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COMPACT FLAT RIEMANNIAN MANIFOLDS

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Let M be a compact flat Riemannian manifold of dimension n, and π its fundamental group. Then we have the following theorem of Bieberbach-Auslander-Kuranishi [1], [2]:

Theorem 1. *The group π is torsion free and satisfies the following exact sequence*

$$1 \to A \to \pi \to \Phi \to 1$$

where A is a finitely generated maximal abelian subgroup of π, and Φ is a finite group. Conversely, every group with the above property is the fundamental group of a compact flat Riemannian manifold of dimension n. The group Φ is the holonomy group of M.

In [3], E. Calabi announced that every compact flat Riemannian manifold with nonzero first betti number can be given by a construction which we shall call the Calabi construction. The purpose of § 1 of this paper is to generalize Calabi's theorem to the case where M has positive semidefinite Ricci tensor and to study the condition under which the Calabi construction is possible. We show that if Φ is cyclic or if the dimension of M is odd and Φ is of odd order, then the first betti number of M is not zero. This will follow from a fixed point theorem.

In [9], A. T. Vasquez proved the following.

Theorem 2. *There is associated with every finite group Φ a positive integer $n(\Phi)$ such that: if M is a compact flat Riemannian manifold with holonomy group Φ, and $\dim M \geq n(\Phi)$, then M is a flat toral extension of another flat manifold of dimension $\leq n(\Phi)$.*

The integer $n(\Phi)$ is not known except for the special case when Φ is a prime order group for which $n(\Phi) = 1$ and when Φ is $Z_2 \times Z_2$ for which $n(\Phi) \leq 6$, cf. [8]. In § 2 of this paper we prove that $n(\Phi)$ can be chosen to be less than or equal to the sum of the indices of maximal cyclic subgroups of Φ. When Φ is of prime order or is $Z_2 \times Z_2$, we obtain the bound stated above. Theorem 2 is reproved by using some elementary methods and hence avoiding results of I. Reiner on integral representation of prime order groups and homology of groups.

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during the preparation of this paper; for example, Lemma 2 is revised according to his suggestion. The author also wishes to thank the referee for comments on the original version of the paper.

1. We begin this section by indicating there is a way to unify the concept of flat toral extension and the Calabi construction. We have the following

Definition. Let $M_1 = R^n/\pi_1$ and M_2 be two compact Riemannian manifolds. Suppose π_1 also acts as a group of isometries on M_2. Then let π_1 act diagonally on $R^n \times M_2$, i.e., $g \in \pi_1$, $m_1 \in R^n$, $m_2 \in M_2$, $g(m_1, m_2) = (gm_1, gm_2)$. The space of orbits is a compact Riemannian manifold with respect to the natural Riemannian structure and will be called an M_2 extension of M_1.

This definition appeared in [9]. When M_2 is a flat torus, Vasquez called in the flat toral extension of M_1. The following proposition is immediate from Calabi's theorem.

Proposition 1. *Let M be a compact flat Riemannian manifold with nonzero first betti number. Then M is an N extension of a flat torus T^r, where N is a compact flat Riemannian manifold with zero first betti number, and r is the first betti number of M.*

We now generalize this proposition.

Theorem 3. *Let M be a compact Riemannian manifold of positive semidefinite Ricci tensor. Then M is an N extension of a flat torus T^r, where N is a compact Riemannian manifold with positive semidefinite Ricci tensor, and r is the first betti number of M.*

Proof. Let $\tilde{M}$ be the universal cover of M with the induced metric from M, and X a nonzero harmonic vector field on M. It is well known that X is a parallel vector field on M; cf. [6]. Lift X to a parallel vector field $\tilde{X}$ on $\tilde{M}$. By de Rham's theorem, $M = R \times M'$ such that $\tilde{X}$ is tangent to the first factor. Here R is the real line and the product is Riemannian.

