) + u(Y)h(U, X) + v(Y)h(V, X) = 0 .$$

From the above two equations it follows that

$$u(X)h(Y, U) - u(Y)h(X, U) + v(X)h(Y, V) - v(Y)h(X, V) = 0 . \tag{3.7}$$

Substituting X for U in (3.7) we have

$$(1-\lambda^2)h(Y,U)=h(U,U)u(Y)+h(U,V)v(Y)\ . \tag{3.8}$$

Similarly, substitution of X for V in (3.7) gives

$$(1-\lambda^2)h(Y,V)=h(U,V)u(Y)+h(V,V)v(Y)\ . \tag{3.9}$$

Taking account of (3.3) at any point where $\lambda^2 \neq 1$, we thus obtain (3.5) and (3.6), which also hold at a point where $\lambda^2=1$ since U and V vanish there. q.e.d.

From this lemma we have immediately

$$H^2U=(\alpha^2+\beta^2)U\ , \tag{3.10}$$

$$H^2V=(\alpha^2+\beta^2)V\ . \tag{3.11}$$

Lemma 3.3. *Let M be a hypersurface of $S^{2n+1}(k)$, $n>1$, satisfying* (3.1), *and the function $\lambda(\lambda^2-1)$ be nonzero almost everywhere. Then*

$$\alpha=0\ , \tag{3.12}$$

$$\operatorname{tr} H^2=2\beta^2-2(n-1)\beta/k\ . \tag{3.13}$$

Proof. Differentiating covariantly

$$fHX=-HfX\ ,$$

we have

$$(\nabla_Y f)HX+f(\nabla_Y H)X=-(\nabla_Y H)fX-H(\nabla_Y f)X\ .$$

Substitution of (2.22) into the above equation gives

$$\begin{aligned}[g(HV,X)Y-g(HX,Y)V]/k+g(HU,X)HY-g(H^2X,Y)U+f(\nabla_Y H)X\\ =-(\nabla_Y H)fX-[g(X,V)HY-g(X,Y)HV]/k\\ -g(X,U)H^2Y-g(HX,Y)HU\ .\end{aligned}$$

Substituting (3.5), (3.6) into the last equation we get

$$\begin{aligned}[\beta u(X)Y-\alpha v(X)Y-g(HX,Y)V]/k+\alpha g(U,X)HY\\ +\beta g(V,X)HY-g(H^2X,Y)U+f(\nabla_Y H)X\\ =-(\nabla_Y H)fX-[v(X)HY-\beta g(X,Y)U+\alpha g(X,Y)V]/k\\ -g(U,X)H^2Y-\alpha h(X,Y)U-\beta g(HX,Y)V\ .\end{aligned}$$

Substituting X, Y for E_i in the last equation and summing for i, we find

$$[2\alpha^2 + 2\beta^2 - 2(n-1)\beta/k - \operatorname{tr}(H^2)]U + 2(n-1)\alpha V/k = 0\ ,$$

because of (2.21), (3.4) and the fact that f is a skew-symmetric linear transformation. Since U and V are linearly independent, (3.12) and (3.13) follow. q.e.d.

By means of Lemmas 3.2 and 3.3, we have immediately

$$HU = \beta V\ , \tag{3.14}$$

$$HV = \beta U\ . \tag{3.15}$$

4. The eigenvalues of the second fundamental tensor

First of all, by differentiating (3.15) covariantly we get

$$(\nabla_X H)V + \lambda H^2 X + HfX/k = (\nabla_X \beta)U + \beta(fHX - \lambda X/k)\ ,$$

or

$$(\nabla_X h)(Y, V) + \lambda g(H^2X, Y) + g(HfX, Y)/k$$
$$= (\nabla_X\beta)u(Y) + \beta g(fHX, Y) - \lambda\beta g(X, Y)/k\ ,$$

which, together with (2.21) and (3.1), implies that

$$(\nabla_X\beta)u(Y) = (\nabla_Y\beta)u(X)\ .$$

Substituting Y for U in the above equation, we get

$$\nabla_X\beta = \rho u(X)\ , \tag{4.1}$$

where ρ is a function.

