J. DIFFERENTIAL GEOMETRY
7 (1972) 563–570

THE CONTACT OF SPACES WITH CONNECTION

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É. Cartan [1] investigated the contact of a space with projective connection with the corresponding projective space. His definition of contact of order r is based on the developments of individual curves by means of a connection, and can be easily extended to the case of two arbitrary "generalized" spaces with connection of the same type. In this paper, we first show the existence of another natural point of view from which the problem of contact may be studied based on the developments by means of successive prolongations of a connection according to Ehresmann [3]. Since the second condition is stronger, we speak of strong and weak contacts of generalized spaces with connection. The comparison of these two points of view leads us to the definition of γ-equivalence of semi-holonomic jets. To treat this problem we introduce an invariant symmetrization of some special semi-holonomic jets. Further, we remark that one can also distinguish between strong and weak deformations of order r for generalized spaces with connection. Finally, we pose a natural generalization of the original problem of É. Cartan by studying the contact of a space with Cartan connection with the corresponding homogeneous space. We treat both strong and weak contacts. Our Propositions 8 and 9 give generalizations of the results of É Cartan [1, pp. 189, 193]. We hope that these results together with the corresponding methods illustrate clearly the fact that prolongations of a connection of first order can be applied to the solution of some natural problems in the general theory of spaces with connection. Further results in this direction can be found in [5] and [6].

We intend to carry out our investigations in a direct geometric form. That is why we introduce a connection of first order on the groupoid associated with a principal fibre bundle and not on a principle fibre bundle itself. Standard terminology and notation of the theory of jets are used throughout the paper; see, e.g., [9]. In addition, j_r^s means the canonical projection of r-jets onto s-jets, $s < r$. Our considerations are carried out in the category C^∞.

1. Preliminaries

Let $P(B, G, \pi)$ be a principal fibre bundle, and PP^{-1} the groupoid associated with P. An element of a connection of first order on PP^{-1} at $x \in B$ can be in-

Communicated by K. Yano May 10, 1971.

troduced as a 1-jet at x of a local mapping of B into PP^{-1} of the form $\rho(y)u^{-1}$, where ρ is a local cross section of P, and $\rho(x) = u$. Denote by $Q^1(PP^{-1})$ the fibre bundle over B of all elements of the connection of first order on PP^{-1}. A connection (of first order) on PP^{-1} is a cross-section $C: B \to Q^1(PP^{-1})$; cf. [3].

Consider a fibre bundle $E(B, F, G, P)$ associated with P. Let C_1, C_2 be two connections on PP^{-1}, and $\mathfrak{S}_1, \mathfrak{S}_2$ be two local cross sections of E over a neighborhood of x. Roughly speaking, we shall say that the pairs $(C_1, \mathfrak{S}_1)$ and $(C_2, \mathfrak{S}_2)$ have contact of order r at x, if the development of $\mathfrak{S}_1$ into E_x by means of C_1 coincides up to order r with that of $\mathfrak{S}_2$ into E_x by means of C_2. Since there are two natural points of view here, starting from the ideas of É. Cartan we can say that $(C_1, \mathfrak{S}_1)$ and $(C_2, \mathfrak{S}_2)$ have contact of order r with respect to curves or weak contact of order r, if for every curve γ on B with $\gamma(0) = x$ we have

$$j_0^r\gamma_1 = j_0^r\gamma_2 \ , \tag{1}$$

where γ_i denotes the development of the curve $\mathfrak{S}_i \circ \gamma$ into E_x by means of the connection C_i, $i = 1, 2$. On the other hand, according to Ehresmann [3] the $(r - 1)$-th prolongation $C_i^{(r-1)}$ of C_i is a semi-holonomic connection of order r on PP^{-1}. (For connections of higher order, see [7].) The prolongation of the partial composition law $(\theta, z) \mapsto \theta \cdot z$, $\theta \in PP^{-1}$, $z \in E$, determines the develop ment of $\mathfrak{S}_i$ into E_x by means of the semi-holonomic element $C_i^{(r-1)}(x)$ of connection of order r, which is a semi-holonomic r-jet $C_i^{(r-1)-1}(x)(\mathfrak{S}_i)$ of B into E_x, [5]. (Note that Ehresmann [3] has used the term "the absolute differential of $\mathfrak{S}_i$ with respect to $C_i^{(r-1)}(x)$" for $C_i^{(r-1)-1}(x)(\mathfrak{S}_i)$.) This point of view suggests

