TIGHT TOPOLOGICAL EMBEDDINGS OF THE MOEBIUS BAND

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1. Introduction and examples

The main result of this paper is as follows. If a Moebius band topologically embedded in a euclidean four-space E^4 does not lie in a hyperplane and is cut by any hyperplane in at most two parts which are homotopy equivalent to a point or a circle, then the band is the union of the triangles $e_i e_{i+1} e_{i+2}$ (i mod 5) of a simplex $e_1 \cdots e_5 \subset E^4$; a more technical formulation is given at the end of this section.

Let $f: M \to E^N$ be an embedding of a topological space M into a *euclidean vector space* E^N. Any hyperplane of E^N has an equation $z = c$, where z is a linear function (covector) and $c \in R$. The parts of M which embed on one side of the hyperplane are ($zf = z \circ f$ is the composition)

$$(zf)_c^- = \{x \in M: zf(x) < c\} \subset M \quad \text{and} \quad (-zf)_{-c}^-.$$

For later use we also define

$$(zf)_c = \{x \in M: zf(x) \leq c\} \subset M.$$

Following T. Banchoff we say that f has the *two-piece property* (TPP) (or is 0-tight) in case

(1) $(zf)_c^-$ is connected or empty for all z, c.

Then every hyperplane cuts M into at most two pieces. If $f(M)$ is convex, then every nonempty $(zf)_c^-$ is contractible for all z, c, and this property is sufficient to make $f(M)$ convex (Corollary 2a). As $(zf)_c^-$ may be equal to $f(M)$, then M has to be contractible itself.

Tightness or having *minimal total absolute curvature* is a property which generalizes convexity. Given the topological space M embedded in E^N, the parts in which any hyperplane cuts M should not be more complicated than strictly necessary. In the above notation this can be roughly defined by the claim that for all z the homology of $(zf)_c^-$ has for increasing c the minimal possible num-

ber of changes defined with suitable multiplicities. In the case of smooth closed manifolds M we get for almost every z a nondegenerate function zf on M, which should have the minimal possible number of critical points; see [1] for details. For connected compact two-dimensional manifolds with non-void boundary we give a simple definition of tight embedding. Such a manifold M has the homotopy type of a wedge of (say) k circles, $k \geq 0$. The embedding $f : M \to E^N$ is said to be *tight* in case $(zf)_c^-$ is empty or homotopy equivalent to a wedge of j circles for some $j \leq k$ all z, c. For the *band* $S^1 \times [0, 1]$ and the *Moebius band* (our main interests in this paper) we have $k = 1$.

Some *definitions* and *notations* to be used are as follows: We call the smallest convex set containing a subset X of euclidean space E^N the *convex hull* $\mathscr{H}(X)$, the linear variety which X spans the *span* $\alpha(X)$, and for compact X the boundary of $\mathscr{H}(X)$ in $\alpha(X)$ the *convex envelope* $\partial\mathscr{H}(X)$. $f : X \to E^N$ is said to be *substantial* (in E^N) in case $f(X)$ is not contained in any hyperplane of E^N.

Examples of tight bands and Moebius bands in E^2 and E^3 are given in Figures 1, 2, 3.

a)   In the plane only the tight band exists and it is always a difference $F \backslash \mathring{D}$ where F and D are compact convex sets with interiors $\mathring{F}$ and $\mathring{D}$ and $D \subset \mathring{F}$. (See Lemma 4.)

b)   In E^3 the substantial tight bands are all obtained as follows (Lemma 8). Let D_1 and D_2 be compact convex sets with nonempty interiors $\mathring{D}_1$ and $\mathring{D}_2$ in different planes $\alpha(D_1)$ and $\alpha(D_2)$ of E^3. Suppose $\mathring{D}_1 \cup \mathring{D}_2$ does not meet $\alpha(D_1) \cap \alpha(D_2)$. Next delete $\mathring{D}_1 \cup \mathring{D}_2$ from $\partial\mathscr{H}(D_1 \cup D_2)$ to obtain $\partial\mathscr{H}(D_1 \cup D_2) \backslash (\mathring{D}_1 \cup \mathring{D}_2)$. If $D_1 \cap D_2 = \emptyset$, then it is a tight band in E^3. Observe that a tight band in E^3 is a developable surface which can be smooth or piecewise linear or neither.

Examples a) and b) are substantial in E^2 and E^3 respectively.

c)   Banchoff found the following tight Moebius band in E^N, $N = 3$ or 4. Let $e_1, \cdots, e_5$ be five points in a general position in E^N. If the union of the five plane triangles $e_i e_{i+1} e_{i+2}$ with i cyclic modulo 5 is an embedded surface, then one checks easily that it is a tight substantially embedded Moebius band. For $N = 4$ it is of course always an embedding. For $N = 3$ see Fig. 2. The example in Fig. 2(b) is generalized in Fig. 3(b) which is obtained as follows:

Take D_1 and D_2 as in Example b), but suppose that D_1 and D_2 meet $\alpha(D_1) \cap \alpha(D_2)$ in *one* and the same point A, and moreover that ∂D_1 contains a straight segment AB and ∂D_2 contains a straight segment AC such that the plane $\alpha(ABC)$ of these two segments cuts the convex body $\mathscr{H}(D_1 \cup D_2)$ in two parts. Let V be a piece (with interior) of the plane triangle ABC bounded by AB, BC and a straight or curved segment from B to C along which V is concave. Then $(V \cup \partial\mathscr{H}(D_1 \cup D_2)) \backslash (\mathring{D}_1 \cup \mathring{D}_2)$ is a tight embedding of the *Moebius band* in E^3.

