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ON THE MATHEMATICAL FOUNDATIONS OF ELECTRICAL CIRCUIT THEORY

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The goal of this note is to derive the differential equations for simple (nonlinear) electrical circuits with resistors, inductors and capacitors. I would like to express my deep indebtedness in these matters to George Oster. Oster supplied me with the basic references to the literature, and conversations on this subject with him were very helpful. The reference Brayton & Moser [4] was also very helpful.

1. A simple electrical circuit provides us first of all with an oriented graph G which will be assumed to be connected but not necessarily planar. This is a one-dimensional cell complex with branches or elements (1-cells) and nodes (0-cells or vertices).

The states of the circuit have two components, the currents through the branches and the voltages across the branches. Let C_j be the vector space of real j-chains of G, C^j the j-cochains of G, $j = 0, 1$. Thus C^j can be thought of as the dual vector space of C_j. As is well-known, the currents in the circuit can be represented as an element i of C_1. Thus $i = \sum_\sigma i_\sigma c_\sigma$ where σ ranges over all the branches, c_σ is the σ-th branch and i_σ is the current through the σ-th branch.

The voltages in the circuit can be represented by an element v of C^1. Thus $v = \sum_\sigma v_\sigma c'_\sigma$ where v_σ is the voltage across the σ-th branch and c'_σ is the cochain which is 1 on c_σ, and 0 on the others.

Let $\mathscr{S} = C_1 \times C^1$. Then $s = (i, v) \in \mathscr{S}$ is a state (unrestricted) of the circuit. Physical laws (Kirchhoff and Generalized Ohm) will constrain the physical states to lie in a submanifold $\sum$ of $\mathscr{S}$ which we proceed to define.

Denote the boundary map by $\partial: C_1 \to C_0$ and coboundary by $\partial^*: C^0 \to C^1$. Thus ∂ is a linear transformation of vector spaces, and ∂^* is its adjoint on the dual spaces.

In this context the Kirchhoff laws KCL, KVL can be expressed as follows (see Branin [2] and the references therein):

$$KCL: i \in \operatorname{Ker} \partial , \qquad KVL: v \in \operatorname{Image} \partial^* .$$

The first condition just expresses the fact that the currents entering a node

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sum to zero while the second asserts the existence of a voltage potential. Note that we do not use "meshes", and also that C_1 has the natural structure of real Cartesian space $\boldsymbol{R}^b$ where b is the number of branches of G; the same holds for C^1.

Let $B_i\colon C_i \times C^i \to \boldsymbol{R}$ be the bilinear pairing of a vector space with its dual, $i = 0, 1$. Then $B_1(i, v) = \sum_{\sigma=1}^{b} i_\sigma v_\sigma$. The Kirchhoff laws yield that any physical state $(i, v) \in C_1 \times C^1$ is restricted to lie in the linear subspace $K = \operatorname{Ker}\partial \times \operatorname{Image}\partial^*$ of $C_1 \times C^1$. The first part of the following well-known proposition is essentially what is called Tellegen's theorem. Cf. Desoer [5], Brayton & Moser [4].

(1.1) Proposition. *B_1 vanishes on K and* $\dim K = \dim C_1$.

Proof. This is standard linear algebra, i.e., if $(i, v) \in K$, then $\partial i = 0$, $v = \partial^* u$ and therefore $B_1(i, v) = B_1(i, \partial^* u) = B_0(\partial i, u) = 0$. Furthermore $\dim K = \dim \operatorname{Ker}\partial + \dim \operatorname{Image}\partial^* = (\dim C_1 - \dim C_0 + 1) + (\dim C_0 - 1) = \dim C_1$.

(1.2) Corollary. *The 1-form $\sum v_\sigma di_\sigma$ on $C_1 \times C^1$ vanishes when restricted to K.*

Since each i_σ is a linear function on $\mathscr{S}$, this corollary is just a restatement of (1.1).

The next step is to introduce the branch elements which a simple electrical circuit gives us. The branches of G in our framework can be classified into exactly three categories, resistance branches, inductor branches and capacitor branches (batteries are included in our definition of resistor).

We let $\mathscr{R}$ denote the real vector space of currents through the resistance branches, so that if $i \in \mathscr{R}$, then $i = \sum i_\rho c_\rho$, summation over the resistance branches. Similarly we define $\mathscr{L}$ as the space of currents through the inductor branches, and $\mathscr{C}$ for the currents through the capacitor branches. Analogously we then have $\mathscr{R}', \mathscr{L}', \mathscr{C}'$, the dual spaces for the voltages across the respective branches.

We suppose that the ρ-th resistance is given by its characteristic, a closed 1-dimensional manifold $\Lambda_\rho \subset \boldsymbol{R} \times \boldsymbol{R} = \{(i_\rho, v_\rho)\}$ for each resistance branch ρ. Let $\Lambda \subset \mathscr{R} \times \mathscr{R}'$ be the product of the Λ_ρ so that $\Lambda = \{(i, v) \in \mathscr{R} \times \mathscr{R}' \mid (i_\rho, v_\rho) \in \Lambda_\rho\}$. Then Λ is a closed submanifold of $\mathscr{R} \times \mathscr{R}'$ with $\dim \Lambda = \dim \mathscr{R}$.

Let $i_R \times v_R\colon \mathscr{S} \to \mathscr{R} \times \mathscr{R}'$ be the natural projection, and π' its restriction to K. (In the future, we will frequently use the symbols $i_L\colon \mathscr{S} \to \mathscr{L}$ etc. for the various projections.)

(1.3) Hypothesis. π' is transverse regular to Λ in the sense of Thom; see, e.g., [1].

This means that if $x \in K$, $y = \pi'(x) \in \Lambda$, then the composition $T_x(K) \xrightarrow{D\pi'(x)} T_y(\mathscr{R} \times \mathscr{R}') \longrightarrow T_y(\mathscr{R} \times \mathscr{R}')/T_y(\Lambda)$ is surjective.

Note that (1.3) is a generic property (i.e., (1.3) will be true for almost all choices of the Λ_ρ), and we will always assume it to be satisfied.

Assuming (1.3) then, let $\sum = \pi'^{-1}(\Lambda)$.

(1.4) Proposition. *$\sum$ is a submanifold of K with* $\dim \sum = \dim (\mathscr{L} \times \mathscr{C}')$.

