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THE SPLITTING THEOREM FOR MANIFOLDS OF NONNEGATIVE RICCI CURVATURE

JEFF CHEEGER & DETLEF GROMOLL

The purpose of this paper is to extend Toponogov's splitting theorem [4], [7] for manifolds of nonnegative sectional curvature to manifolds of nonnegative Ricci curvature. By use of the extension we are able to show that our results on the structure of the fundamental group in the compact case and on locally homogeneous spaces, proved in [4] for manifolds of nonnegative sectional curvature, remain valid for manifolds of nonnegative Ricci curvature. In addition, we sharpen a result of Milnor on the rate of growth of the fundamental group in the noncompact case. As a final application, we show under fairly general circumstances (in particular, if M is locally irreducible) that the holonomy group of an arbitrary compact riemannian manifold is compact. By way of explanation, we remark that Berger has shown that the holonomy group of M is compact if M is locally irreducible and the Ricci curvature of M does *not* vanish identically. The case $\mathrm{Ric}_M \equiv 0$ is precisely the one we are able to handle. The last application was suggested during a conversation between the authors and L. Charlap.

Let M be a complete riemannian manifold. Recall that a ray (respectively a line) in M is a geodesic $\gamma\colon [0, \infty) \to M$ (respectively $\gamma\colon (-\infty, \infty) \to M$) each segment of which is minimal. With each ray γ in M we associate a function g_γ as follows: Let $g_t(x) = \overline{x, \gamma(t)} - t$ for $t \geq 0$ where the bar denotes metric distance. The function g_t is continuous, but not differentiable on the cut locus of $\gamma(t)$. It follows easily from the triangle inequality that the family g_t is uniformly equicontinuous. For fixed x, the function $t \to g_t(x)$ is decreasing on $[0, \infty)$ and bounded below by $-\overline{x, \gamma(0)}$. Hence, for $t \to \infty, g_t$ converges uniformly on compact sets to a continuous function g_γ.

Theorem 1. *If M has nonnegative Ricci curvature, then for any ray γ in M the function g_γ is superharmonic.*

Here superharmonic means that given any connected compact region D in M with smooth boundary ∂D, one has $g_\gamma \geq h_\gamma$ on D where h_γ is the continuous function on D which is harmonic on $\operatorname{int} D$ with $h_\gamma \,|\, \partial D = g_\gamma \,|\, \partial D$. Since this is true for all connected domains, a standard argument gives that if $h_\gamma(y) = g_\gamma(y)$ for $y \in \operatorname{int} D$, then $g_\gamma \equiv h_\gamma$ on D. If, moreover, the sectional curvature of M is

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nonnegative, then g_γ is a convex function; see [4]. Before giving the proof of Theorem 1 we will show how it implies our main result.

Theorem 2. *Let M be a complete manifold of nonnegative Ricci curvature. Then M is the isometric product $\overline{M} \times R^k$ where $\overline{M}$ contains no lines and R^k has its standard flat metric.*

Proof. By induction, it suffices to show that if M contains a line, then M splits isometrically as $M' \times R$.

Let γ be a line in M. Consider the rays $\gamma_+ = \gamma \,|\, [0, \infty)$ and γ_- with $\gamma_-(t) = \gamma(-t)$ and the corresponding superharmonic functions $g_+ = g_{\gamma+}, g_- = g_{\gamma_-}$. Now since γ is a line, for any t, s we have by the triangle inequality that

$$(1) \qquad \overline{x, \gamma(t)} - t + \overline{x, \gamma(-s)} - s \geq 0$$

with equality holding along $\gamma[-s, t]$. Hence

$$(2) \qquad g_+ + g_- \geq 0$$

with equality holding along γ. Taking D as above to be an arbitrary connected region containing $y \in \gamma(-\infty, \infty)$ in its interior we have

$$(3) \qquad g_+(y) + g_-(y) = 0 \, .$$

Let h_+, h_- be the continuous functions on D which are harmonic on int D with $h_+ \,|\, \partial D = g_+ \,|\, \partial D$ and $h_- \,|\, \partial D = g_- \,|\, \partial D$. Since $h_+ + h_- \,|\, \partial D$ is nonnegative, we have also that $h_+(y) + h_-(y) \geq 0$ by the minimum principle for harmonic functions. Now $g_+ \geq h_+$ and $g_- \geq h_-$, so we must have $g_+(y) = h_+(y)$ and $g_-(y) = h_-(y)$. Then on D, $g_+ \equiv h_+$ and $g_- \equiv h_-$. Since D is arbitrary, we have shown that g_+, g_- are differentiable and harmonic on M.