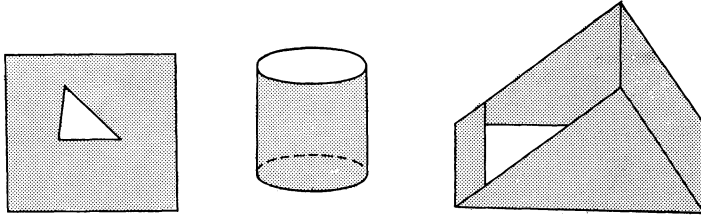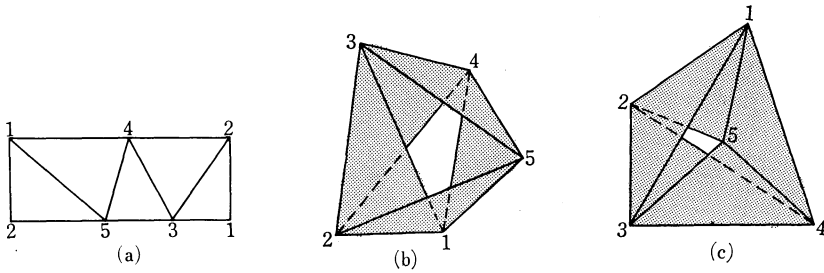The example in Fig. 2(c) is generalized in Fig. 3(c) and is obtained as follows:
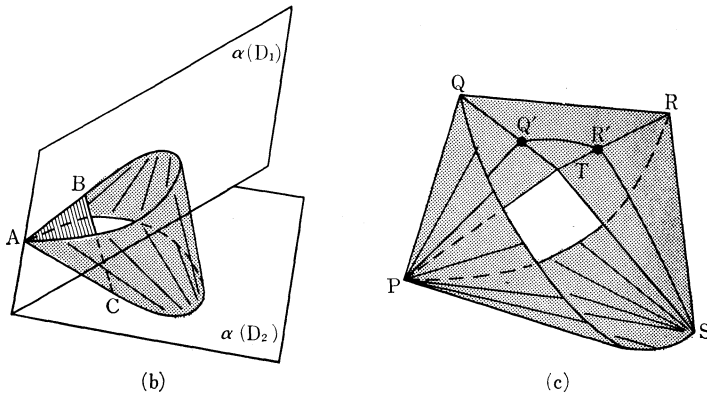
Fig. 1

Fig. 2

Fig. 3

Take D_1 and D_2 as in Example b), but suppose that D_1 and D_2 meet $\alpha(D_1) \cap \alpha(D_2)$ in a line segment QR, and moreover that ∂D_1 and ∂D_2 contain straight line segments QP and RS respectively. First make the piece $\partial \mathcal{H}(D_1 \cup D_2) \setminus (\mathring{D}_1 \cup \mathring{D}_2)$, which is the union of the line segment QR and an embedded 2-disc. Next take a point T in the interior of $\mathcal{H}(\{P, Q, R, S\})$ and add the triangles TPQ, TQR and TRS to obtain the required tight Moebius band. A further slight generalization seems to be obtainable by first taking Q' in the interior of QT and R' in the interior of RT and then deleting suitable

convex curved triangles PTQ', $Q'TR'$ and $R'TS$ from the above straight triangles. In this manner all tight Moebius bands seem to be obtained. This may be easy to prove but the author did not do it. At any rate it follows easily that there is no smooth tight Moebius band in E^3. The main, more interesting, result of this paper is:

Theorem. *If $f: M \to E^4$ is a topological tight substantial embedding of the compact Moebius band in euclidean four-space, then the image is the union of the five 2-simplices $e_i e_{i+1} e_{i+2}$, i modulo 5, of some 5-simplex $e_1 e_2 e_3 e_4 e_5$ in E^4.*

2. General lemmas on top-sets

For a *compact connected* set $f: M \to E^N$ and orthogonal unit covectors $z_1, z_2,$ $\cdots, z_N$, we define:

a *top-set* or *top¹-set*: $M(z_1) = \{x \in M : z_1 f(x) = \sup_{y \in M} z_1 f(y)\}$;

a *top²-set*: $M(z_1, z_2) = \{x \in M(z_1) : z_2 f(x) = \sup_{y \in M(z_1)} z_2 f(y)\}$;

a *top^i-set*: $M(z_1, \cdots, z_i) = \{x \in M(z_1, \cdots, z_{i-1}) :$

$$z_i f(x) = \sup_{y \in M(z_1, \ldots z_{i-1})} z_i f(y)\} \ .$$

A *top*-set* is a top^k-set for some $k \geq 0$. M will sometimes be called a top⁰-set. If the span of a top*-set is a k-dimensional euclidean space, then we call it an E^k-top*-set.

Lemma 1. *If the embedding $f: M \to E^N$ has the two-piece property ($=$TPP), then every top*-set of M also has the TPP.*

Proof. It is sufficient to prove the conclusion for top¹-sets, because we can then apply the result to a top¹-set to obtain the conclusion for a top²-set, etc.