Proof. That $\sum$ is a submanifold is a standard fact of differential topology (the implicit function theorem) as well as the fact that $\operatorname{codim}_K \sum = \operatorname{codim}_{\mathscr{R}\times\mathscr{R}} \Lambda$; see [1]. Then we have from (1.1) that $\dim \sum = \dim C_1 - \dim \mathscr{R} = \dim (\mathscr{L} \times \mathscr{C}')$ since $\operatorname{codim}_K \sum = \dim \mathscr{R}$. We call $\sum$ the *manifold of states* (physical) of the network.

Next we wish to define a symmetric bilinear form I over $\sum$ which will come from the inductance and capacitance. Thus for each $x \in \sum$, I_x will be a symmetric bilinear form on the tangent space $T_x(\sum)$. For this we first define a form J over $\mathscr{L} \times \mathscr{C}'$ by $J_{(i,v)} = -\sum L_\lambda(i_\lambda)di_\lambda^2 + \sum C_\gamma(v_\gamma)dv_\gamma^2$. Here the first sum is over the inductor branches and the second over the capacitor branches. L_λ is a smooth positive function of $i_\lambda \in \boldsymbol{R}$, the *inductance* in the λ-th branch and given in advance. Similarly, C_γ is the capacitance in the γ-th branch and is also a smooth positive function on $\boldsymbol{R}$ given in advance.

The above J on $\mathscr{L} \times \mathscr{C}'$ is a smooth nondegenerate symmetric form, i.e., an indefinite metric. Let $\pi\colon \sum \to \mathscr{L} \times \mathscr{C}'$ be the restriction of the natural projection $i_L \times v_C\colon \mathscr{S} \to \mathscr{L} \times \mathscr{C}'$, and let $I = \pi^*J$ so that I is a symmetric form defined over $\sum$ from J via π. If at $x \in \sum$, the derivative $D\pi(x)\colon T_x(\sum) \to T_{\pi(x)}(\mathscr{L} \times \mathscr{C}')$ is an isomorphism then (and only then) the form I_x on $T_x(\sum)$ will be nondegenerate. Recall from (1.4) that $\dim T_x(\sum) = \dim T_{\pi(x)}(\mathscr{L} \times \mathscr{C}')$.

The next step in our development is to define a certain closed 1-form w on $\sum$, and using this define the equations of motions for the states.

To this end, let $h'\colon \mathscr{S} \to \boldsymbol{R}$ be the composition $\mathscr{S} \xrightarrow{i_C \times v_C} \mathscr{C} \times \mathscr{C}' \xrightarrow{\text{duality pairing}} \boldsymbol{R}$, and h be the restriction of h' to $\sum$. Thus $h'(i, v) = \sum_\gamma i_\gamma v_\gamma$ summation over the capacitance branches.

Next let η_1 be the 1-form on $\mathscr{R} \times \mathscr{R}'$ defined by $\eta_1 = \sum v_\rho di_\rho$, and use the symbol η for the 1-form on $\sum$ induced by the map $\pi''\colon \sum \to \mathscr{R} \times \mathscr{R}'$ previously defined.

(1.5) Proposition. *η is a closed 1-form on $\sum$.*

Proof. It is sufficient to show that $d\eta_1$ vanishes when restricted to Λ since π'' factors through Λ. But $d(v_\rho di_\rho) = 0$ for each ρ; it is a 2-form on a one-dimensional manifold Λ_ρ.

Define $w = \eta + dh$ on $\sum$. Thus w is a closed 1-form on $\sum$.

(1.6) Main theorem. *The equations of motion for the network are $I(dx/dt, Y) = w(Y)$ for all $Y \in T(\sum)$.*

This is an equation for a curve of states in time, $t \to x(t)$, which must be satisfied. So $x\colon [a, b] \to \sum$, and at $x(t) \in \sum$ we have $I_{x(t)}(x'(t), Y) = w_{x(t)}(Y)$ for all $Y \in T_{x(t)}(\sum)$.

(1.7) Remark. Let U be the open subset of $\sum$ where $D\pi(x)\colon T_x(\sum) \to T_{\pi(x)}(\mathscr{L} \times \mathscr{C}')$ is an isomorphism. Then on U, I defines an isomorphism between vector fields and 1-forms. Thus on U, there is a vector field X which corresponds to w under this correspondence. This X when integrated gives the passage of a state in time on U. That is, X on U can be thought of as the ordinary differential equation (nonsingular) for the circuit. However, for the full picture, one must keep all of $\sum$.

Of course if $H^1(\sum) = 0$ or even if w is cohomologous to zero, we can write $w = dP$ for some smooth function $P: \sum \to R$ and the right hand side of our equation simplifies accordingly. For the usual electric circuits $H^1(\Lambda_\rho) = 0$ for each ρ, so $H^1(\Lambda) = 0$ and $w = dP$; the equation becomes $I(dx/dt, Y) = dP(Y)$ for all Y.

Our starting point for the proof of (1.6) is (1.2) which we can write

$$\sum_\rho v_\rho di_\rho + \sum_\lambda v_\lambda di_\lambda + \sum_\gamma v_\gamma di_\gamma = 0$$

as a 1-form on $\sum \subset K$. Here according to our usual convention the first sum is over resistance branches, the second over inductor branches and the third over capacitor branches. Each of these sums can be interpreted as 1-forms on $\sum$ as induced by maps:

$$\begin{tikzcd} & & & \mathscr{R} \times \mathscr{R}' \\ \sum \arrow[r] & K \arrow[r] & \mathscr{S} \arrow[ur] \arrow[r] \arrow[dr] & \mathscr{L} \times \mathscr{L}' \\ & & & \mathscr{C} \times \mathscr{C}' \end{tikzcd}$$

By the Leibniz rule we have

$$d(\sum i_\gamma v_\gamma) = \sum v_\gamma di_\gamma + \sum i_\gamma dv_\gamma \ .$$

Putting this into the first equation, we obtain

$$-\sum v_\lambda di_\lambda + \sum i_\gamma dv_\gamma = \sum v_\rho di_\rho + d \sum i_\gamma v_\gamma = w \ .$$

We also have the following basic relations of circuit theory (Faraday, etc.):

$$v_\lambda = L_\lambda(i_\lambda) di_\lambda/dt \ , \qquad i_\gamma = C_\gamma(v_\gamma) dv_\gamma/dt \ .$$

Making another substitution we obtain

$$-\sum L_\lambda(i_\lambda)(di_\lambda/dt)di_\lambda + \sum C_\gamma(v_\gamma)(dv_\gamma/dt)dv_\gamma = w \ ,$$

which is just another way of writing the equation of (1.6).