We shall identify the fundamental group $\pi_1(M)$ with the group of covering transformations of $\tilde{M}$. Let g be an arbitrary element of $\pi_1(M)$. We claim that the projection of the action of g on R is a translation. In fact, if not, the projection will be a reflection on R and hence fix a point o in R. Let $\tilde{X}_0$ be the vector defined by $\tilde{X}$ at o. Then $\tilde{X}_0$ is mapped under the projection to $-\tilde{X}_0$. Since $\tilde{X}$ is invariant under g, this implies $\tilde{X}_0 = 0$ and therefore $\tilde{X} = 0$ which is a contradiction.

Now simple induction shows that $\tilde{M}$ is a Riemannian product of R^r and a manifold M'' with positive semidefinite Ricci tensor such that the projection of covering transformations on R^r are translations and the translation vectors span R^r. Let G be the subgroup of $\pi_1(M)$ which acts trivially on R^r. It remains to prove that M''/G is compact. In fact, since the commutator subgroup $[\pi_1(M), \pi_1(M)]$ is contained in G, we have

$$\operatorname{rank}(\pi_1(M)/G) \leq \operatorname{rank}(\pi_1(M)/[\pi_1(M), \pi_1(M)]) = \operatorname{rank}(H_1(M_1, Z)) = r ,$$

which together with the fact that the translations span R^r concludes that rank $(\pi_1(M)/G) = r$. It then follows that $M = (R^r \times (M''/G))/(\pi_1(M)/G)$ is a fibre space over the torus $R^r/\pi_1(M)/G$ with fibre M''/G. Thus M''/G is a closed subset of a compact space and is hence compact.

Corollary 1. *Every compact four dimensional Riemannian manifold with zero Ricci tensor and nonzero first betti number is flat.*

This follows from the fact that every three dimensional Ricci flat Riemannian manifold is flat.

The following corollary is due to Chern and Milnor, using the Gauss-Bonnet theorem.

Corollary 2. *Every compact four dimensional Riemannian manifold with positive semidefinite Ricci tensor has nonnegative Euler number, and its Euler number is zero iff its compact orientable covering manifold is an N extension of the circle where N is a three dimensional compact manifold with nonnegative Ricci tensor.*

Proof of the Corollary. If the Euler number is nonpositive, then the Poincaré duality shows that the first betti number of the manifold is nontrivial. The theorem then says that its compact orientable covering manifold is an N extension of the circle where N is a three dimensional compact manifold with nonnegative Ricci tensor. This in turn implies that the manifold has a nonzero vector field and the Euler number of the manifold is zero, cf. [5] and [10].

This corollary shows that the classification of four dimensional manifolds with positive semidefinite Ricci tensor and zero Euler number depends on the classification of three dimensional ones. This was verified by Calabi when the manifolds have zero curvature.

Remark. The referee points out to us that Theorem 3 is a special case of a recent work of Cheeger and Gromoll.

Now let us assume M is flat. We first study the condition under which the first betti number of M is not zero. To this end, we have to estimate the size of $[\pi_1, \pi_1]$.

It is well known that M is covered by R^n, the Euclidean space of dimension n, and that π, considered as a group of covering transformations on R^n, is a discrete subgroup of the group of Euclidean motions. Every element in π will be represented in the form (A, a) where A is the orthogonal part and a is the translation vector of the element. The group formed by the orthogonal part of elements of π is exactly the holonomy group of M. Suppose A has order m. Then it is easy to check that (A, a) acts freely on R^n iff the vector $A^{m-1}a + A^{m-2}a + \cdots + a \neq 0$. (Note that (A, a) acts freely if $\{(A, a)\}$ is torsion free and discreate.)

Let (A, a), (B, b) be two arbitrary elements in π. Then by direct computation, we know

$$[(A, a), (B, b)] = (ABA^{-1}B^{-1}, c) \ ,$$

where $c = -ABA^{-1}B^{-1}b - ABA^{-1} + Ab + a$.