Suppose that f has TPP and that contrary to the conclusion,

$$M(z_1) \ \cap \ (z_2 f)_c^-$$

does have at least two components for some orthogonal unit covectors z_1 and z_2 and a real number c. Then there is $c' < c$ with c' near to c such that the compact set $\bar{M}' = M(z_1) \cap (z_2 f)_{c'}$ as well as the set $M' = M(z_1) \cap (z_2 f)_c^-$ has at least two components. Now take an open neighborhood U of $\bar{M}'$ in M such that $U = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$, and U_1 and U_2 contain different parts of $\bar{M}'$. For our convenience we assume that $c' = 0$ and also that z_1 takes the value 0 on $M(z_1)$.

The function z_2/z_1 assumes on the compact set $M \setminus U$ a positive maximal value which we call $2/\varepsilon$; its values vary in $[-\infty, 2/\varepsilon]$ and are strictly bounded above by $1/\varepsilon$. We then find the inclusions:

$$M' \subset [(-z_1 + \varepsilon z_2)f]_0^- \subset [(-z_1 + \varepsilon z_2)f]_0 \subset U \ ,$$

and so $[(-z_1 + \varepsilon z_2)f]_0^-$ inherits the property of having at least two components from M' and U. But this contradicts TPP for $M \subset E^N$.

Lemma 2. *Suppose* $f: M \to E^N$ *has the property that* $(zf)_c^-$ *is p-connected for all z and c. Then for any* E^j-*top*-set* M' *we have: if* $j \leq p + 1$, *then* M' *is convex; if* $j = p + 2$, *then* $\partial \mathcal{H} M' \subset M'$.

Proof. For $j = 1$, and $p \geq 0$, the conclusion holds because the case $p \geq 0$ implies TPP, and Lemma 1 gives that $M' \subset E^1$ is a connected set, so a segment, and hence convex. We now consider an induction step and suppose the conclusions to be true for E^k-top*-sets in case $k < j$ with $1 \leq j \leq p + 2$. Let M' be an E^j-top*-set. All top^1-sets of M' in E^j are then convex sets in M'. Since $\partial \mathcal{H} M'$ is the union of these convex hulls of the top-sets of M', $\partial \mathcal{H} M' \subset M'$. If $j = p + 2$, then the proof is finished. If $j \leq p + 1$, then $\partial \mathcal{H} M'$ is a $(j\text{-}1)$-sphere. If now $\mathcal{H} M' = M'$, then the proof is also finished. So suppose $\mathcal{H} M'$ contains a point m not in $M' \subset M$. As M is compact, there is a small ball $B(m, 2\delta)$ in E^N with centre m and radius, say 2δ, which does not meet M. $\partial \mathcal{H} M'$ cannot be contracted inside

$$T \backslash B \overset{\text{def}}{=\!=\!=} \{v \in E^N : \text{distance } (v, \alpha(M')) \leq \delta\} \backslash B(m, 2\delta) \ .$$

Consequently, there is a neighborhood U of M' in M inside T, in which $\partial \mathcal{H} M'$ cannot be contracted either.

Now suppose $M' = M(z_1, \cdots, z_l)$ and $z_1 = \cdots = z_l = 0$ for points in M'. Then

$$M' = M(z_1, \cdots, z_{l-1}) \cap (z_l j)_0 \subset U \ ,$$

and by the methods used in Lemma 1 we obtain $\varepsilon_1 > 0$ such that

$$M' \subset M(z_1, \cdots, z_{l-2}) \cap [(-z_{l-1} + \varepsilon_1 z_l)f]_0 \subset U \ .$$

Repeating this process we first find

$$M' \subset M(z_1, \cdots, z_{l-1}) \cap [(-z_{l-2} + \varepsilon_2(-z_{l-1} + \varepsilon_1 z_l)f]_0^- \subset U \ ,$$

and after repetition finally: $M' \subset (zf)_0 \subset U$ for a suitable z. Then there is $c > 0$ such that $M' \subset (zf)_c^- \subset U$, and $(zf)_c^-$ is not p-connected because $\partial \mathcal{H} M'$ does not contract in $(zf)_c^-$. This contradicts the assumptions.

Corollary 2a. *Suppose the embedding* $f: M \to E^N$ *for a compact* M *has the property that* $(zf)_c^-$ *is contractible for all z and c. Then* M *is convex.*

Corollary 2b. *Suppose the topological substantial embedding* $f: S^n \to E^N$ *has* $(zf)_c^-$ $(n-1)$-*connected for all z and c. Then* $N = n + 1$, *and* $f(S^n)$ *is the boundary of a convex* $(n+1)$-*body.*

(Compare [1, Th. 4] and the theorem of Chern-Lashof in [2].)

Proof. By Lemma 2, all E^j-top*-sets are convex for $j \leq n$. If there is an

E^{n+1}-top*-set M', then $\partial \mathcal{H} M'$ is an n-sphere embedded (!) in $S^n = f(S^n)$, and therefore equal to that n-sphere; hence the proof is finished.

If all top^1-sets are E^j-top*-sets for some $j \leq n$ (hence convex), then $f(S^n)$ contains the $(N - 1)$-sphere $\partial \mathcal{H} f(S^n)$; and this is impossible for $N > n + 1$.