Definition 1. If

$$C_1^{(r-1)-1}(x)(\mathfrak{S}_1) = C_2^{(r-1)-1}(x)(\mathfrak{S}_2) \ , \tag{2}$$

then the pairs $(C_1, \mathfrak{S}_1)$ and $(C_2, \mathfrak{S}_2)$ are said to have the strong contact of order r at x.

Using the idea of prolongations of a connection, we can restate (1) in the following form, which will be more convenient for our later investigations. We recall that $T_1^r(B)$ denotes the fibre bundle of all 1^r-velocities on B, i.e., of all r-jets of $\boldsymbol{R}$ into B with source 0.

Definition 2. The pairs $(C_1, \mathfrak{S}_1)$ and $(C_2, \mathfrak{S}_2)$ are said to have weak contact of order r at x, if

$$C_1^{(r-1)-1}(x)(\mathfrak{S}_1 X) = C_2^{(r-1)-1}(x)(\mathfrak{S}_2 X) \tag{3}$$

for every $X \in T_1^r(B)$, $\beta X = x$.

Obviously, (2) implies (3) justifying our terminology,

Definition 3. Let V and W be two manifolds, and let $Y_1, Y_2 \in \bar{J}^r(V, W)$, $\alpha Y_1 = \alpha Y_2 = v$. The jets Y_1, Y_2 are said to be equivalent with respect to curves or γ-equivalent, if $Y_1 X = Y_2 X$ for every $X \in T_1^r(B)$ with $\beta X = v$.

Thus we can say that the pairs $(C_1, \mathfrak{S}_1)$ and $(C_2, \mathfrak{S}_2)$ have weak contact of order r at x if and only if the semi-holonomic jets $C_1^{(r-1)-1}(x)(\mathfrak{S}_1)$ and $C_2^{(r-1)-1}(x)(\mathfrak{S}_2)$ are γ-equivalent. It is easy to see that two holonomic jets are γ-equivalent only if they coincide. In particular, if the connections C_1 and C_2 are integrable, then there is no difference between strong and weak contacts. However, in general, we have another situation for arbitrary semi-holonomic jets.

2. γ-equivalence of some semi-holonomic jets

Let V and W be two manifolds, and put

$$\bar{J}^{r,r-1}(V, W) = \{X \in \bar{J}^r(V, W)\,;\ j_r^{r-1}X \in J^{r-1}(V, W)\}\ . \tag{4}$$

Then we have the following exact diagram of vector bundles over $V \times W$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & J^r(V, W) \arrow[dl] \arrow[dd] & T(W) \otimes S^r T^*(V) \arrow[l] \arrow[dd] & 0 \arrow[l] \\
0 & J^{r-1}(V, W) \arrow[l] & & & \\
 & & \bar{J}^{r,r-1}(V, W) \arrow[ul] & T(W) \otimes \otimes^r T^*(V) \arrow[l] & 0 \arrow[l]
\end{tikzcd} \tag{5}$$

The tensor symmetrization gives a splitting $s_0\colon T(W) \otimes \otimes^r T^*(V) \to T(W) \otimes S^r T^*(V)$, and s_0 determines canonically a splitting $s\colon \bar{J}^{r,r-1}(V, W) \to J^r(V, W)$. Indeed, in any exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & B_1 \arrow[dl, "\varphi"'] \arrow[dd, "i"] & B_2 \arrow[l] \arrow[dd] & 0 \arrow[l] \\
0 & A \arrow[l] & & & \\
 & & C_1 \arrow[ul, "\psi"] & C_2 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

a splitting $s_2\colon C_2 \to B_2$ determines a splitting $s_1\colon C_1 \to B_1$ as follows. For any $x \in C_1$, take an element $y \in B_1$ satisfying $\varphi(y) = \psi(x)$. Then $x - i(y) \in C_2$, and we define $s_1(x) = y + s_2(x - i(y))$.

