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AN INDEX THEOREM FOR WIENER-HOPF OPERATORS ON THE DISCRETE QUARTER-PLANE

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1. Introduction

Let T^j be the j-dimensional torus, represented as j-tuples of complex numbers with modulus equal to one. Letting $L^2(T^j)$ be the usual Hilbert space of square-integrable complex functions on T^j with respect to normalized Haar measure, we consider the subspace $H^2(T^j)$ consisting of functions in $L^2(T^j)$ which are boundary values of analytic functions in the j-disc $\{(z_1, \cdots, z_j): |z_k| < 1\}$. It is well-known that $\{z_1^{n_1} \cdots z_j^{n_j}: n_k \geq 0\}$ forms an orthonormal basis for $H^2(T^j)$. We denote by P^j the orthogonal projection from $L^2(T^j)$ onto $H^2(T^j)$. Note that $P_r{}^j \equiv P^j \oplus \cdots \oplus P^j$ (r times) is the orthogonal projection from $L^2(T^j)_r \equiv L^2(T^j) \oplus \cdots \oplus L^2(T^j)$ (r times) onto $H^2(T^j)_r \equiv H^2(T^j) \oplus \cdots \oplus H^2(T^j)$ (r times).

Now for $\phi(z_1, \cdots, z_j)$ a $r \times r$ matrix-valued complex continuous function on T^j, we define a bounded operator on $H^2(T^j)_r$ by

$$W_\phi f = P_r{}^j(\phi f) .$$

These are the Wiener-Hopf operators. We note that the Fourier transform takes $L^2(T^j)_r$ onto $L^2(Z^j)_r$ and $H^2(T^j)_r$ onto $L^2((Z^+)^j)_r$. Hence, the W_ϕ are unitarily equivalent via the Fourier transform to certain matrix convolution operators on the discrete semigroup $(Z^+)^j$.

In this paper, we consider the C^*-algebra $\mathscr{A}_r{}^j$ of operators on $H^2(T^j)_r$ generated by all the W_ϕ. Our main result is a "canonical form" for the case $j = 2$ which gives the index of A whenever A is a Fredholm operator in $\mathscr{A}_r{}^2$.

Our analysis depends upon the fact the structure of $\mathscr{A}_r{}^1$ is rather completely understood [1], [3]. In particular, W_ϕ in $\mathscr{A}_r{}^1$ is Fredholm if and only if determinant $(\phi) \neq 0$ and

$$\text{index}\,(W_\phi) = -\text{winding number (determinant } (\phi)) .$$

The situation in $\mathscr{A}_r{}^2$ is quite different. It was shown in [4], [7] that a W_ϕ in $\mathscr{A}_1{}^2$ is a Fredholm operator if and only if ϕ is non-vanishing and homotopic in $C(T^2, C - 0)$ to the constant 1 (here, $C(X, Y) = Y^X$ denotes the space of continuous functions from X to Y with the appropriate matrix supremum norm

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topology). Hence, the index of Fredholm W_ϕ in $\mathscr{A}_1{}^2$ is always zero. On the other hand, there are Fredholm operators in $\mathscr{A}_1{}^2$, not of the form W_ϕ, which have arbitrary index [4]. The situation in $\mathscr{A}_r{}^2$ for $r \geq 2$ is again distinctive. We shall see that there are Fredholm W_ϕ in $\mathscr{A}_2{}^2$ with arbitrary index.

2. Preliminary results

Henceforth, we restrict our attention to T^j for $j = 1, 2$. Let $\mathscr{K}_r{}^j$ be the algebra of all compact operators on $H^2(T^j)_r$. Further, let G_r be the group of invertibles in $\mathscr{A}_r{}^1$, and let K_r be the group of all elements in G_r which have the form $I + K$ for K in $\mathscr{K}_r{}^1$. Finally, for GL_r the complex $r \times r$ general linear group, let H_r be the subgroup of $GL_r{}^T$ whose elements have determinants with winding number zero.