We claim that the first betti number of M is not zero iff the group π has a nonzero common fixed point. In fact, if x is a nonzero fixed point of Φ, then

$$\begin{aligned}&\langle x, -ABA^{-1}B^{-1}b - ABA^{-1}a + Ab + a\rangle \\ &\quad = -\langle x, b\rangle - \langle x, a\rangle + \langle x, b\rangle + \langle x, a\rangle = 0 ,\end{aligned}$$

where $\langle\ ,\ \rangle$ is the dot product of R^n, and (A, a), (B, b) are two arbitrary elements in π. This implies that the lattice generated by the translation vectors of $[\pi, \pi]$ is of dimension less than or equal to $n - 1$. Hence $\pi/[\pi, \pi]$ is of rank at least one.

From the remark that every transformation in Φ has a nonzero fixed point, we have immediately

Proposition 2. *Suppose the holonomy group of a compact flat Riemannian manifold is cyclic. Then the first betti number of the manifold is not zero.*

We now prove

Theorem 4. *If the holonomy group of an odd dimensional compact flat Riemannian manifold is of odd order, then the first betti number of this manifold is not zero.*

Proof. From the remark above, we need only to prove

Lemma 1. *Let Φ be an odd order group acting orthogonally on an odd dimension Euclidean space. Then Φ has a nonzero simultaneous fixed point in it.*

Proof of the Lemma. The famous theorem of Feit-Thomson asserts that Φ is solvable. Let

$$1 \lhd \Phi_1 \lhd \Phi_2 \lhd \cdots \lhd \Phi_m = \Phi$$

be the derived series of Φ, and A be an arbitrary element in Φ_1. Then from the fact that there exists no nonzero vector x such that $Ax = -x$, it follows that the fixed point space of A is odd dimensional since A is of odd order and the dimension of the Euclidean space is odd. Let B be another element in Φ_1. Then B leaves invariant the fixed point space of A which is odd dimensional since $AB = BA$. Hence A, B have a simultaneous fixed point. (Note that B is of odd order and therefore is a rotation.) The simultaneous fixed point space of A and B is still of odd dimension. Continuing in this way, we know that the simultaneous fixed point space F_1 of Φ_1 is odd dimensional. Now let C be an arbitrary element in Φ_2, and A an arbitrary element in Φ_1. Then $AC = CA'$ for some A' in Φ_1. For all x in F_1, we have

$$ACx = CA'x = Cx ,$$

which means that C leaves F_1 invariant. Let F_2 be the fixed point space of C

and Φ_1. F_2 is also odd dimensional. Let D be another element in Φ_2. Then $DC = CDA''$ for some A'' in Φ_1. Since

$$CDx = CDA''x = DCx = Dx$$

for all x in F_2, D leaves F_2 invariant. The fixed point space of C, D and Φ_1 is odd dimensional. Continuing in this way, we prove that the fixed point space of Φ is odd dimensional and is therefore nonvoid.

Remark. Theorem 4 can be generalized in the following manner. Let Φ be a solvable group. Then M has a cover $\tilde{M}$ such that $\tilde{M}$ is a compact flat Riemannian manifold with nonzero betti number and that the group of covering transformations is isomorphic to the group Φ/Φ^2 where Φ^2 is the group generated by the square of the elements of Φ and is of course the products of Z_2's.

The proof depends on the fact that the square of an orthogonal transformation is a rotation in any invariant subspace. This generalization cannot be improved in the sense that there are examples for which Φ is $Z_2 \times Z_2$ and yet the first betti number is not zero.

The referee points out to us that a proof of Lemma 1 avoiding the Feit-Thompson theorem can be found by using a theorem of Burnside (see, e.g., W. Feit, *Characters of finite groups*, Benjamin, New York, 1967, p. 68).

2. Now let us proceed to find a bound for $n(\Phi)$.

Lemma 2. *Suppose the holonomy group Φ of a compact flat manifold M is cyclic, then M is a flat torus extension of a circle.*

Proof. First let A be a generator of Φ. Let a be such that (A, a) is an element in π. Let V_1 be the fixed point space of A, and V_2 the orthogonal complement of V_1. Let Z^n be the group of translations of π. Then for all z in Z^n, $z = z_1 + z_2$ with z_1 in V_1 and z_2 in V_2. (Here we identify the group of translations with their translation vectors.) Thus $Az - z = (A - I)z_2$. Since $Az - z$ is a lattice point in V_2, and $A - I$ restricted to V_2 is nonsingular and leaves the lattice point invariant, z_2 is a rational combination of the lattice points in V_2. By subtraction, z_1 is also a rational combination of the lattice points in V_1. Actually, we know that there exists an integer m such that mz_1 is a lattice point for all z in Z^n. Hence the set of all elements of the form z_1 forms a lattice in V_1. Let $z_1^1, z_1^2, \cdots, z_1^m$ be a basis of this lattice.