Definition 4. For every $X \in \bar{J}^{r,r-1}(V, W)$, the holonomic r-jet $s(X)$ is called the symmetrization of X.

Proposition 1. *Every $A \in \bar{J}^{r,r-1}(V, W)$ is γ-equivalent to its symmetrization $s(A)$.*

Proof. The assertion is proved by means of the corresponding expressions in some local coordinates. So we may suppose directly that A is a jet of $\boldsymbol{R}^n$ into $\boldsymbol{R}^m$ with source, target 0 and some coordinates

$$a_i^\alpha, \cdots, a_{i_1\cdots i_{r-1}}^\alpha, a_{i_1\cdots i_r}^\alpha, \qquad \alpha = 1, \cdots, m, \quad \text{and} \quad i, i_1, \cdots = 1, \cdots, n,$$

where $a_{i_1i_2}^\alpha, \cdots, a_{i_1\cdots i_{r-1}}^\alpha$ are symmetric in all subscripts. Then the coordinates of $s(A)$ are $a_i^\alpha, \cdots, \alpha_{i_1\cdots i_{r-1}}^\alpha, a_{(i_1\cdots i_r)}^\alpha$, where the parenthese denote symmetrization. Let $x_1^i, \cdots, x_r^i$ be the coordinates of a 1^r-velocity X on $\boldsymbol{R}^n$ at 0, and let $y_1^\alpha, \cdots, y_r^\alpha$ be the coordinates of AX. Then it is easy to see that the only expression containing $a_{i_1\cdots i_r}^\alpha$ in the formula for y_r^α is $a_{i_1\cdots i_r}^\alpha x_1^{i_1} \cdots x_1^{i_r}$. Hence $AX = s(A)X$ follows from $a_{i_1\cdots i_r}^\alpha x_1^{i_1} \cdots x_1^{i_r} = a_{(i_1\cdots i_r)}^\alpha x_1^{i_1} \cdots x_1^{i_r}$.

Proposition 1 gives a sufficient condition for a semi-holonomic r-jet to be γ-equivalent to a holonomic r-jet. In general, this condition is not necessary. Since $\bar{J}^{2,1}(V, W) = \bar{J}^2(V, W)$, every semi-holonomic 2-jet is γ-equivalent to a holonomic 2-jet. For $r = 3, 4$ we have the following two propositions.

Proposition 2. *A semi-holonomic* 3-*jet* X *is* γ-*equivalent to a holonomic* 3-*jet if and only if* j_3^2X *is holonomic.*

Proposition 3. *A jet* $A \in \bar{L}_{m,n}^4$ *with coodinates* $a_i^\alpha, a_{ij}^\alpha, a_{ijk}^\alpha, a_{ijkl}^\alpha$ *is* γ-*equivalent to a holonomic* 4-*jet if and only if*

$$a_{ij}^\alpha = a_{ji}^\alpha, \qquad a_{(ij)k}^\alpha = a_{k(ij)}^\alpha. \tag{6}$$

Proof of Propositions 2 *and* 3. We shall start with proving Proposition 3 by direct calculation, since we shall deduce Proposition 2 as an auxiliary result. Suppose that A is γ-equivalent to a holonomic 4-jet B with coordinates $b_i^\alpha, b_{ij}^\alpha, b_{ijk}^\alpha, b_{ijkl}^\alpha$ symmetric in all subscripts. Then for every 1^4-velocity $X = (x^i, y^i, z^i, t^i)$ on $\boldsymbol{R}^n$ at 0 we have $AX = BX$, and therefore from the definition of the composition of semi-holonomic jets it follows easily that the underlying 3-jets j_4^3A and j_4^3B are γ-equivalent if and only if

$$\begin{aligned} a_i^\alpha x^i &= b_i^\alpha x^i, \\ a_{ij}^\alpha x^i x^j + a_i^\alpha y^i &= b_{ij}^\alpha x^i x^j + b_i^\alpha y^i, \\ a_{ijk}^\alpha x^i x^j x^k + 2a_{ij}^\alpha y^i x^j &+ a_{ij}^\alpha x^i y^j + a_i^\alpha z^i \\ &= b_{ijk}^\alpha x^i x^j x^k + 3b_{ij}^\alpha y^i x^j + b_i^\alpha z^i. \end{aligned} \tag{7}$$