We recall that in $\mathscr{A}_1{}^j$ [2], $\|W_\phi\| = \|\phi\|_\infty$ and $W_\phi^* = W_{\bar{\phi}}$. It follows that W_z and W_w generate $\mathscr{A}_1{}^2$. We also recall that $\mathscr{A}_r{}^1 = \{W_\phi + K \colon \phi \text{ continuous}, K \in \mathscr{K}_r{}^1\}$ and the map

$$\sigma(W_\phi + K) = \phi$$

gives a $*$-homomorphism from $\mathscr{A}_r{}^1$ onto the algebra $M_r{}^T$ of $r \times r$ matrix-valued continuous functions on T. The kernel of σ is precisely $\mathscr{K}_r{}^1$.

Now to each element of $\mathscr{A}_r{}^2$ by the analysis of [4] there corresponds a pair $(\sigma_1(z), \sigma_2(z))$ in $C(T, \mathscr{A}_r{}^1) \oplus C(T, \mathscr{A}_r{}^1)$ satisfying

$$(*) \qquad \sigma[\sigma_1(z)](w) = \sigma[\sigma_2(w)](z) .$$

Further, it is easy to check that *any* pair $(\sigma_1(z), \sigma_2(z))$ satisfying $(*)$ corresponds to an element of $\mathscr{A}_r{}^2$. Finally, the map constructed in [4] which sends A to $(\sigma_1(A)(z), \sigma_2(A)(z))$ is a $*$-homomorphism from $\mathscr{A}_r{}^2$ onto

$$\sum_r = \{(\sigma_1(z), \sigma_2(z)) \text{ in } C(T, \mathscr{A}_r{}^1) \oplus C(T, \mathscr{A}_r{}^1) \colon (*) \text{ holds}\}$$

with kernel $\mathscr{K}_r{}^2$. We recall that if W_w, W_z are the generators of $\mathscr{A}_1{}^2$, then the map $A \to (\sigma_1(A)(z), \sigma_2(A)(z))$ is determined by

$$\begin{aligned} \sigma_1(W_w)(z) &= S , &\qquad \sigma_2(W_w)(z) &= zI ; \\ \sigma_1(W_w)(z) &= zI , &\qquad \sigma_2(W_z)(z) &\equiv S , \end{aligned}$$

where S is the "unilateral shift" generating $\mathscr{A}_1{}^1$. Note that S is just "W_z in $\mathscr{A}_1{}^1$", an unfortunate notational ambiguity.

By our previous remarks, an element A of $\mathscr{A}_r{}^2$ is a Fredholm operator (i.e., A has closed range and finite-dimensional kernel and cokernel) if and only if $(\sigma_1(A)(z), \sigma_2(A)(z))$ is in $C(T, G_r) \oplus C(T, G_r)$. The main result of this paper expresses the index of Fredholm A in $\mathscr{A}_r{}^2$ in terms of $(\sigma_1(A)(z), \sigma_2(A)(z))$. Recall that

$$\text{index}\,(A) = \text{dimension}\,(\ker A) - \text{dimension}\,(\text{coker}\, A)$$

is a continuous homomorphism from the semigroup of Fredholm operators in $\mathscr{A}_r{}^2$ to Z.

We require some preliminary observations. Notice that

$$\{1\} \longrightarrow K_r \xrightarrow{i} G_r \xrightarrow{\sigma} H_r \longrightarrow \{1\} .$$

is an exact sequence of topological groups and $G_r \xrightarrow{\sigma} H_r$ is a principal fibre bundle with fibre K_r. Now noting that K_r, G_r, H_r are all arcwise connected, the homotopy exact sequence for bundles gives

$$\cdots \longrightarrow \pi_2(H_r) \longrightarrow \pi_1(K_r) \xrightarrow{i_{\#r}} \pi_1(G_r) \xrightarrow{\sigma_{\#r}} \pi_1(H_r) \longrightarrow 0 ,$$

where the group $\pi_n(X)$ is the set of path components of X^{S^n} [5], with S^n the n-dimensional sphere.