The final part of this section is devoted to showing how energy and power fit naturally into our framework. Compare, e.g., Valkenburg [12, p. 23]. First we define a real valued function W on $\sum$, the total energy stored in all the inductive and capacitive elements, or energy for short.

Let $W_L\colon \mathscr{L} \to \boldsymbol{R}$ be the function $W_L(i) = \int_{0,\Gamma}^{i} \sum_{\lambda} L_\lambda(i_\lambda) i_\lambda di_\lambda$ where Γ is any path from 0 to i in $\mathscr{L}$. This function is well-defined since clearly the integral is independent of Γ.

Similarly define $W_C\colon \mathscr{C}' \to \boldsymbol{R}$ by $W_C(v) = \int_{0,\Gamma}^{v} \sum_{\gamma} C_\gamma(v_\gamma) v_\gamma dv_\gamma$. Then let $W\colon \sum \to \boldsymbol{R}$, $W = W_L \circ i_L + W_C \circ v_C$ be the energy of the network where $i_L\colon \sum \to \mathscr{L}$ is the restriction of the projection $i_L\colon \mathscr{S} \to \mathscr{L}$, etc.

The power, or the power in all the resistance elements, can be represented by another function $P_R\colon \sum \to \boldsymbol{R}$ as the composition

$$\sum \longrightarrow \mathscr{S} \xrightarrow{i_R \times v_R} \mathscr{R} \times \mathscr{R}' \to \boldsymbol{R} .$$

The last map is just the pairing of a vector space with its dual.

(1.8) Theorem. *On the part of* $\sum$ *where* X *is defined via* (1.7), $X \cdot W = -P_R$.

Here $X \cdot W$ is differentiation of the function W with respect to X or in other words with respect to the natural action of X on W.

The theorem expresses mathematically that the energy decreases along the trajectories according to the power dissipated in the resistors. For example, if there are no resistors one has $X \cdot W = 0$ or conservation of energy (but these are not Hamiltonian systems, however).

Proof of (1.8). By Tellegen's theorem, we can write $P_R + P_L + P_C = 0$ on K or on $\sum$ where P_L, P_C are defined via $\mathscr{L} \times \mathscr{L}'$, $\mathscr{C} \times \mathscr{C}'$ respectively, similarly to P_R. Then it is sufficient to prove that $X \cdot W_L = P_L$, $X \cdot W_C = P_C$ on $\sum$, where W_L, W_C can be thought of as functions on $\sum$. We prove the first, the second being similar.

For this we project a solution curve $\phi_t(x)$ of X into $\mathscr{L} \times \mathscr{L}'$ via the restriction p of $i_L \times v_L\colon \mathscr{S} \to \mathscr{L} \times \mathscr{L}'$ to $\sum$. It is sufficient to check

$$\frac{d}{dt} W_L(p\phi_t(x))\Big|_{t=0} = P_L(p\phi_t(x))\Big|_{t=0} = P_L(px) .$$

Using the definition of W_L and noting for each λ that

$$L(i_\lambda) i_\lambda di_\lambda = L(i_\lambda) i_\lambda \frac{di}{dt} dt = i_\lambda v_\lambda dt ,$$

this last is easily checked.

2. We give some simple examples here of electrical circuits, largely to show how the framework of the preceding section can act as a unifying force.

Example 1. This is a simple *RLC* series circuit with the resistance characteristic current controlled. That is to say, we have the situation of Fig. 1 with

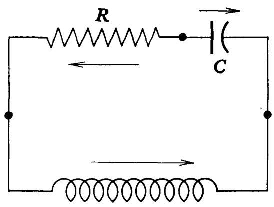

Fig. 1

the orientation as depicted and Λ_ρ is described by $\{(i_\rho, f(i_\rho))\} \subset \mathscr{R} \times \mathscr{R}$ where $f\colon \boldsymbol{R} \to \boldsymbol{R}$ is a real smooth function.

Here our $\sum$ is diffeomorphic to $\mathscr{L} \times \mathscr{C}' = \{(i_\lambda, v_\gamma)\}$ by $\pi\colon \sum \to \mathscr{L} \times \mathscr{C}'$, with π the restriction of $i_L \times v_C\colon \mathscr{S} \to \mathscr{L} \times \mathscr{C}'$. In fact it can be easily seen using the definition of $\sum$ and Kirchhoff's laws that an inverse to π is given by

$$\mathscr{L} \times \mathscr{C}' \to \textstyle\sum \subset \mathscr{R} \times \mathscr{L} \times \mathscr{C} \times \mathscr{R}' \times \mathscr{L}' \times \mathscr{C}' ,$$

$$(i_\lambda, v_\gamma) \to (i_\lambda, i_\lambda, -i_\lambda, f(i_\lambda), -f(i_\lambda) + v_\gamma, v_\gamma) .$$

Thus our basic equations get transferred to $\mathscr{L} \times \mathscr{C}'$. Here the forms I and J are essentially the same and letting $(x, y) = (i_\lambda, v_\gamma)$, where $I = -L(x)dx^2 + C(y)dy^2$. Then $w = \eta + dh = v_\rho di_\rho + d(i_\gamma v_\gamma) = f(x)dx - d(xy)$ and $w = dP$ where $P(x, y) = -xy + \int_0^x f(t)dt$. Of course P is well-defined up to a constant.

Now one deduces for the basic equations (see Remark (3) below):

$$L(x)\frac{dx}{dt} = -\frac{\partial P}{\partial x}(x, y) = y - f(x) , \qquad C(y)\frac{dy}{dt} = \frac{\partial P}{\partial y}(x, y) = -x .$$

Thus in the case $L \equiv 1$ and $C \equiv 1$, the equations are $dx/dt = y - f(x)$, $dy/dt = -x$. This is exactly Lienard's equation as a first order equation (cf. Hartman [6, p. 179], Lefschetz [8, p. 250]).

Remarks. (1) For $f(x) = x^3 - x$, this becomes Van der Pol's equation.

(2) Of course a battery can be included, by incorporating it into the resistance term.