If there is an E^j-top*-set with $n + 1 < j < N$, then this E^j-top*-set has the same relevant properties as $f(S^n)$. By repeating this argument a finite number of times we reach the conclusion of Corollary 2.

Our next lemma is of a different kind.

Lemma 3. *Let $f: M \to E^N$ be a* TPP *embedding of a compact connected n-manifold with nonvoid boundary ∂M and interior $\overset{\circ}{M}$. If for some z and c, $\emptyset \neq (zf)_c \subset \overset{\circ}{M}$ or $M(z) \subset \overset{\circ}{M}$, then there are w and e such that some component of $(wf)_e^-$ has the homotopy type of the wedge $M \vee S^{n-1}$.*

By the definitions we have:

Corollary 3a. *If $f: M \to E^N$ is a tight embedding of a band or a Moebius band, then $M(z) \subset \overset{\circ}{M}$ and $(zf)_c \subset \overset{\circ}{M}$ are impossible for every z and c.*

Corollary 3b. *If f is as in Corollary 3a, then (clearly) $H_2(M, (zf)_c^-) = 0$ and hence i_* is injective in the exact sequence*

$$H_2(M) = 0 \longrightarrow H_2(M, (zf)_c^-) = 0 \longrightarrow H_1((zf)_c^-) \overset{i^*}{\longrightarrow} H_1(M) .$$

Proof of Lemma 3. If $M(z) \subset \overset{\circ}{M}$, then for some c, $M(z) \subset (-zf)_c \subset \overset{\circ}{M}$. Hence we only have to consider the case $\emptyset \neq (zf)_c \subset \overset{\circ}{M}$. In that case for $c' - c > 0$, but small, also $(zf)_{c'} \subset \overset{\circ}{M}$. Almost every unit covector $-w$ attains on the N-dimensional convex set $\mathcal{H}((zf)_{c'})$ its minimal value $-e$ at a unique point y of $(zf)_{c'}$ and for $-w$ near to z this point lies in the interior of $(zf)_{c'}$ and by TPP, then $(-wf)_{-e}$ is that interior point y of M. Then $(wf)_e^- = M \setminus \{y\}$ is homotopy equivalent to $M \vee S^{n-1}$. (To see this assume y "near" ∂M.)

3. Bands and Moebius bands

From now on $f: M \to E^N$ is a tight embedding of a band or a Moebius band M in E^N.

Lemma 4. *Every E^2-top*-set M' of f is a compact convex set F or a difference $F \setminus \overset{\circ}{D}$ obtained by deleting from F the interior of a compact convex set $D \subset F$.*

Remark. In the last case we call M' an *essential top*-set* because, as we will see, it carries an essential 1-cycle of M.

Proof. Let $F = \mathcal{H}(M')$. By Lemma 2 (with $p = 0$) we know $\partial F \subset M'$. If the whole convex set F is not in M', then $\partial F(\subset M')$ does not contract in M' and not even in some small open neighborhood U of M' in M. Then as in Lemma 2 one finds z and c such that the circle $\partial F \subset (zf)_c^-$ does contract neither in $(zf)_c^-$ nor in M because i_* in (2) is injective. Therefore ∂F represents a nonzero element of $H_1(M)$. If the open plane set $F \setminus M'$ has more than one com-

ponent, then we would find analogously two independent elements of $H_1((zf)_c^-)$, and $(zf)_c^-$ could not have the homotopy type of a point or circle. Hence $F \backslash M'$ is connected. Next suppose it is not convex. Let $z_{k+1} = 0$ be the equation of a line which contains the points p, q and r in this order, such that p and r are in $F \backslash M'$ and q is in M'. As p and r can be connected by an arc in $F \backslash M'$, it is seen that not both parts $z_{k+1} \geq 0$ and $z_{k+1} \leq 0$ of M' (which contain both q, and each one "half" of ∂F) can be connected. Hence for example the part M'' with equation $z_{k+1} \leq 0$ of M' has at least two components. Then f is not TPP by Lemma 1, and we have a contradiction. Consequently, $F \backslash M' = \mathring{D}$ is convex and Lemma 4 is proved.

Lemma 5. *Every E^3-top*-set $M' = M(z_1, \cdots, z_k)$ has exactly two essential top-sets.*

Proof. If no top-set of M' is essential, then all top-sets are convex and their union is $\partial \mathscr{H} M'$, a 2-sphere contained in M. This is impossible. Hence M' has *at least one essential top-set*, say $M'' = M(z_1, \cdots, z_{k+1}) = F \backslash \mathring{D}$ in the notation of Lemma 4. Suppose M' has no other essential top-set. Then the other top-sets of M' are all convex, and fill $\partial \mathscr{H} M' \backslash F$, an open disc in $M' \subset M$. Thus z_{k+1} takes a minimal value in an interior set in $M' \subset M$, which must be the top-set $M(-z_{k+1}) \subset \mathring{M}$. This contradicts Corollary 3a. Hence there are *at least two essential top-sets* on M'.