We find directly that (7) is equivalent to $a_i^\alpha = b_i^\alpha$, $a_{ij}^\alpha = b_{ij}^\alpha$, $a_{(ijk)}^\alpha = b_{ijk}^\alpha$, thus proving Proposition 2. Assume that (7) is satisfied. Then A is γ-equivalent to B if and only if $a_{ijkl}^\alpha x^i x^j x^k x^l + a_{ijk}^\alpha y^i x^j x^k - a_{ijk}^\alpha x^i x^j y^k = b_{ijkl}^\alpha x^i x^j x^k x^l$, which proves Proposition 3.

3. Strong and weak deformations of spaces with connection

An exact definition of a space with Cartan connection was first given by Ehresmann [2]. Some further problems in differential geometry suggest to us the following more general

Definition 5. Let $P(B, G)$ be a principal fibre bundle, $E(B, F, G, P)$ a fibre bundle associated with P, C a connection (of first order) on PP^{-1}, and $\mathfrak{S}$ a cross section of E. Then the quadruple $\mathscr{S} = \mathscr{S}(P(B, G), F, C, \mathfrak{S})$ is called a generalized space with connection.

If the structure group acts transitively on the standard fibre, then our definition is equivalent to that of a space with connection given by Švec [8].

An element $z \in E$ determined by a pair (u, s), $u \in P$, $s \in F$, is denoted by $z = \{(u, s)\}$. If $P(B, G)$ and $\bar{P}(\bar{B}, G)$ are two principle fibre bundles with the same structure group, then the associated bundles $E(B, F, G, P)$ and $\bar{E}(\bar{B}, F, G, \bar{P})$ is said to be of the same type. A diffeomorphism $f: E_x \to \bar{E}_{\bar{x}}$ is called an isomorphism, if there exist some elements $u \in P_x$ and $\bar{u} \in \bar{P}_{\bar{x}}$ such that $f = \bar{u}u^{-1}$, where u and $\bar{u}$ are considered as a mapping $u: F \to E_x$, $s \mapsto \{(u, s)\}$ and a mapping $\bar{u}: F \to \bar{E}_{\bar{x}}$, $s \mapsto \{(\bar{u}, s)\}$ respectively. Further, let $(\psi, \psi_0): P(B, G) \to \bar{P}(\bar{B}, G)$ be an isomorphism of principle fibre bundles (we shall also say that ψ is an isomorphism over ψ_0). Then ψ induces a mapping $\hat{\psi}: E \to \bar{E}$ given by $\hat{\psi}(\{u, s)\}) = \{(\psi(u), s)\}$, $u \in P$, $s \in F$. Moreover, if $\bar{C}$ is a connection on $\bar{P}\bar{P}^{-1}$, then $\psi^*\bar{C}$ will denote the induced connection of PP^{-1}.

We should remark that there are two natural definitions of the deformation of order r of the generalized spaces with connection. If G acts on F transitively, then the weak deformation coincides with the deformation introduced by Švec [8].

Definition 6. Let $\mathscr{S}(P(B, G, \pi), F, C, \mathfrak{S})$ and $\bar{\mathscr{S}}(\bar{P}, (\bar{B}, G, \bar{\pi}), F, \bar{C}, \bar{\mathfrak{S}})$ be two generalized spaces with connection of the same type. A diffeomorphism $\varphi_0: B \to \bar{B}$ is called a strong (or weak) deformation of order r, if there exists a principal fibre bundle isomorphism $\varphi: P \to \bar{P}$ over φ_0 such that $\hat{\varphi}\mathfrak{S} = \bar{\mathfrak{S}}$ and that the pairs $(C, \mathfrak{S})$ and $(\varphi^*\bar{C}, \mathfrak{S})$ have strong (or weak) contact of order r at every $x \in B$.