Lemma. *For $(\sigma_1(z), \sigma_2(z))$ invertible in $\sum_r$ we have*

$$\sigma_{\#r}[\sigma_i(z)]_{G_r} = [\sigma(\sigma_i(z))]_{H_r} = 0 .$$

Proof. It suffices to show that $\sigma[\sigma_i(z)](w)$ is homotopic to 1 in $GL_r{}^{T^2}$. Now suppose $(\sigma_1(z), \sigma_2(z))$ is invertible in $\sum_r$. Then

$$f_i(z, w) = \text{determinant}\ \sigma[\sigma_i(z)](w)$$

has winding number zero for each fixed z. But

$$f_1(z, w) = f_2(w, z) ,$$

so $f_i(z, w)$ has winding number zero for each fixed w. An easy argument now shows that f_i is homotopic to the constant 1 in $C(T^2, C - 0)$.

Let SL_r be the subgroup of GL_r consisting of matrices with determinant identically one. Consider the exact sequence

$$\{1\} \longrightarrow SL_r{}^{T^2} \longrightarrow GL_r{}^{T^2} \xrightarrow{\det} (C - 0)^{T^2} \longrightarrow \{1\} .$$

Since det (determinant) has an obvious cross-section, it is only necessary to check that $SL_r{}^{T^2}$ is arcwise-connected. But this is an easily established topological fact.

3. Main theorem

We can now prove the main result.

Theorem. *Let A be a Fredholm operator in $\mathscr{A}_r{}^2$ with symbol pair $(\sigma_1(z), \sigma_2(z))$. Then there is a path $(\sigma_1(z)_t, \sigma_2(z)_t)$ of invertible elements in $\sum_r$ such that $\sigma_1(z)_0 = \sigma_1(z)$, $\sigma_2(z)_0 = \sigma_2(z)$ and such that*

$$\sigma_1(z)_1 = \begin{pmatrix} z^m & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \\ & & & & \ddots \end{pmatrix}, \quad \sigma_2(z)_1 = \begin{pmatrix} z^n & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \\ & & & & \ddots \end{pmatrix}$$

for integers n and m, and the index of A is given by index $A = -(m + n)$.

(Note: In general the path $(\sigma_1(z)_t, \sigma_2(z)_t)$ is not unique and, for $r > 1$, neither m nor n is uniquely determined.)

Proof. By the Lemma, $\sigma_{\#r}[\sigma_1] = 0$. Hence, there is a homotopy $\sigma_1(z)_t$ in G_r with $0 \leq t \leq \frac{1}{2}$ such that $\sigma_1(z)_0 = \sigma_1(z)$ and $\sigma_1(z)_{1/2}$ is in $C(T, K_r)$. Consider $\sigma[\sigma_1(z)_t](.)$, which is a homotopy between $\sigma[\sigma_1(z)](.)$ and 1 in $C(T, H_r)$. Now $C(T, H_r) \subset C(T^2, GL_r)$, so $\sigma[\sigma_1(.)_t](.)$ is a homotopy between $\sigma[\sigma_1(.)](.)$ and 1 in $C(T^2, GL_r)$. Now for each fixed z, $\det \sigma[\sigma_1(.)_t](z)$ is an arc in $C(T, C - 0)$ joining $\det \sigma[\sigma_1(.)](z)$ to 1. Hence, the winding number of $\det \sigma[\sigma_1(.)_t](z)$ is 0 for each z, and so $\sigma[\sigma_1(.)_t](z)$ is in $C(T, H_r)$.

Now we have $\sigma[\sigma_2(z)](w) = \sigma[\sigma_1(w)](z)$ by (*) so that $\sigma[\sigma_2(z)](.) = \sigma[\sigma_1(.)_0](z)$. The "second covering homotopy theorem" [9] now applies to give the existence of an arc $\sigma_2(z)_t$ in $C(T, G_r)$ such that $\sigma_2(z)_0 = \sigma_2(z)$ and

$$\sigma[\sigma_2(z)_t](.) = \sigma[\sigma_1(.)_t](z) .$$

Thus, $(\sigma_1(z)_t, \sigma_2(z)_t)$ is a path of invertible elements in $\sum_r$ such that $\sigma_1(z)_0 = \sigma_1(z)$, $\sigma_2(z)_0 = \sigma_2(z)$ and $\sigma_1(z)_{1/2}$ is in $C(T, K_r)$. Hence

$$1 \equiv \sigma[\sigma_1(z)_{1/2}](w) \equiv \sigma[\sigma_1(w)_{1/2}](z) \equiv \sigma[\sigma_2(z)_{1/2}](w)$$

implies that $\sigma_2(z)_{1/2}$ is in $C(T, K_r)$ as well.