Now take two essential top-sets of M': $M_1 = F_1 \backslash \mathring{D}_1$ and $M_2 = F_2 \backslash \mathring{D}_2$. If the *embedded* circle ∂F_i does not represent a generator of M, then it must represent twice a generator and M must be a Moebius band. Thus ∂F_i bounds a Moebius band in M so that we can find some covector z which has constant value on ∂F_i and has a minimal value c such that $(zf)_c \subset \mathring{M}$ contrary to Corollary 3a. Therefore ∂F_1 and ∂F_2 carry a generator of $H_1(M)$, and we can fix a generator and use it to define an orientation in the circles ∂F_1 and ∂F_2 which are embedded in different planes $\alpha(F_1)$ and $\alpha(F_2)$ in $E^3 = \alpha(M')$. If ∂F_1 and ∂F_2 meet, then they meet in a connected part of the line of intersection of these M'-supporting planes. Therefore in M we find that $\partial F_1 \cup \partial F_2$ "bounds" either an embedded annulus V or an embedded disc V. V lies in E^3 because otherwise some linear function z which vanishes on E^3 would determine a compact set $(zf)_c$ in the interior of $V(\subset \mathring{M})$ contradicting Corollary 3a.

Next connect ∂F_1 and ∂F_2 in E^3 but outside $\mathscr{H} M'$ by a cylinder (handle) W to obtain a closed embedded surface $V \cup W$ in E^3. This surface is embedded in E^3, and therefore is orientable. Since it cannot be a Klein bottle, it is a torus, and it follows that ∂F_1 and ∂F_2 induce opposite orientations on the 2-sphere $\partial \mathscr{H} M'$ via the top-set discs F_1 and F_2. Now, if there would be at least three essential top-sets on M', they would give rise to three mutually opposite orientations on $\partial \mathscr{H} M'$, which is impossible. Hence Lemma 5 is proved.

Lemma 6. *Let M' be an E^3-top*-set of M with essential top-sets $F_1 \backslash \mathring{D}$ and $F_2 \backslash \mathring{D}_2$ in the notation of Lemma 4. Then*

$$\mathcal{H}(D_1 \cup D_2) = \mathcal{H}(M') = \mathcal{H}(F_1 \cup F_2) ,$$
$$\partial\mathcal{H}(D_1 \cup D_2) \setminus (\mathring{D}_1 \cup \mathring{D}_2) \subset M' .$$

Proof. By the other lemmas the second statement follows from the first. Clearly

$$\mathcal{H}(D_1 \cup D_2) = \mathcal{H}(D_1 \cup \partial D_2) \subset \mathcal{H}(M') .$$

If $\mathcal{H}(M') \neq \mathcal{H}(D_1 \cup D_2)$, then there is a linear function z which takes its minimal value c on a compact set in the interior of $\partial\mathcal{H}M' \setminus (D_1 \cup D_2) \subset \mathring{M}$. This contradicts Corollary 3a.

Lemma 7. *Every tight substantial embedding* $f \colon M \to E^N$, $N \geq 3$, *has at least two essential* E^2-top*-sets.

Proof. First observe that every E^j-top*-set M', with $j \geq 3$, does have a nonconvex top-set M'', because otherwise $M' \subset M$ contains the $(j-1)$-sphere $\partial\mathcal{H}M'$. If M'' is such an E^m-top*-set, then $m < j$. If also $m > 2$, we replace M' by M'', and repeat the argument. Whence we can continue and find an essential E^2-top*-set $M(z_1, \cdots, z_{k+1}) = M'' = F \setminus \mathring{D}$ (as in Lemma 4), top-set of some top*-set $M' = M(z_1, \cdots, z_k)$.

If M'' is the only nonconvex top-set of $M' = M(z_1, \cdots, z_k)$, then z_{k+1} takes a minimal value in the interior of the open disc $\partial\mathcal{H}M' \setminus F \subset \mathring{M}$. This contradicts Corollary 3a again, and hence Lemma 7 follows.

4. M-interior points on ∂F, and tight bands

For tight $f \colon M \to E^N$ we know that no top*-set is in $\mathring{M}$. Consider an essential E^2-top^k-set $M' = F \setminus \mathring{D}$. The top-sets of ∂F are top*-sets of f, which are either points, hence on ∂M, or straight segments whose endpoints are top*-sets hence on ∂M again. So, if a point $y \in \partial F$ is in $\mathring{M}$ ("is M-interior"), then it must be in the interior of a straight line segment on ∂F. Next let $s \subset \partial F$ be an open M-interior segment on ∂F with closure $\bar{s}$ whose endpoints p and q are on ∂M. The closure $\bar{s}$ cannot cut M into two parts, because otherwise we then find z and c such that $(-zf)_{-c}$ contains $\bar{s}$ but not much more of M, and $(zf)_c^-$ is not connected, contradicting TPP.

Assume now that M is a band and also that $M = f(M)$ is *not in a 2-plane*. Then s connects the two parts $\partial_1 M$ and $\partial_2 M$ of the boundary ∂M. Going back and forth we see that there must be an even number of such M-interior segments $s_1, \cdots, s_{2p}$ on the 1-sphere ∂F, whose union divides M into $2p$ parts. At most one (and at least one) of these parts can have points outside the span $\alpha(F)$ of F, again because otherwise we find some $(zf)_c^-$ with at least two components. Only the two segments s_i bounding that part (which we call V) can in fact be on the boundary ∂F, because the others are interior points in $\alpha(F) \cap M = F \setminus \mathring{D}$, Consequently $p \leq 1$. Observe also that $M \setminus V \subset \alpha(F)$. By considering s_1 and s_2 on the band M, we see that one of the two parts of the boundary,

say $\partial_1 M \subset M \backslash V \subset \alpha(F)$, is completely contained in F in case $p = 1$ as well as in case $p = 0$.