4. The problem of contact

Analogously, we can introduce

Definition 7. Let $\mathscr{S}$ and $\bar{\mathscr{S}}$ be two generalized spaces with connection of the same type. $\mathscr{S}$ and $\bar{\mathscr{S}}$ are said to admit strong (or weak) contact of order r at $x \in B$ and $\bar{x} \in \bar{B}$, if there are some neighborhoods U of x and $\bar{U}$ of $\bar{x}$ and a principal fibre bundle isomorphism $(\varphi, \varphi_0): \pi^{-1}(U) \to \bar{\pi}^{-1}(\bar{U})$ such that the pairs $(C, \mathfrak{S})$ and $(\varphi^*\bar{C}, \hat{\varphi}^{-1}\bar{\mathfrak{S}})$ have strong (or weak) contact of order r at x.

We recall the definitions of the semi-holonomic contact elements. Let V and W be two manifolds, and let $\dim V = n$, $X \in \bar{J}^r(V, W)$, $\alpha X = v$, $\beta X = w$. By the contact element $k(X)$ of the first kind determined by X we mean the set XhL_n^r of n^r-velocities $h \in H_v^r(V)$ on W at w. (For comparison, by the contact element $\mathscr{K}(X)$ of the second kind determined by X we mean the set $X\bar{h}\bar{L}_n^r$ of n^r-velocities $\bar{h} \in \bar{H}_v^r(V)$ on W at w.) Every n^r-velocity of $k(X)$ is called a representative of $k(X)$. Further, let M be a manifold. Then two r-jets $X \in \bar{J}^r(M, E_x)$ and $\bar{X} \in \bar{J}^r(M, \bar{E}_{\bar{x}})$ are said to be congruent if there exists an isomorphism $f\colon E_x \to \bar{E}_{\bar{x}}$ such that $fX = \bar{X}$, and two contact elements on E_x and $\bar{E}_{\bar{x}}$ are said to be congruent if some of their representatives are congruent.

Proposition 4. *Two generalized spaces with connection $\mathscr{S}$ and $\bar{\mathscr{S}}$ admit strong contact of order r at x and $\bar{x}$ if and only if the contact elements $k(C^{(r-1)-1}(x)(\mathfrak{S}))$ and $k(\bar{C}^{(r-1)-1}(\bar{x})(\bar{\mathfrak{S}}))$ of first kind are congruent.*

Proof. Our assertion is a simple consequence of the following

Lemma. *Let $\mathscr{S}$ be a generalized space with connection, and $(\varphi, \varphi_0)\colon P(B, G) \to \bar{P}(\bar{B}, G)$ be a principal fibre bundle isomorphism, so that $f(x) = \varphi(u)u^{-1}$, $u \in P_x$, is a well-determined isomorphism of $E_x$onto $\bar{E}_{\bar{x}}$ for every $x \in B$, where $\bar{x} = \varphi_0(x)$, and $\bar{E} = \bar{E}(\bar{B}, F, G, \bar{P})$. Let $\bar{C} = (\varphi^{-1})^*C$ be the induced connection, and let $\bar{\mathfrak{S}} = \hat{\varphi}\mathfrak{S}$, $\Phi(x) = j^r_x\varphi_0$. Then*

$$\bar{C}^{(r-1)-1}(\bar{x})(\bar{\mathfrak{S}}) = f(x)(C^{(r-1)-1}(x)(\mathfrak{S}))\Phi(x)$$

for every r and every $x \in B$.