(3) Formally the equations are derived as follows from the main theorem of § 1. Suppose on $\boldsymbol{R}^2 = \{(x_1, x_2)\}$, $I = -L(x_1)dx_1^2 + C(x_2)dx_2^2$, $I(X, Y) = dP(Y)$ for all Y. Let $Y_1 = (1, 0)$, $Y_2 = (0, 1)$. Then $-L(x_1)X_1 = I(X, Y_1) = dP(Y_1) = (\partial P/\partial x_2)(x_1, x_2)$, and similarly $C(x_2)X_2 = I(X, Y_2) = (\partial P/\partial x_2)(x_1, x_2)$ where $X = (X_1, X_2)$.

(4) The power dissipated in the resistor is $-P_R(x, y) = -xf(x)$, and the energy of § 1 is given by $W(x, y) = x^2 + y^2$ assuming $L \equiv C \equiv 1$.

Let us look at the phase portrait of the above system of differential equations in the (x, y)-plane, under the assumptions there exist positive constants c, k with

(a) $\qquad xf(x) > c|x| \quad \text{for} \quad |x| > k \qquad$ (natural assumption),

(b) $\qquad f'(0) < 0 \qquad$ (nonlinearity assumption).

Proposition. *Under the above assumptions (including $L \equiv 1, C \equiv 1$), orbits starting with large energy tend towards a periodic orbit, there is one zero of the vector field, a source, and there is a cross-section to the flow in the strongest sense.*

First it is clear that $x = 0, y = -f(0)$ is the unique zero, and the vector field has $\begin{pmatrix} -f'(0) & -1 \\ 1 & 0 \end{pmatrix}$ as its matrix of first partial derivatives at $(0, -f(0))$. The eigenvalues are $\lambda = \frac{1}{2}(-f'(0) \pm \sqrt{f'(0)^2 - 4})$ and will both have positive real part since $f'(0) < 0$. Thus the singularity is a source.

Using the theorem of § 1 that $X \cdot W = -P_r$, remark (4) of this example and our assumption (a), one can check the following lemma.

Lemma. *There is some disk D in $\boldsymbol{R}^2$ defined by $W(x, y) \leq K$ such that every orbit $0(t)$ has an associated t_0 so that $\{0(t) \mid t \geq t_0\} \subset D$.*

This gives us our proposition except for the cross-section Q which we define by $Q = \{(x, y) \in \boldsymbol{R}^2 \mid x = 0,\ y \geq -f(0)\}$. Then $\partial Q =$ the zero, and the vector field at every other point of Q is perpendicular to Q (i.e., horizontal). Furthermore, fairly direct, and well-known arguments yield that every non-trivial orbit meets Q and after leaving Q returns again to Q. Thus we define $T\colon Q \to Q$ by taking this point of first return to obtain a diffeomorphism which is the identity on ∂Q, which contains all of the qualitative information of the system. Furthermore T expands away from ∞ and ∂Q. This finishes the proof of the proposition and our discussion of Example 1.

Example 2. A simple *RLC* circuit in parallel with voltage controlled characteristic as in Fig. 2 with $i_\rho = f(v_\rho)$.

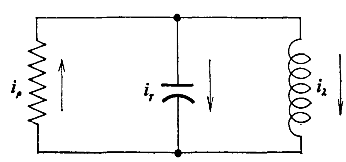

Fig. 2

We have here a diffeomorphism $\mathscr{L} \times \mathscr{C}' \to \sum$ defined by (compare Example 1)

$$(i_\lambda, v_r) \to (f(v_r), i_\lambda, f(v_r) - i_\lambda, -v_r, v_r, v_r) .$$

Then $w = -v_r df(v_r) + d(v_r(f(v_r) - i_\lambda))$, and $w = d((f(y) - x)y) - y\, df(y)$ if $(x, y) = (i_\lambda, v_r)$. So

$$P(x, y) = -xy + \int_0^y f(t)dt\ , \qquad I = -L(x)dx^2 + C(y)dy^2\ ,$$

and the equations are

$$L(x)dx/dt = -\partial P/\partial x = y\ , \qquad C(y)dy/dt = \partial P/\partial y = -x + f(y)\ .$$

These equations are essentially the same as in Example 1. Behind this is a duality, see for example Desoer & Kuh [5].

Example 3. We will see in detail how an example from Brayton [3, p. 14] fits into our framework. This is the circuit in the following Fig. 3.

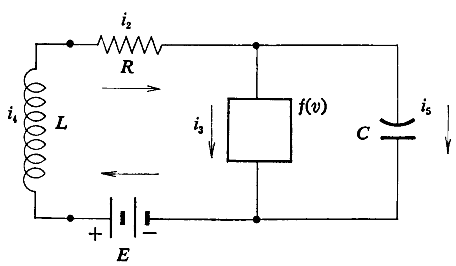

Fig. 3

Here L, the resistance R in branch 2, and C are all linear. E is the constant voltage source and can be thought of as a resistance with characteristic $(i_1, v_1) = (i_1, E)$ and the resistance in branch 3 is voltage controlled with characteristic defined by the function f on $\boldsymbol{R}$.

In this case the map $\pi\colon \sum \to \mathscr{L} \times \mathscr{C}'$ is a diffeomorphism, and in fact an inverse is given by

$$(i_4, v_5) \to (i_4, i_4, f(v_5), i_4, i_4 - f(v_5), -E, Ri_4, v_5, E - Ri_4 - v_5, v_5)\ ,$$
$$\mathscr{L} \times \mathscr{C}' \to \sum \subset (\mathscr{R} \times \mathscr{L} \times \mathscr{C}) \times (\mathscr{R}' \times \mathscr{L}' \times \mathscr{C}')\ ,$$

where we have used the natural $\boldsymbol{R}^3$ structure on $\mathscr{R}$ ond $\mathscr{R}'$. Also we have used Kirchhoff laws which in this case read: $i_1 = i_2 = i_4 = i_3 + i_5$ and $v_3 = v_5$, $v_1 + v_2 + v_3 + v_4 = 0$; it is easy to see that we have exhibited an inverse to π. Let $(x, y) = (i_4, v_5)$, and we get for J or I even, $I = -L\, dx^2 + C\, dy^2$. We obtain the "mixed potential" P as follows: $dP(x, y) = w = dh + \eta = d((x - f(y))\cdot y) - E\, dx + R\, x\, dx + y\, df(y)$,

$$P(x, y) = xy - E\,x + R\frac{x^2}{2} - \int_0^y f(u)\, du\ .$$

The differential equations are

$$L\, dx/dt = -y - Rx + E\ , \qquad C\, dy/dt = x - f(y)\ .$$

Furthermore we have expressions as follows for the work and power:

$$W(x, y) = Lx^2 + Cy^2 , \qquad P_R = Rx^2 - Ex + yf(y) .$$

For the function f considered by Brayton, one has the fact that there is a disk in the (x, y)-plane such that every orbit stays in the disk for large enough time. This is similar to the case of Example 1 and uses the fact that $X \cdot W = -P_R$.