Each element of $C(T, K_r)$ is homotopic in $C(T, K_r)$ to an element of the form

$$\begin{pmatrix} z^m & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \\ & & & & \ddots \end{pmatrix}$$

[8]. This is the construction which shows that $\pi_1(K_r) = Z$. Further, *any* pair of elements in $C(T, K_r)$ automatically satisfies (*). The construction of the desired path is now obvious.

To prove the index theorem, we recall that by a standard result for Fredholm operators, A has the same index as *any* Fredholm operator with symbol pair $(\sigma_1(z)_1, \sigma_2(z)_1)$. The desired result then follows from the facts that the index of a product of Fredholm operators is the sum of the indices and that the operator $W_w + (I - W_w)W_z W_{\bar{z}}$ is Fredholm with index -1 and symbol pair

$$\sigma_1(z) = I\ , \qquad \sigma_2(z) = \begin{pmatrix} z & & & \\ & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \\ & & & & \ddots \end{pmatrix}.$$

4. Some examples

We consider some particular Wiener-Hopf operators in $\mathscr{A}_2^{\,2}$. Letting

$$\phi(z, w) = \begin{pmatrix} z^n & -w^m \\ \bar{w}^m & \bar{z}^n \end{pmatrix}$$

it follows from the criterion of [4] discussed above that W_ϕ, which has symbols

$$\sigma_1(z) = \begin{pmatrix} z^n I & -S^m \\ S^{*m} & \bar{z}^n I \end{pmatrix}, \qquad \sigma_2(z) = \begin{pmatrix} S^n & -z^m I \\ \bar{z}^m I & S^{*n} \end{pmatrix},$$

is a Fredholm operator. In fact, W_ϕ is just the "smash product" of S^n with S^{*m} described in [6]. It can be seen by direct computation that this operator has index $-mn$. However, it is instructive to follow the proof of the Theorem.

It is easy to see that σ_1 and σ_2 are both homotopic to 1 in $C(T, G_2)$. We write down the homotopies

$$\sigma_1(z)_t = \begin{cases} \begin{pmatrix} z^n I & -(1-2t)S^m \\ (1-2t)S^{*m} & \bar{z}^n I \end{pmatrix} & 0 \leq t \leq \frac{1}{2}\ , \\ \begin{pmatrix} z^n I \cos 2\pi(t-\frac{1}{2}) & I \sin 2\pi(t-\frac{1}{2}) \\ -I \sin 2\pi(t-\frac{1}{2}) & \bar{z}^n I \cos 2\pi(t-\frac{1}{2}) \end{pmatrix} & \frac{1}{2} \leq t \leq \frac{3}{4}\ , \\ \begin{pmatrix} I \sin 2\pi(t-\frac{3}{4}) & I \cos 2\pi(t-\frac{3}{4}) \\ -I \cos 2\pi(t-\frac{3}{4}) & I \sin 2\pi(t-\frac{3}{4}) \end{pmatrix} & \frac{3}{4} \leq t \leq 1\ , \end{cases}$$

$$\sigma_2(z)_t = \begin{cases} \begin{pmatrix} (1-2t)S^n & -z^m I \\ \bar{z}^m I & (1-2t)S^{*n} \end{pmatrix} & 0 \leq t \leq \frac{1}{2}\ , \\ \begin{pmatrix} I \sin \pi(t-\frac{1}{2}) & -z^m I \cos \pi(t-\frac{1}{2}) \\ \bar{z}^m I \cos \pi(t-\frac{1}{2}) & I \sin \pi(t-\frac{1}{2}) \end{pmatrix} & \frac{1}{2} \leq t \leq 1\ . \end{cases}$$

Note that for $t > 0$, in general, $\sigma[\sigma_1(z)_t](w) \neq \sigma[\sigma_2(w)_t](z)$ so the pair $(\sigma_1(z)_t, \sigma_2(z)_t)$ is not the "symbol" of any operator in $\mathscr{A}_2^{\,2}$.