Proof. If $C(x) = j^1_x(\rho(y)u^{-1})$, where $\rho(y)$ is a local cross section of P, and $\rho(x) = u$, then we can express $\mathfrak{S}(y)$ in the form $\mathfrak{S}(y) = \{(\rho(y), s(y))\}$, and we have $C^{-1}(x)(\mathfrak{S}) = j^1_x u(s(y))$. On the other hand, $\bar{C}(\bar{x}) = j^1_{\bar{x}}(\varphi_0(y) \mapsto \varphi(\rho(y))\varphi(u)^{-1})$, $\bar{\mathfrak{S}}(\varphi_0(y)) = \{\varphi(\rho(y)), s(y))\}$ and $\bar{C}^{-1}(\bar{x})(\bar{\mathfrak{S}}) = j^1_{\bar{x}}(\varphi_0(y) \mapsto \varphi(u)(sy)))$ $= \varphi(u)u^{-1}j^1_x(\varphi_0(y) \mapsto u(s(y)))$, which proves our lemma for $r = 1$. For $r > 1$, our lemma is a simple consequence of the recurrence formula $C^{(r-1)-1}(x)(\mathfrak{S})$ $= C^{-1}(x)(C^{(r-2)-1}(y)(\mathfrak{S}))$ obtained in [5, Corollary 1].

Two contact elements on E_x and $\bar{E}_{\bar{x}}$ are said to be γ-congruent, if there are some of their representatives X and $\bar{X}$ and an isomorphism $f\colon E_x \to \bar{E}_{\bar{x}}$ such that $\bar{X}$ and fX are γ-equivalent. Quite analogously to Proposition 4, we can obtain

Proposition 5. *Two generalized spaces with connection $\mathscr{S}$ and $\bar{\mathscr{S}}$ admit weak contact of order r at x and $\bar{x}$ if and only if the contact elements $k(C^{(r-1)-1}(x)(\mathfrak{S}))$ and $k(\bar{C}^{(r-1)-1}(\bar{x})(\bar{\mathfrak{S}}))$ of first kind are γ-congruent.*

5. The contact of a space with Cartan connection with the corresponding homogeneous space

We have remarked in [6] that a space with Cartan connection can be introduced as a generalized space with connection satisfying the following additional conditions:

a) G acts on F transitively,

b) $\dim B = \dim F$,

c) $C^{-1}(x)(\mathfrak{S})$ is regular for every $x \in B$.

The homogeneous space F can be endowed with a canonical structure of a space with Cartan connection [4]; this space is denoted by $\mathscr{F}$.

Proposition 6. *A space $\mathscr{S}(P(B, G), F, C, \mathfrak{S})$ with Cartan connection admits at $x \in B$ strong contact of second order with $\mathscr{F}$ if and only if the torsion form $\tau(x)$ of $\mathscr{S}$ at x vanishes.*

Proof. Theorem 1 of [5] asserts that $C'^{-1}(x)(\mathfrak{S})$ is holonomic if and only if $\tau(x)$ vanishes. Since G acts on F transitively, $k(C'^{-1}(x)(\mathfrak{S}))$ is congruent with a contact element of second order determined by the identity transformation of F if and only if it is holonomic. Applying Proposition 4. we thus obtain our assertion.

The isotropic group H_x^r of order r of E_x at $\mathfrak{S}(x)$ is defined by

$$H_x^r = \{g \in G_x ; j^r_{\mathfrak{S}(x)} g = j^r_{\mathfrak{S}(x)} \mathrm{id}_{E_x}\} ,$$

where G_x is the group of all isomorphisms of E_x. The curvature form $\Omega(x)$ of C at $x \in B$ can be considered as an element of $\mathfrak{g}_x \otimes \Lambda^2 T_x^*(B)$, [5], and we have introduced in [6] the torsion form $\tau^r(x)$ of order r of $\mathcal{S}$ at x as the projection of $\Omega(x)$ into $(\mathfrak{g}_x / \mathfrak{h}_x^r) \otimes \Lambda^2 T_x^*(B)$, $\tau^0(x) = \tau(x)$ being the usual torsion form of $\mathcal{S}$ (naturally, $\mathfrak{g}_x$ and $\mathfrak{h}_x^r$ denote the Lie algebras of G_x and H_x^r respectively). By Theorem 1 of [6], we obtain immediately a sufficient condition for strong contact of $\mathcal{S}$ with $\mathcal{F}$.