Furthermore, if in addition there is just one zero, one can easily check that this zero is an attractor and that there is a 1-dimensional cross-section.

Note that up to now all the examples satisfy the condition that $\pi \colon \sum \to \mathscr{L} \times \mathscr{C}'$ is a (global) diffeomorphism. Compare this to the Brayton-Moser hypothesis [4] that the currents through the inductors and the voltages across the capacitors determine all currents and voltages in the circuit via Kirchhoff's law. In fact it is a reasonable interpretation that the *B-M* hypothesis means exactly that $\pi \colon \sum \to \mathscr{L} \times \mathscr{C}'$ has a well-defined inverse and hence is a diffeomorphism. In this case, the derivation of Example 1 generalizes to give the equations:

$$L_\lambda(i_\lambda) di_\lambda / dt = -\partial P / \partial i_\lambda , \qquad \text{each } \lambda, \text{ an inductor branch,}$$
$$C_\gamma(v_\gamma) dv_\gamma / dt = -\partial P / \partial v_\gamma , \qquad \text{each } \gamma, \text{ a capacitor branch.}$$

These are the *B-M* equations (up to a sign; Brayton and Moser seem to use an unusual sign convention).

Remark 1. These equations contain as special cases, all examples to this point. If there are no inductors, then one has a gradient dynamical system.

Remark 2. As with Brayton and Moser, these equations admit an easy extension to the case of mutual inductance, capacitance.

Example 4. This circuit has just two elements, a linear capacitor and a nonlinear current controlled resistor as given in Fig. 4.

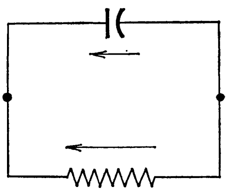

Fig. 4

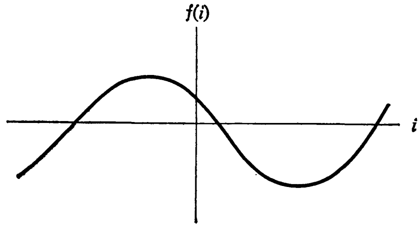

Fig. 5

Here $i, f(i)$ are the current and voltage in the resistor where we assume that f has the qualitative properties indicated in Fig. 5. Let i be a parameterization of $\sum$ so that

$$\sum = \{(i, -i, f(i), f(i)) \in \mathscr{R} \times \mathscr{C} \times \mathscr{R}' \times \mathscr{C}'\} .$$

Then $dP(i) = -d(if(i)) + f(i)di, P(i) = -if(i) + \int_0^i f(j)dj$, and the equation can be read off from the main theorem of § 1 as

$$C(df(i))^2(X, Y) = dP(Y) ,$$

or

$$Cf'(i)^2\, di/dt = dP/di = -if'(i) ,$$

or finally

$$C\, di/dt = -i/f'(i) .$$

Note that this equation does not fit into the framework of the B-M equations, $\pi: \sum \to \mathscr{L} \times \mathscr{C}'$ is not a diffeomorphism and in fact the equation is singular where $f'(i) = 0$. So we have obtained a singular first order, ordinary differential equation on the 1-dimensional manifold $\sum$ or a vector field as in Fig. 6 where the singularities (not the zero) are marked with an x.

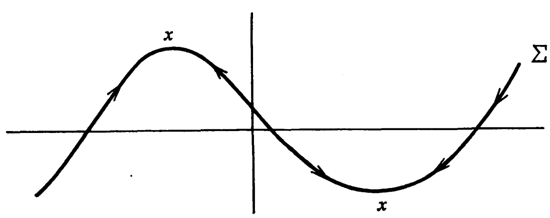

Fig. 6

What happens at the singularity is undetermined by the mathematics, i.e., by the differential equation.

However, one can give a prescription for what happens at these singularities which is consistent with experiment and can be justified by circuit theory. This has to do with the theory of "relaxation oscillations" and proceeds as follows. At a singularity, the state jumps instantaneously to the part of $\sum$ described by the dotted arrows j_i, j_2 in the following Fig. 7.

Thus, at least after a while, the state oscillates, the oscillation including two portions which take no time and two passages along the manifold.

To justify this interpretation of what happens at the jumps, we proceed as follows according to a suggestion of C. Desoer and some ideas in the literature [7], [9]. R. E. Kalman first mentioned to me that this study was connected with relaxation oscillations (in connection with Example 5). We introduce into the circuit of Fig. 4, an inductor in series with inductance L, to obtain the

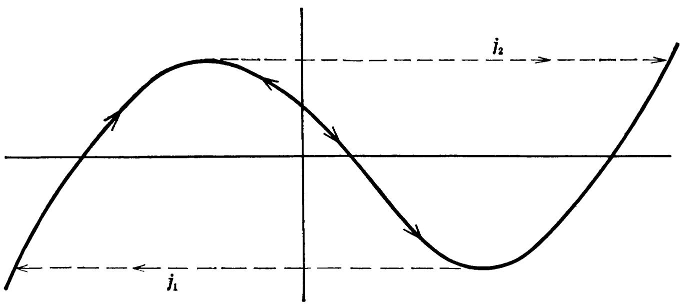

Fig. 7

circuit of Example 1, (see Fig. 1). The differential equations for this example are nonsingular and thus so to speak we have regularized the differential equations of the original circuit. We may in fact assume that L is as small as we want, so that there is a good physical justification in introducing L.

The new differential equations are

$$L\, di/dt = +v_r - f(i)\ , \qquad C\, dv_r/dt = -i\ ,$$

where i is the current through the inductor (or through the resistor, which is the same thing) and v_r is the voltage across the capacitor. Thus if $L = 0$, then $v_r = f(i), dv_r/dt = f'(i)di/dt$ and $dv_r/dt = -i/C$ so we recover the equations of Example 4.