Now let $a = a_1 + a_2$ with a_1 in V_1 and a_2 in V_2. Since a_1 is fixed by A, we have $A^{p-1}a + A^{p-2}a + \cdots + a = A^{p-1}a_2 + A^{p-2}a_2 + \cdots + a_2 + pa_1$. The first sum $A^{p-1}a_2 + A^{p-2}a_2 + \cdots + a_2$ lies in V_2, and is fixed by A if p is the order of A; hence it must be zero. The fact that $A^{p-1}a + A^{p-2}a + \cdots + a$ is not zero implies that a_1 is not zero.

The last argument also shows that pa_1 is the first multiple of a_1 which lies in $(Z^n)_1$. Let k be the largest integer such that $pa_1 = kz_1$ for some $z \in Z^n$. Then $z_1 = \sum_{i=1}^{m} e_i z_1^i$ where the greatest common divisor of the e_i's is one. By changing the basis $\{z_1^1, \cdots, z_1^m\}$ if necessary, we may assume $z_1 = z_1'$. Now let G be the group generated by $(Z^n)_1$ and a_1. Since p and k are relatively prime, by replacing A, if necessary, by another generator of Φ and by adding to the corresponding a, a vector of the form $z \in Z^n$, we may assume $\{a_1, z_1^2, z_1^3, \cdots, z_1^m\}$ forms a basis of G. Let V_1' be the vector space generated by $z^2, z^3, \cdots, z^m$, and V_1'' be the line in V_1 perpendicular to V_1'. The projection of a_2 on V_1'', called a_3, is obviously nontrivial. Let T_2 be the flat torus obtained by taking the quotient of the sum $V_1'' \oplus V_2$ with respect to the lattice points in it. Note that the quotient is compact by the above construction. Let T_1 be the circle obtained by taking the quotient of V_1'' with respect to the action of a_3. Then we claim that M is a T_2 extension of T_1. In fact, the action of a_3 on T_2 may be taken to be the projected action of (A, a) on T_2. The only thing we have to check is that the group generated by (A, a) and the lattice points in the sum $V_1'' \oplus V_2$ is the group π_1 itself. This follows from the fact that $\{a_1, z_1^2, z_1^3, \cdots, z_1^m\}$ forms a basis of $(Z^n)_1$.

Corollary. *Under the assumption of Lemma* 2 *the characteristic algebra vanishes in dimension greater than* 1.

Proof. This follows easily from Lemma 2.7 of [9].

Remark. When Φ is of prime order, Lemma 2 was proved in [1], but our method is more elementary and constructive. Lemma 2 cannot be generalized to the case where Φ is abelian; in fact, if Φ is $Z_2 \times Z_2$, the first betti number of M may not be zero and hence may not be a flat toral extension of any torus, cf. [5].

We can now improve the main theorem of [1].

Theorem 5. *Let Φ be a finite group. Then there exists an integer $n(\Phi)$ such that if M is any compact flat Riemannian manifold with holonomy group isomorphic to Φ, then M is a flat toral extension of some compact flat Riemannian manifold of dimension less than or equal to $n(\Phi)$. Furthermore, $n(\Phi)$ can be chosen to be less than or equal to the sum of the indices of maximal cyclic subgroups of Φ.*

Proof. Let the n dimensional euclidean space R^n, be the universal covering manifold of M, and Z^n the group of translation in π. Let $A_1, A_2, \cdots, A_n$ be a set of elements in Φ such that they generate pairwise distinct maximal cyclic subgroups of Φ. Let a_i be vectors such that (A_i, a_i) is an element in π, and n_i be the order of A_i for each i. According to Lemma 2, one can make appropriate choices such that there exists an $n - 1$ dimensional subspace R_{A_i} of R^n such that the projected action of (A_i, a_i) on the orthogonal complement of R_{A_i} is a translation and that the group Z^n is the group generated by $(A_i, a_i)^{n_i}$ and the lattice points lying in R_{A_i}. Let M_1 be the intersection of all subspaces of the form BR_{A_i}, where B runs over the holonomy group of M, and i from 1 to m. Let M_2 be the orthogonal complement of M_1. It is clear that every element in the holonomy group leaves M_1 and M_2 invariant.