We prove next that $\|\operatorname{grad} g_+\| \equiv 1$ and the integral curves of grad g_+ are geodesics. For fixed x, y we have

$$(4) \qquad \begin{aligned} |g_t(x) - g_t(y)| &= |\overline{\gamma(t), x} - t - \overline{\gamma(t), y} + t| \\ &= |\overline{\gamma(t), x} - \overline{\gamma(t), y}| \leq \overline{x, y} \, . \end{aligned}$$

Letting $t \to \infty$, we obtain $|g_+(x) - g_+(y)| \leq \overline{x, y}$. It follows that $\|\operatorname{grad} g_+\| \leq 1$. On the other hand, given x let σ_i denote a minimal geodesic from x to $\gamma(i)$. Let i_j be a sequence such that $\sigma_{i_j}'(0) \to \sigma'(0)$. Then for all y on σ, we have $|g_+(x) - g_+(y)| = \overline{x, y}$. It follows that $\|\operatorname{grad} g_+\| = 1$ and that σ is the integral curve of grad g_+ through x.

Finally, set grad $g_+ = N$ and let $E_1, \cdots, E_{n-1}$, N be an orthonormal frame in a neighborhood of x which is parallel along σ. Then $\nabla_N N = 0$ and at x,

$$
\begin{aligned}
\operatorname{Ric}(N) &= \sum_{i=1}^{n-1} \langle R(E_i, N)N, E_i\rangle \\
&= \sum_{i=1}^{n-1} \langle \nabla_{E_i}\nabla_N N - \nabla_N \nabla_{E_i} N - \nabla_{[E_i,N]}N, E_i\rangle \\
&= \sum_{i=1}^{n-1} -\langle \nabla_N \nabla_{E_i} N, E_i\rangle - \langle \nabla_{\nabla_{E_i}N} N, E_i\rangle \qquad (5)\\
&= \sum_{i=1}^{n-1} -N\langle \nabla_{E_i} N, E_i\rangle - \sum_{1\le i,j\le n-1} \langle \nabla_{E_i}N, E_j\rangle\langle \nabla_{E_j}N, E_i\rangle \\
&= \sum_{i=1}^{n-1} N\langle N, \nabla_{E_i}E_i\rangle - \|\nabla N\|^2 = -N(\Delta g_+) - \|\nabla N\|^2 = -\|\nabla N\|^2 .
\end{aligned}
$$

Since $\operatorname{Ric}_{(N)} \geq 0$ it follows that $\operatorname{Ric}_{(N)} \equiv \|\nabla N\| \equiv 0$, which means that N is parallel. Hence by the de Rham decomposition theorem, M splits off a line locally isometrically. The splitting is easily seen to be global and is given by the level surfaces and integral curves of g_+. This completes the proof of Theorem 2.

The proof of Theorem 1 requires some lemmas.

Lemma 1. *If $F\colon M \to R$ is any differentiable function and p is not a critical point, then the Laplacian $\Delta F(x)$ is given by $-m(F) + N(N(F))$ where m is the mean curvature vector of the level surface through x and* $N = \operatorname{grad} F/\|\operatorname{grad} F\|$.

Proof. Let $E_1, \cdots, E_{n-1}, N$ be a frame field in a neighborhood of p. Then

$$\Delta F(p) = \sum_i E_iE_i(F) - \nabla_{E_i}E_i(F) + N(N(F)) - \nabla_N N(F) . \tag{6}$$

Since $\{E_i\}$ and $\nabla_N N$ are tangent to the level surfaces and $\sum \nabla_{E_i}E_i = m$,

$$E_iE_i(F) = 0 , \tag{7}$$

$$- \sum \nabla_{E_i}E_i(F) = -m(F) , \tag{8}$$

$$\nabla_N N(F) = 0 , \tag{9}$$

and the lemma follows.