Next, covariant differentiation of (3.14) yields

$$(\nabla_X H)U + HfHX - \lambda HX/k = (\nabla_X\beta)V + \beta\lambda HX + \beta fX/k\ ,$$

or

$$(\nabla_X h)(Y, U) + g(HfHX, Y) - \lambda g(HX, Y)/k$$
$$= (\nabla_X\beta)g(Y, V) + \beta\lambda g(HX, Y) + \beta g(fX, Y)/k\ ,$$

which implies that

$$2g(HfHX, Y) = \rho(u(X)v(Y) - u(Y)v(X)) + 2\beta g(fX, Y)/k\ . \tag{4.2}$$

Lemma 4.1. *Let W be an eigenvector of the second fundamental tensor H perpendicular to the plane spanned by U and V. Then the eigenvalue corresponding to W is $\sqrt{-\beta/k}$ or $-\sqrt{-\beta/k}$. Consequently, β is nonpositive since $\sqrt{-\beta/k}$ is an eigenvalue of a symmetric linear transformation.*

Proof. First we prove that if W is an eigenvector of H corresponding to γ, then fW is an eigenvector of H corresponding to $-\gamma$. From (3.1), we have

$$fHW = -HfW ,$$

so that

$$HfW = -fHW = -\gamma fW .$$

Next, substituting X for W in (4.2) and making use of the above equation and get

$$\gamma^2 fW = -\beta fW/k ,$$

from which it follows that

$$\gamma = \sqrt{-\beta/k} \quad \text{or} \quad \gamma = -\sqrt{-\beta/k} .$$

On the other hand, it is easily seen from (3.14) and (3.15) that the vectors $U + V$ and $U - V$ are eigenvectors of H corresponding to β and $-\beta$ respectively. q.e.d.

Now let r be the multiplicities of the eigenvalue β. Then from Theorem 3.1 and Lemma 4.1 it follows that

$$\text{(4.3)} \qquad \operatorname{tr} H^2 = 2r\beta^2 - 2(n-r)\beta/k ,$$

which, together with (3.13), implies that

$$\text{(4.4)} \qquad (r-1)\beta\left(\beta + \frac{1}{k}\right) = 0 .$$

When $\beta = 0$, the hypersurface is totally geodesic. When $\beta = -1/k$, (4.3) is reduced to

$$\text{(4.5)} \qquad \operatorname{tr} H^2 = 2n/k^2 ,$$

and thus the hypersurface is $S^n \times S^n$ because of Chern-do Carmo-Kobayashi's result [2]. Hence we have

Theorem 4.2. *Let M be a hypersurface of $S^{2n+1}(k)$ in E^{2n+2} such that condition* (3.1) *is satisfied and $\lambda(\lambda^2 - 1)$ does not vanish almost everywhere. If the multiplicities of the eigenvalue β of H, which corresponds to the eigenvector $U + V$, is not* 1, *then M is a great sphere S^{2n} or $S^n \times S^n$.*

We consider now the last mean curvature H_{2n} of the hypersurface, which by definition is of the form

$$\text{(4.6)} \qquad H_{2n} = (-1)^{2n-1}\beta^{n+1}/k^{n-1} .$$

If $|H_{2n}| \leq k^{-2n}$, then $|\beta| \leq 1/k$, and consequently

$$\operatorname{tr} H^2 \leq 2n/k^2 . \tag{4.7}$$

If M is compact, by the result of Simons [4], $\operatorname{tr} H^2$ can take only two values, namely, 0 and $2n/k^2$. By combining this with Chern-do Carmo-Kobayashi's result, we hence reach

Theorem 4.3. *Let M be a compact hypersurface of $S^{2n+1}(k)$ such that two linear transformations f and H are anti-commutative and the function $\lambda(1-\lambda^2)$ does not vanish almost everywhere. If the last mean curvature H_{2n} of M satisfies the inequality*

$$|H_{2n}| \leq k^{-2n} ,$$

then M is a great sphere or $S^n \times S^n$.

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