We first note that if R_1 and R_2 are two linear subspaces of R^n such that each one of them is spanned by the lattice points lying inside it, then $R_1 \cap R_2$ is also so. This follows from the fact that the dimension of the intersection of two sublattices is equal to that of $R_1 \cap R_2$ as seen by viewing the R_i's as rational vector spaces. From this remark, we know that M_1 is spanned by the lattice points lying inside it. Call the group formed by these lattice points G_1.

Now it is clear from the definitions that G_1 is a normal subgroup of π. Let $G_2 = \pi/G_1$. We claim that G_2 acts freely and properly discontinuously on M_2 by the projected action of π. In fact, let $\{A_i\}$ be a sequence of elements in π and (x, y) be an arbitrary point on R^n with $x \in M_1$ and $y \in M_2$. Then $A_i(x, y) = (A_i^1x, A_i^2y)$ where A_i^1, A_i^2 are isometries of M_1 and M_2 respectively. Since G_1 has a compact quotient on M_1, there are elements $\{B_i\}$ in G_1 such that $B_iA_i(x, y) = (A_i^1x + b_i, A_i^2y)$ with $\{A_i^1x + b_i\}$ lying in a compact set. Now if $\{A_i^2y\}$ converges in M_2, then $B_iA_i(x, y)$ converges in $M_1 \oplus M_2$ by passing to a subsequence if necessary. The fact that π acts properly discontinuously implies that a subsequence of $\{B_iA_i\}$ is constant eventually. Hence a subsequence of A_i^2 is constant eventually, and G_2 acts properly discontinuously on M_2. Now let us prove that G_2 acts freely on M_2. In fact, let (A, a) be an arbitrary element in π. Suppose A contains in the maximal cyclic group generated by A_i. Then (A, a) can be written in the form $(A_i, a_i)^r \cdot z$ where r is an integer and z is a lattice point in R_{A_i}. If the projected action of such an element on the orthogonal complement of R_{A_i} is trivial, then r is zero by the definition of R_{A_i}. Similar argument shows that if (A, a) acts trivially on M_2, it is a lattice point in R_{A_i} for all i, i.e., an element in G_1. This means G_2 acts freely on M_2.

It is now clear that M is an M_1' extension of M_2', where M_1' is the flat torus M_1/G_1 and M_2' is the manifold M_2/G_2. It remains to compute the dimension of M_2. For each i, we may consider the holonomy group acting on the set of all BR_{A_i}'s. Since the isotropy group contains the maximal cyclic group generated by A_i, the number of distinct BR_{A_i}'s is less than or equal to the index of this cyclic group. The theorem now follows from the fact that each R_{A_i} is of codimension 1 and hence the codimension of the intersection of them is exactly the number defined in the theorem.

Corollary. *Let Φ be a finite group. If M is a compact flat Riemannian manifold with holonomy group isomorphic to Φ, then the characteristic algebra of M vanishes in dimension greater than the sum of the indices of the maximal cyclic subgroups of Φ.*

Remark. When Φ is cyclic, the bound in Theorem 5 is of course best possible. But the author does not know how sharp it is in the other cases. When the group Φ is $Z_2 \times Z_2$, the bound is 6 which was also obtained by Vasquez in this special case.

It would be interesting to know the condition under which a flat Riemannian manifold is a flat manifold extension of another flat manifold. Lemma 2 shows that the classification of n dimensional flat Riemannian manifolds with cyclic holonomy group depends only on the study of the isometric action of such a cyclic group on R^{n-1}, which preserves the lattice points.

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