By Lemma 7 there are at least two E^2-top*-sets, say $F_1 \backslash \mathring{D}_1$ and $F_2 \backslash \mathring{D}_2$, in which we now find the two parts of ∂M: $\partial_1 M$ and $\partial_2 M$ respectively. Then $M \cup F_1 \cup F_2$ is an embedded two-sphere, and has the two-piece property because M is tight (compare Corollary 2a). Thus this set and therefore also M are in a 3-dimensional space E^3. With Lemma 4 we hence reach

Lemma 8. *All tight bands in euclidean spaces are those described in the examples of § 1 under* a) *and* b).

5. Tight Moebius bands in E^4

In this section $f: M \to E^N$ is a tight substantial embedding of the Moebius band and $N \geq 4$.

Lemma 9. *If $F \backslash \mathring{D}$ is an E^2-top*-set of a tight Moebius band in E^N, then ∂F consists of one straight open arc s of M-interior points and one closed arc $\bar{t} = \partial F \backslash s$ in the boundary ∂M: $\partial F = L = s \cup \bar{t}$, $s \cap \bar{t} = \emptyset$. (See Fig. 4.)*

Proof. If ∂F has no M-interior points, then $\partial F = \partial M$, and some covector z, which is constant on ∂F, attains its minimal value $c < 0$ on M on a set $(zf)_c \subset \mathring{M}$ contradicting Corollary 3a. We now follow the above arguments for D to obtain that the M-interior points of ∂F form a set of straight line segments $s_1, \cdots,$ s_{2p-1}; the number of the segments now is odd. Moreover, the arguments of D also give $p = 1$, which proves Lemma 9.

We next will study *the various intersection possibilities* between $L_1 = \partial F_1$ and $L_2 = \partial F_2$ concerning two different essential E^2-top*-sets $F_1 \backslash \mathring{D}_1$ and $F_2 \backslash \mathring{D}_2$. The embedded 1-spheres L_1 and L_2 carry the essential cycle of the Moebius band and so they meet: $L_1 \cap L_2 \neq \emptyset$. In the notation of Lemma 9 let

$$L_1 = s_1 \cup \bar{t}_1 , \qquad t_1 = \operatorname{int} \bar{t}_1 ,$$
$$L_2 = s_2 \cup \bar{t}_2 , \qquad t_2 = \operatorname{int} \bar{t}_2 .$$

Clearly, $s_1 \cap \bar{t}_2 \subset \mathring{M} \cap \partial M = \emptyset$, and also $\bar{t}_1 \cap s_2 = \emptyset$. Next we prove

Lemma 10. $s_1 \cap s_2 = \emptyset$.

Corollary 10'. $L_1 \cap L_2 = \bar{t}_1 \cap \bar{t}_2 \neq \emptyset$.

Proof. Suppose $s_1 \cap s_2 \neq \emptyset$. Suppose also, for some $i \geq 0$,

$$s_1 \cup s_2 \subset M(z_1, \cdots, z_{i-1}) \quad \text{and} \quad s_1 \subset M(z_1, \cdots, z_i) ,$$

and let $z_1, \cdots, z_i$ take the value 0 on $M(z_1, \cdots, z_i)$. Then $z_1, \cdots, z_{i-1}$ take value 0 at all points of s_2, and z_i takes only values ≤ 0 on $M(z_1, \cdots, z_{i-1})$ but the value 0 on $s_1 \cap s_2$.

Then $z_i = 0$ at all points of (the straight line segment) s_2. Applying this for $i = 0$ (clearly true), etc., we find that s_2 meets s_1 in a line segment. As both have their endpoints in M, s_1 and s_2 then *coincide*, i.e., $s_1 = s_2$, which leads to a

contradiction as follows. Clearly $\bar{t}_1 \cup \bar{t}_2 = \partial M$, and as the planes $\alpha(F_1)$ and $\alpha(F_2)$ have $s_1 = s_2$ in common, they span an E^3. This E^3 contains $\partial M = \bar{t}_1 \cup \bar{t}_2$. As $N \geq 4$, M does not lie in that E^3, and we find a covector z with constant values on E^3 and minimal value c on $(zf)_c \subset \mathring{M}$, contradicting Corollary 3a. Hence Lemma 10 is proved.

The union $\bigcup_z \mathscr{H}M(z)$ of the convex hulls of all top-sets is the convex envelope $\partial \mathscr{H}M \subset E^N$. The non-essential top-sets are equal to their convex hulls, and are contained in M. The essential top-sets have convex hulls which are convex parts of linear spaces E^j, $2 \leq j \leq N - 1$. *These convex hulls together with M have to fill homeomorphically $\partial \mathscr{H}M$ of dimension $N - 1 \geq 3$.* In case all top-sets are E^j-top-sets with $j \leq 2$, then M together with a finite number of convex hulls of essential E^2-top-sets is not enough to fill $\partial \mathscr{H}M$. Hence there are at least four different E^2-top*-sets: $F_i \backslash \mathring{D}_i$ and boundaries $L_i = s_i \cup \bar{t}_i$, $i = 1, \cdots, 4$.