Our theorem implies that there is a $\tilde{\sigma}_2(z)_t$ so that $\tilde{\sigma}_2(z)_0 = \sigma_2(z)$ and $\tilde{\sigma}_2(z)_t$ is in $C(T, G_2)$ for all t, $0 \le t \le 1$ with

$$\sigma[\tilde{\sigma}_2(z)_t](w) = \sigma[\sigma_1(w)_t](z) \ ,$$

so that $(\sigma_1(z)_t, \tilde{\sigma}_2(z)_t)$ is the "symbol" of some Fredholm operator A_t in $\mathscr{A}_2^2$ for $0 \le t \le 1$. We now explicitly construct $\tilde{\sigma}_2(z)_t$ as follows: Let P_n be the projection on the kernel of S^{*n}. Then, with $Q_n = I - P_n$,

$$\tilde{\sigma}_2(z)_t = \begin{cases} \begin{pmatrix} S^n & -z^m P_n \\ 0 & S^{*n} \end{pmatrix} + \begin{pmatrix} 0 & -(1-2t)z^m Q_n \\ (1-2t)\bar{z}^m I & 0 \end{pmatrix} & 0 \le t \le \frac{1}{2} \ , \\ \begin{pmatrix} S^n \cos 2\pi(t-\frac{1}{2}) & -z^m P_n + Q_n \sin 2\pi(t-\frac{1}{2}) \\ -I \sin 2\pi(t-\frac{1}{2}) & S^{*n} \cos 2\pi(t-\frac{1}{2}) \end{pmatrix} & \frac{1}{2} \le t \le \frac{3}{4} \ , \\ \begin{pmatrix} (-z^m P_n + Q_n) \sin 2\pi(t-\frac{3}{4}) & (-z^m P_n + Q_n) \cos 2\pi(t-\frac{3}{4}) \\ -I \cos 2\pi(t-\frac{3}{4}) & I \sin 2\pi(t-\frac{3}{4}) \end{pmatrix} & \frac{3}{4} \le t \le 1 \ . \end{cases}$$

Note that

$$\tilde{\sigma}_2(z)_1 = \begin{pmatrix} -z^m P_n + (I - P_n) & 0 \\ 0 & I \end{pmatrix} ,$$

so $[\tilde{\sigma}_2(z)_1]_{K_2} = mn$ in $\pi_1(K_2)$, and the index of W_ϕ is $-mn$.

This example has an interesting consequence. Since $i_{\#2}(mn) = i_{\#2}[\tilde{\sigma}_2(z)_1]_{K_2} = [\tilde{\sigma}_2(z)_1]_{G_2} = 0$, it follows that $i_{\#2} \equiv 0$. If $(\sigma_1(z), \sigma_2(z))$ is invertible in $\sum_2$, then we already know that $\sigma_{\#2}[\sigma_1(z)]_{G_2} = \sigma_{\#2}[\sigma_2(z)]_{G_2} = 0$ from the Lemma. But $\rightarrow \pi_2(H_r) \rightarrow \pi_1(K_r) \xrightarrow{i_{\#r}} \pi_1(G_r) \xrightarrow{\sigma_{\#r}} \pi_1(H_r) \rightarrow 0$ is exact, so $[\sigma_1(z)]_{G_2} = [\sigma_2(z)]_{G_2} = 0$.

The case $r = 1$ is quite different, because $\pi_2(H_1) = 0$ and $\pi_1(K_1) = Z$ so $i_{\#1} \not\equiv 0$. In fact, $G_1 \xrightarrow{\sigma} H_1$ has a global cross-section $(\phi \rightarrow W_\phi)$, and so G_1 is homeomorphic with $K_1 \times H_1$ from bundle theory [9], whence $\pi_1(G_1) = Z \times Z$ since $\pi_1(H_1) = Z$. On the other hand, $i_{\#2} \equiv 0$ easily implies that $i_{\#r} \equiv 0$ for $r \ge 2$ which further implies that $G_r \xrightarrow{\sigma} H_r$ does *not* have a global cross-section for $r > 1$.

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