Proposition 7. *If the torsion form of order $r - 1$ of $\mathcal{S}$ vanishes in a neighborhood of $x \in B$, and the torsion form of order r of $\mathcal{S}$ vanishes at x, then $\mathcal{S}$ admits at x strong contact of order $r + 2$ with $\mathcal{F}$.*

Corollary. *A space with Cartan connection without torsion* (*of order* 0) *admits at x strong contact of third order with the corresponding homogeneous space if and only if its torsion form of first order at x vanishes.*

Further, we shall treat weak contact. By Propositions 1, 2 and 5, we deduce immediately

Proposition 8. *A space with Cartan connection admits at every point weak contact of second order with the corresponding homogeneous space. A necessary and sufficient condition for $\mathcal{S}$ to admit at x weak contact of third order with $\mathcal{F}$ is that the torsion form* (*of order* 0) *of $\mathcal{S}$ at x vanish.*

Proposition 9. *Let $\mathcal{S}(P(B, G), F, C, \mathfrak{S})$ be a space with Cartan connection without torsion* (*of order* 0). *Then $\mathcal{S}$ admits at x weak contact of fourth order with $\mathcal{F}$ if and only if the torsion form of first order of $\mathcal{S}$ at x vanishes.*

Proof. If $\tau^1(x) = 0$, then $C''^{-1}(x)(\mathfrak{S}) = j_4^3 C'''^{-1}(x)(\mathfrak{S})$ is holonomic by Theorem 1 of [6]. By Proposition 1, $C'''^{-1}(x)(\mathfrak{S})$ is γ-equivalent to a holonomic 4-jet, and we conclude by Proposition 5 that $\mathcal{S}$ admits at x weak contact of fourth order with $\mathcal{F}$. Conversely, let $C'''^{-1}(x)(\mathfrak{S})$ be γ-equivalent to a holonomic 4-jet, and let a_i^m, a_{ij}^m, a_{ijk}^m, a_{ijkl}^m be its coordinates in some local coordinate systems on B and E_x. Then (6) holds.

Moreover, since $\mathcal{S}$ has no torsion, $C'^{-1}(y)(\mathfrak{S})$ is holonomic for every $y \in B$. Let $C(x) = j_x^1 \rho(y) u^{-1}$. Then, by [5, Corollary 1], $C''^{-1}(x)(\mathfrak{S}) = j_x^1 \Sigma(y)$, where $\Sigma(y) = u\rho^{-1}(y)(C'^{-1}(y)(\mathfrak{S}))$ is a local cross section of $J^2(B, E_x)$, which implies

$$a_{ijk}^m = a_{jik}^m . \tag{8}$$

From (6) and (8) we deduce easily that a^m_{ijk} are symmetric in all subscripts. Hence $C''^{-1}(x)(\mathfrak{S})$ is holonomic and, in our case, this is equivalent to $\tau^1(x) = 0$.

References

[1] É. Cartan, *Leçons sur la théorie des espaces à connexion projective,* Paris, 1937.
[2] C. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable,* Colloque Topologie, Bruxelles, 1950, 29–55.
[3] ——, *Sur les connexions d'ordre supérieur,* Atti Quinto Congr. Un. Mat. Ital. (Pavia-Torino, 1955) Rome, 1956, 344–346.
[4] S. Kobayashi, *Theory of connections,* Ann. Mat. Pura Appl. **43** (1957) 119–194.
[5] I. Kolář, *On the torsion of spaces with connection,* Czechoslovak Math. J. **21** (**96**) (1971) 136-148.
[6] ——, *Higher order torsions of spaces with Cartan connection,* Cahiers Topologie Géom. Différentielle **12** (1971) 137–146.
[7] N. van Quê, *Du prolongement des espaces fibrés et des structures infinitésimales,* Ann. Inst. Fourier (Grenoble) **17** (1967) 157–223.
[8] A. Švec, *Cartan's method of specialization of frames,* Czechoslovak Math. J. **16** (1966) 552–599.
[9] P. Ver Eecke, *Géométrie différentielle, Fasc. I: Cacul des Jets,* Sao Paulo, 1967.

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