Now to justify our prescription of the jumps one looks at the phase portrait of the above as $L \to 0$.

For this we refer to the literature [7], [9]. The idea is that $\sum$ of Example 4 can be imbedded naturally in the (i, v_r)-plane of Example 1: $i \to (i, f(i) = v_r)$, and one can imbed the cycle of Example 4 including the jumps also (in fact just look at Fig. 7). Then given any neighborhood of the cycle, by taking L small enough, the unique cycle of Example 1 (for the choice of f) falls into this neighborhood, with the time taken along the jump part arbitrarily small. The proof is not difficult.

Example 5. The circuit we discuss here has some features of Example 4 but is more complicated. It and Example 4 both fit naturally into the framework of § 1 and the differential equations there apply. In both of these examples, the map $\pi\colon \sum \to \mathscr{L} \times \mathscr{C}'$ has a singular derivative and the Brayton-Moser framework does not fit. The indefinite metric I is degenerate and this leads to singularities of the differential equation as a vector field on $\sum$ (singularity in the sense that the vector field is not defined there). A regularization is developed in Example 6 for Example 5.

The circuit then is exactly that of Example 1 except that the resistor characteristic is not assumed to be current-controlled and the orientation is different. See Fig. 8. We do assume that the resistor is voltage controlled so that its

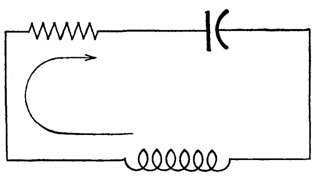

Fig. 8

characteristic is the graph of a real function $f(v_\rho) = i_\rho$. In this case one has a global coordinate system for $\sum$ given by the restriction of the projection $v_R \times v_c \colon \mathscr{S} \to \mathscr{R}' \times \mathscr{C}'$ to $\sum$. An inverse $\mathscr{R}' \times \mathscr{C}' \to \sum \subset \mathscr{S}$ is given by

$$(x, y) = (v_\rho, v_\tau) \to (f(v_\rho), f(v_\rho), f(v_\rho), v_\rho, -v_\rho - v_\tau, v_\tau) .$$

Again this is checked from the definitions and Kirchhoff's laws. Then $w = \eta + dh = v_\rho di_\rho + d(i_\tau v_\tau) = xdf(x) + d(y\, f(x))$ and $w = dP$ where $P(x, y) = (x + y)f(x) - \int_0^x f(t)dt$. In this case I has the form

$$\begin{aligned} I &= -L(i_\lambda)di_\lambda^2 + C(v_\tau)dv_\tau^2 = -L(f(x))(df(x))^2 + C(y)dy^2 \\ &= C(y)dy^2 - L(f(x))(df(x)/dx)^2dx^2 . \end{aligned}$$

Note that this form is degenerate when $(df/dx)(x) = 0$. This fact prevents one from using the B-M (i_λ, v_τ) as coordinates for $\sum$.

Our equations for the circuit now become:

$$L(f(x))(df/dx)^2dx/dt = -(\partial P/\partial x)(x, y) , \qquad C(y)dy/dt = (\partial P/\partial y)(x, y) .$$

Thus if $C \equiv L \equiv 1$, we obtain

$$(df/dx)(x)dx/dt = -(x + y) , \qquad dy/dt = f(x) .$$

One can write down the energy and power as $W = i_\lambda^2 + v_\tau^2 = f(x)^2 + y^2$ and $P_R = i_\rho v_\rho = xf(x)$ respectively.

One can expect that $df(x)/dx = 0$ for certain isolated x if the resistance is sufficiently nonlinear. In this case the equations for the circuit have a 1-dimensional set of singularities.

Remark. The same equations are valid for the dual circuit, that of Example 2 where the resistance is current controlled.

We now proceed to study the phase portrait in the (x, y)-plane of these equations when the characteristic has the qualitative behavior described by Fig. 9.

There will be three zeros of this vector field given by $f(x) = 0$, $x + y = 0$ or $(x_1, -x_1), (x_2, -x_2), (x_3, -x_3)$ where x_i are the three zeros of f. The local behavior of the flow in the neighborhood will in general be determined by the eigenvalues of the matrix of partical derivatives

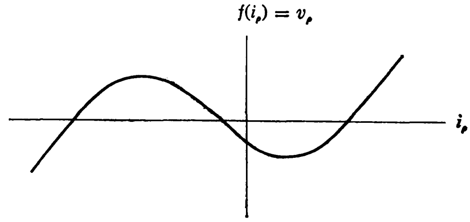

Fig. 9

$$\begin{pmatrix} a & -1/f'(x) \\ f'(x) & 0 \end{pmatrix} \quad \text{with} \quad a = \frac{xf''(x) - f'(x)}{f'(x)^2}$$

for $x =$ some x_i. This is an easy calculation from the basic equations. Then the eigenvalues have the form $\lambda = \frac{1}{2}(a \pm \sqrt{a^2 - 4})$.

Typically under these conditions if $x_1 < x_2 < x_3$, one might expect that $|a| < 2$ and $a(x_1) < 0$, $a(x_2) > 0$, $a(x_3) < 0$, in which case the first and third zeros are sinks while the second is a source. We will make this assumption in what follows.

Next let x', x'' be the values of x where $f'(x) = 0$. The vector field will be undefined then at the two vertical lines through x', x''.