With Lemma 10 we find that on the Moebius band the arcs $t_1, \cdots, t_4$ (in ∂M!) cannot be completely disjoint. Hence we may assume $t_1 \cap t_2 \neq \emptyset$ in this case.

Now suppose there does exist an E^j-top-set with $j \geq 3$. By the proof of Lemma 7 it contains at least two essential E^2-top*-sets $F_i \backslash \mathring{D}_i$ and boundaries $L_i = s_i \cup \bar{t}_i$, $i = 1, 2$. Suppose $t_1 \cap t_2 = \emptyset$. Then $L_1 \cap L_2 = \bar{t}_1 \cap \bar{t}_2$ is one point, say T, and all the convex hulls of top-sets that contain T are certainly not enough to fill $\partial \mathscr{H}M$. So there must be *other* essential E^j-top-sets. If all of these sets have $j = 2$, there must be many; if one has $j \geq 3$ it must carry two new essential E^2-top*-sets $F_i \backslash \mathring{D}_i$ with $\partial F_i = L_i = s_i \cup \bar{t}_i$, $i = 3, 4$. So in all cases we find L_i, $i = 1, \cdots, 4$, mutually different, and we may definitely assume $t_1 \cap t_2 \neq \emptyset$.

We now study the case where *the open set $t_1 \cap t_2$ is nonempty.* (See Fig. 5.) Then $\bar{t}_1 \cap \bar{t}_2$ is a segment which lies in the intersection of planes $\alpha(F_1) \cap \alpha(F_2)$ and hence on a line. At each end of the straight line segment $\bar{t}_1 \cap \bar{t}_2$, at least one of L_1 and L_2 continues inside $\mathring{M}$, and the other one of L_1 and L_2 continues with t_1 or t_2 respectively since by Lemma 9 there is exactly one straight line segment s_1 available on L_1 and one s_2 available on L_2 for that purpose. So we have the situation on the Moebius band given in Fig. 5. $M \backslash (L_1 \cup L_2)$ consists
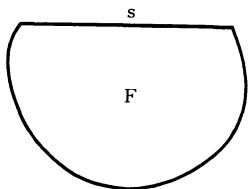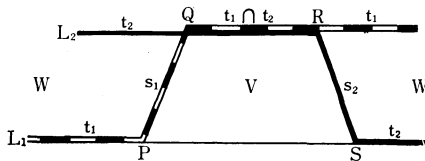

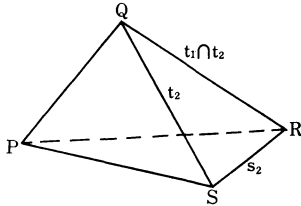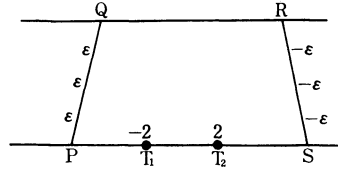Fig. 4   Fig. 5

Fig. 6

Fig. 7

of two components V and W with $\partial \overline{W} \subset L_1 \cup L_2 \subset E^3$. E^3 is now the span E^3 $= \alpha(L_1 \cup L_2) = \alpha(\{P, Q, R, S\})$.

We next show

Lemma 11. *W is completely contained in the 3-space* $\alpha(P, Q, R, S) = E^3$.

Proof. Suppose $T \in W$, and $T \notin E^3 \subset E^N$. Then some covector z, with constant values in E^3 and T, attains its minimal value c in the set $(zf)_c \subset \mathring{W} \subset \mathring{M}$, contradicting Corollary 3a.

Lemma 12. $M \cap E^3$ *is an* E^3-*top*-set, and* $N = 4$.

Proof. Suppose the covector z vanishes in E^3 and W, and takes positive as well as negative values on M, which is on V. The set of points in the closure $\overline{V}$, on which z takes its maximal (minimal) value on $\overline{V}$, does not meet the broken line $PQRS$, is not in the interior of M by Corollary 3a, and therefore is not in the interior of V by tightness. Hence a positive value as well as a negative value is taken on the remaining part of ∂V, a segment from P to S. Suppose then that P, T_1, T_2 and S are points in this order on that segment at which z takes values say 0, -2, 2, 0 respectively (0, 2, -2, 0 can be taken care of analogously). Thus z' near z and $\varepsilon > 0$ can be easily found such that:

On the straight line segment PQ: $z' = \varepsilon$; on the straight line segment RS: $z' = -\varepsilon$; moreover $z' < 0$ in T_1; and $z' > 0$ in T_2. See Fig. 7.

Moreover, it cannot be true that the part $z' < 0$ as well as the part $z' > 0$ of M are connected. Thus f is not tight, and we can conclude that the function z takes on V only values ≤ 0 (or ≥ 0). Therefore $z = 0$ is the equation of a supporting hyperplane.

If $N \geq 5$, then substantiality implies that we can find two points in M, which together with E^3 do not lie in a hyperplane, and we can find a z as above for which $z = 0$ separates these two points and therefore does not support; a contradiction. Consequently $N \leq 4$, and for $N = 4$, $z = 0$ is the equation of the top-set $M \cap E^3$. Hence the Lemma is proved in view of our assumption $N \geq 4$.

In the top set $M \cap E^3$ we have the convex sets F_1 and F_2: F_1 with boundary $\partial F_1 = L_1 = s_1 \cup \bar{t}_1$ consisting of straight line segments $s_1 = PQ$, $\bar{t}_1 \cap \bar{t}_2 = QR$ and a possibly curved part RP; F_2 with boundary $\partial F_2 = L_2 = s_2 \cup \bar{t}_2$ consisting of straight line segments $s_2 = SR$, $\bar{t}_1 \cap \bar{t}_2 = RQ$ and a possibly curved part QS. (See Fig. 8.)
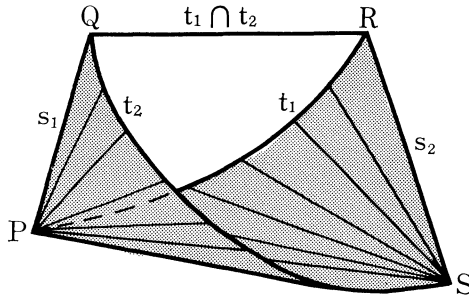
Fig. 8

By Lemmas 6 and 11, we obtain an open topological disc

$$\partial \mathcal{H}(F_1 \cup F_2) \setminus (F_1 \cup F_2) \subset M ,$$

which has the same boundary as W and hence equals $W \subset E^3$. The set of all convex hulls of all top sets of M, which contain the point S, together with M is not enough to fill the 3-sphere $\partial \mathcal{H} M$ homeomorphically. Therefore there must be some essential top*-set not containing S and not contained in $\alpha(P, Q, R, S)$. If this is an E^3-top*-set, it will contain two essential E^2-top*-sets, and at least one of them is not in the 3-plane $\alpha(P, Q, R, S)$.

Hence we obtain in any case an essential E^2-top*-set not containing S and not lying in $\alpha(P, Q, R, S)$. Call this one $F_3 \setminus \mathring{D}_3$, and let $\partial F_3 = L_3 = s_3 \cup \bar{t}_3$. Then $S \notin \bar{t}_3, \bar{t}_3 \not\subset \bar{t}_1 \cup \bar{t}_2$. As $\bar{t}_3 \cap \bar{t}_1 \neq \emptyset$, we find on the boundary ∂M that $\bar{t}_3 \cap \bar{t}_1$ must contain the whole segment PR of $\bar{t}_1$, which lies in the intersection of the two different 2-planes $\alpha(F_1)$ and $\alpha(F_2)$ and hence *is a straight line segment.* (See Fig. 6.) With t_1 and t_3 we now find a broken line $QPRT$ analogous to the broken line $PQRS$ with respect to t_1 and t_2, and in $\alpha(Q, P, R, T)$ we find an E^3-top-set analogous to the one in $\alpha(PQRS)$. The convex hulls of all top sets, which contain P, together with M are not enough to fill the 3-sphere $\partial \mathcal{H} M$. Hence there must be some essential top set not containing P. If this is an E^3-top-set, it contains two essential E^2-top-sets one of which is different from $F_3 \setminus \mathring{D}_3$. If there are only E^2-top*-sets, there must be many to fill $\partial \mathcal{H} M$ and we again choose one different from $F_3 \setminus \mathring{D}_3$.

Call the new E^2-top*-set $F_4 \setminus \mathring{D}_4$ with $\partial F_4 = L_4 = s_4 \cup t_4$. Then as above we conclude that $\bar{t}_4$ must contain the segment SQ of t_2 and therefore *also that the segment is straight.* Moreover, L_1 and L_2 are triangle boundaries, and $\overline{W}$ *is the union of the plane 2-simplices QPS and PSR.* From the intersection properties between $\bar{t}_i$ and $\bar{t}_j$ for $i, j = 1, \cdots, 4$ and the fact that if they would overlap too much they would lie in the same 2-plane, we now have the mutual situation on ∂M, which is divided in 5 segments, as in Fig. 9. The symbol t_i stands in the middle of the double segment t_i which covers $2/5$ of the circle for $i = 1, \cdots, 4$. Finally we consider all convex hulls of top sets which contain R and alone

do not fill $\partial\mathscr{H}M$ sufficiently. So there must be either some other E^3-top-set which contains two essential E^2-top*-sets one of which is not $F_4\backslash\mathring{D}_4$ and could be called $F_5\backslash\mathring{D}_5$, or some new E^2-top*-set to be called $F_5\backslash\mathring{D}_5$. In either case we find the missing link t_5 covering the interval PTS of ∂M.

By symmetry each of the five E^3-top-sets now obtained is of the kind describ-ed in $\alpha(P,Q,R,S)$. Besides the 2-simplices QPS and PSR which we had already, we first have the new 2-simplices PQT and QTR, and finally have the 2-simplex RTS to complete the Moebius band. Hence our theorem is proved.
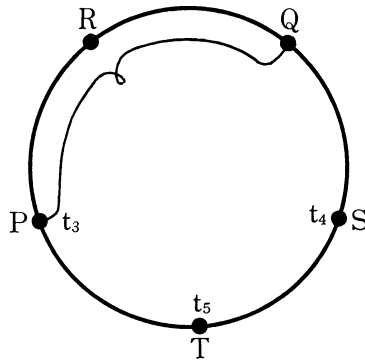

Fig. 9

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