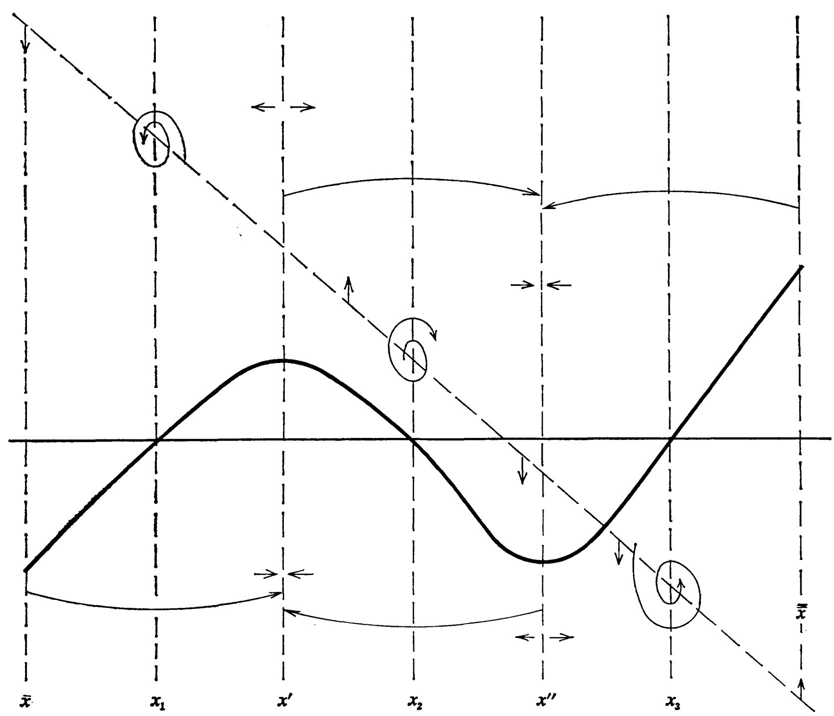

Fig. 10

One knows exactly the qualitative behavior of the vector field along the curves $x + y = 0$, $f(x) = 0$, as well as $f'(x) = 0$. This follows from the form of the differential equations. Putting all of this information together, we can obtain the phase portrait as partially depicted in Fig. 10.

This leaves open the question of what happens as a trajectory runs into a singular line $f'(x) = 0$. We give a prescription for this with a justification in the next section.

Example 6. We follow a suggestion of C. Desoer to regularize Example 5 by adding a (small) capacitance C' as in Fig. 11.

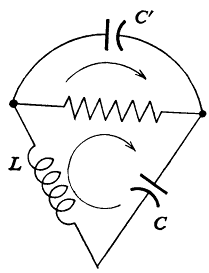

Fig. 11

The goal will be to give a prescription for what happens in Example 5 at the singularities (undefined places of the vector field) and to provide some information on the phase portrait of Example 6. We emphasize that the resistor in Example 6 is the same non-linear one as in Example 5.

Proposition. *As $C' \to 0$ in Example 6, one obtains Example 5 with the following "prescription". When a trajectory hits the line $x = x''$, the state jumps instantaneously to the line $x = \bar{x}$ keeping the same y value. When a trajectory hits the line $x = x'$, the state jumps instantaneously to the line $x = \bar{\bar{x}}$, again keeping the same y value.*

In what follows, clarification will be added to this proposition. The first step is to write down the equation for Example 6. As usual, we find a coordinate chart on $\sum$, this time as a map $\mathscr{L} \times \mathscr{C}' \to \sum \subset \mathscr{S}$ which is an inverse to $\pi\colon \sum \to \mathscr{L} \times \mathscr{C}'$ and given by:

$$(i_\lambda, v_\gamma, v'_\gamma) \to (f(v'_\gamma), i_\lambda, i_\lambda, i_\lambda - f(v'_\gamma), v'_\gamma, -v_\gamma - v'_\gamma, v_\gamma, v'_\gamma) ,$$

where i_λ is of course the current through L, and v_γ, v'_γ is the voltage across the capacitor C, C' respectively. Here we have used the Kirchhoff laws $i_\rho + i'_\gamma = i_\gamma = i_\lambda$, $v_\lambda + v_\gamma + v_\rho = 0$, $v_\rho = v'_\gamma$.

Then one checks easily that $P = i_\lambda(v_\gamma + v'_\gamma) - \displaystyle\int_0^{v'_\gamma} f(v'_\gamma)dv'_\gamma$. The equations are

$$L\, di_\lambda/dt = -(v_\gamma + v'_\gamma) , \quad C\, dv_\gamma/dt = i_\lambda , \quad C'\, dv'_\gamma/dt = i_\lambda - f(v'_\gamma) .$$

If $C' = 0, L = C = 1$, then $i_\lambda = f(v'_\tau)$, $di_\lambda/dt = f'(v'_\tau)dv'_\tau/dt$, $v'_\tau = v_\rho$ and

$$dv_\tau/dt = f(v_\rho) \ , \qquad f'(v_\rho)dv_\rho/dt = -(v_\tau + v_\rho) \ .$$

This checks; these are the equations of Example 5. We let $(x, y, z) = (v'_\tau, v_\tau, i_\lambda)$, and $L = C = 1$. So we have

$$dx/dt = (z - f(x))/C' \ , \quad dy/dt = z \ , \quad dz/dt = -(x + y) \ .$$

Energy and power have the form $W = z^2 + y^2 + C'x^2$, $P_R = xf(x)$.

Now we consider the surface S imbedded in $\boldsymbol{R}^3$, $(x, y, z) = (x, y, f(x))$. This surface we think of as Example 5 imbedded in Example 6, and the question is what happens on S as $C' \to 0$.

Now we leave to the reader to carry out this process in detail to prove the previous proposition. One chases along trajectories for C' very small, and keeps sharp account of how the signs of components of our vector field change.

Now we discuss the more general problem of the phase portrait of Fig. 11, and we do not necessarily assume that C' is small.

My feeling is that one can expect the qualitative behavior of this differential equation in 3-space to be rather complicated. One property is that for the kind of resistor characteristic we have been considering, there will be three zeros of the vector field. The study of the local behavior of the system about these zeros should be rather straightforward.

Furthermore, using (1.8), our expression for W and P_R and the form of these differential equations, one can check that the states eventually contract onto some fixed, compact subset of our 3-dimensional phase space; thus one knows the qualitative behavior of this differential equation at ∞ of $\boldsymbol{R}^3$.

Finally we remark that there is a very well-behaved cross-section Q in this example, defined by

$$Q = \{(x, y, z) \in \boldsymbol{R}^3 \,|\, x + y > 0, z = f(x)\}$$

with associated diffeomorphism $T\colon Q \to Q$. It should not be a problem of too great difficulty to study the behavior of T on the boundary ∂Q of Q. The general problem then reduces to the study of T. I would think that this should be a relevant, interesting and challenging problem, to study the qualitative properties of this transformation T, say for f of the type we have been considering. It might be useful to impose other conditions such as f, say of a generic type. Can one use (1.8) to obtain further information? Is the fact that this system is of gradient type for an indefinite metric of some use?

Example 7. Consider the circuit of Fig. 12. Note that the map $i_L \times v_C\colon \mathscr{S} \to \mathscr{L} \times \mathscr{C}'$ restricted to K has image in a one-dimensional subspace of $\mathscr{L} \times \mathscr{C}'$, $\mathscr{L} \times \mathscr{C}'$ being of dim 2. This is because $\mathscr{C}'$ is zero, and $i_L \times v_C$ amounts to

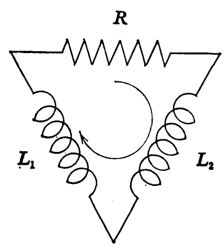

Fig. 12

i_L: Ker $\partial \to \mathscr{L}$. But dim Ker $\partial = 1$. Thus no matter what the characteristic of $\mathscr{R}$ is, or L_1, L_2 are, the differential equations of (1.6) are degenerate everywhere on $\sum$ for this Example 7.

Let's look briefly at the generalization of this *forced degeneracy*.

Consider generally the map $i_L \times v_C \colon \mathscr{S} \to \mathscr{L} \times \mathscr{C}'$ restricted to K. This is a product

$$\operatorname{Ker} \partial \times \operatorname{Im} \partial^* \to \mathscr{L} \times \mathscr{C}'$$

of the restriction of projections $i_L \colon \operatorname{Ker} \partial \to \mathscr{L}$, $v_C \colon \operatorname{Im} \partial^* \to \mathscr{C}'$. Thus the condition for *forced degeneracy* is precisely "either $i_L \colon \operatorname{Ker} \partial \to \mathscr{L}$ or $v_C \colon \operatorname{Im} \partial^* \to \mathscr{C}'$ fails to be surjective". In the case of *forced degeneracy*, the differential equations are everywhere ill-defined on $\sum \subset K$ independent of the characteristics.

There is one sufficient criterion for the above necessary and sufficient condition for forced degeneracy. Namely, if either

$$\dim \mathscr{L} > \dim \operatorname{Ker} \partial \quad \text{or} \quad \dim \mathscr{C}' > \dim \operatorname{Im} \partial^* ,$$

then $i_L \times v_C \to \mathscr{L} \times \mathscr{C}'$ will fail to be surjective. This last criterion can be easily verified by a counting procedure, e.g., as in the original example of Fig. 11. More generally, $\dim \mathscr{L} =$ the number of inductors, $\dim \operatorname{Ker} \partial = \#$ branches $- \#$ nodes $+ 1$, so if $\#$ inductors $> \#$ branches $- \#$ nodes $+ 1$ then one has forced degeneracy. A dual statement can be made for the capacitors, namely, $\#$ capacitors $> \#$ nodes $- 1$ implies forced degeneracy.

3. This section consists essentially of a number of remarks related to the preceding sections.

Theorem (1.6) suggests that it might be worthwhile to consider abstractly dynamical systems of that type. The ordinary gradient systems of a function with respect to a (positive definite) Riemannian metric are well understood (see, e.g., [11 § I.2] or [10]). The equations of (1.6) involve extensions of these in several ways. For example, consider this.

(3.1) Problem. What can one say about the dynamical systems which are gradient systems of a function with respect to a nondegenerate indefinite metric, say on a compact manifold? That is to say, let M be a compact manifold with a nondegenerate symmetric form defined over it (e.g., a Lorentz manifold), and $f: M \to R$ a smooth function. Then grad f is the vector field corresponding under this form to the 1-form df. What special properties do such systems have?

Now (1.6) motivates one to consider a further extension of (3.1). This consists of replacing df by a closed 1-form w on M. We will call such a system grad w where grad w is the vector field corresponding to w under the nondegenerate metric.

Moe Hirsch pointed out to me the facts in this paragraph. We construct examples of systems of this generalized gradient type. Let a compact manifold M have a vector field X which is a suspension as defined in [11, § II.1]. Then one has a canonical map $\pi: M \to S^1$, S^1 the circle. Now take any Riemannian metric on M such that $\|X(x)\| = 1$ for each $x \in M$ and $\pi^{-1}(q)$ is orthogonal to X for each $q \in S^1$, and let w_0 be the canonical 1-form on S^1. It can be checked that $X = \text{grad}\,(\pi^* w_0)$. One also has a converse construction. Suppose $X =$ grad w, $dw = 0$ and X is never zero on a compact manifold M. Consider $[w]$ in $H^1(M, \boldsymbol{R})$ and take an approximation w_1 with $[w_1] \in H^1(M, \boldsymbol{Q})$, $\boldsymbol{Q}$ the rationals. Let p be a positive integer so that $[p, w_1] \in H^1(M, \boldsymbol{Z})$, $\boldsymbol{Z}$ the integers. Then there is a smooth function $f: M \to S^1$ generated by $p\, w_1$ (Bruschlinsky). If the approximation w_1 is good enough, $f^{-1}(q)$ will be transversal to X for each $q \in S^1$. Thus $f^{-1}(q)$ is a cross-section. Is the following true?

(3.2). Suppose $X = \text{grad}\,(w)$ is the gradient of a closed 1-form with respect to a Riemannian metric on a compact manifold M. Suppose further that w is not cohomologous to zero and that X is well-behaved in the sense it satisfies the conditions of (2.2) of [11]. Then X has a closed orbit, not a point, which is asymptotically stable (i.e., a sink).

The conclusion would seem to be relevant to electrical circuit problems since a stable periodic solution is of direct physical interest. However the hypothesis on the behavior of X is strong.

The following would seem to be interesting problems.

(3.3) Problem. Can one always regularize the equations of (1.6) by adding arbitrarily small inductors and capacitors to the circuit appropriately? How? By regularizing we mean obtaining new equations which have the property $\pi: \sum \to \mathscr{L} \times \mathscr{C}'$ is a diffeomorphism (or at least a local diffeomorphism), e.g., as in Example 4 and Examples 5, 6.

(3.4) Problem. Suppose the power P_R of a circuit can be written in the form $P_R = \pi_\rho P_\rho$ where each $P_\rho: \Lambda_\rho \to R$ has the property $P_\rho(i_\rho, v_\rho) \geq C_\rho(i_\rho^2 + v_\rho^2)$ for large (i_ρ, v_ρ) with some positive constant C_ρ. Suppose also that $\pi: \sum \to \mathscr{L} \times \mathscr{C}'$ is a diffeomorphism. Then (under possibly additional conditions) does there exist a compact set $U \subset \sum$ with the property, given $x \in \sum$ is there a t_0 such that the orbit $\phi_t(x) \subset U$ if $t \geq t_0$? In other words is there a compact set of attraction for the system? The idea would be to use (1.8).

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