Lemma 2. *Let M have nonnegative Ricci curvature. Then for $p \in M$ and x not on the cut locus of p, we have $\Delta \rho_p(x) \leq (n-1)/\rho_p(x)$ where $\rho_p(x) = \overline{x, p}$.*

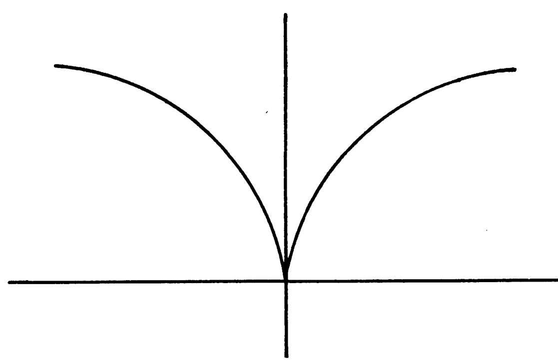
Proof. Let $\sigma\colon [0, l] \to M$ be the minimal geodesic of lenghth $l = \overline{x, p}$ from p to x, and $\{J_i\}$ be the unique Jacobi fields vanishing at $\sigma(0)$ such that $J_i(l) = E_i(l)$ where $E_1, \cdots, E_{n-1}, N = \sigma'$ are a parallel frame field along σ. Then we have

$$
\begin{aligned}
\frac{n-1}{l} &= \int_0^l \sum_{i=1}^{n-1} \left\langle \nabla_N\left(\frac{t}{l}E_i\right), \nabla_N\left(\frac{t}{l}E_i\right)\right\rangle \\
&\geq \int_0^l \sum_{i=1}^{n-1} \left\langle \nabla_N\left(\frac{t}{l}E_i\right), \nabla_N\left(\frac{t}{l}E_i\right)\right\rangle - \left\langle R\left(N, \frac{t}{l}E_i\right)\frac{t}{l}E_i, N\right\rangle .
\end{aligned} \tag{10}
$$

By the fundamental inequality for the index form, (10) yields

$$
\begin{aligned}
\frac{n-1}{l} &\geq \int_0^l \sum_{i=1}^{n-1} \langle \nabla_N J_i, \nabla_N J_i \rangle - \langle R(N, J_i) J_i, N \rangle \\
&= \int_0^l \sum_{i=1}^{n-1} \langle \nabla_N J_i, \nabla_N J_i \rangle + \langle \nabla_N \nabla_N J_i, J_i \rangle \\
&= \int_0^l \sum_{i=1}^{n-1} N \langle \nabla_N J_i, J_i \rangle = - \int_0^l \sum_{i=1}^{n-1} N \langle \nabla_{J_i} J_i, N \rangle \\
&= - \sum_{i=1}^{n-1} \langle \nabla_{E_i} E_i, N \rangle = - \sum_{i=1}^{n-1} \nabla_{E_i} E_i(\rho_p) = \Delta \rho_p \,.
\end{aligned} \tag{11}
$$

Before going any further with the details we will try to give an intuitive explanation of what is really going on. Lemma 2 certainly suggests that g_γ, which is the limit of the functions $g_t = \rho_{\gamma(t)} - t$, should have nonpositive Laplacian and hence be superharmonic. The difficulty is that the functions g_t are not differentiable on the cut locus of $\gamma(t)$, and g_γ may not be differentiable anywhere. Even if g_γ were differentiable almost everywhere with $\Delta g_\gamma \leq 0$, we might have the situation depicted below.



$y = f(x) = x^{2/3}$,
$f'' < 0$ almost everywhere, but f is not superharmonic.

This example also suggests the method for overcoming the above difficulty, namely, to look at the local behavior of the gradient near the points of non-differentiability; see c′) in the proof of Theorem 1.

Lemma 3. *For g_t as above and any compact set $K \subset M$, there exist a sequence of C^∞-functions g_t^i and a constant L such that on K,*

a) $g_t^i \xrightarrow{\text{unif}} g_t$,

b) $\|dg_t^i\| < L$ *for all* i,

and on any compact subset of K, on which dg_t exists,

c) $dg_t^i \xrightarrow{\text{unif}} dg_t$.

Proof. Let $U_1, \cdots, U_N$ be a coordinate covering of K, and $\{\varphi_i\}$ a partition of unity subordinate to $\{U_i\}$. By the triangle inequality, the functions $\varphi_i g_t$ satisfy $|\varphi_i g_t(x) - \varphi_i g_t(y)| \leq \overline{x, y}$. Then there exists a constant L_1 such that on any U_i, $|\varphi_i g_t(x) - \varphi_i g_t(y)| \leq L_1 \|x - y\|_i$ where $\| \ \|_i$ denotes Euclidean distance on U_i. The theorem now follows by applying standard approximation techniques (convolution with an approximate identity) to the functions $\varphi_i g_t$.

Proof of Theorem 1. Let D be a connected region with smooth boundary ∂D, and $f_y(x)$ be the Greens function for Δ on D with singularity at $y \in \operatorname{int} D$ and $f|\partial D \equiv 0$. Then f_y satisfies

α) for fixed y, $\Delta f_y(x) \equiv 0$ on $\operatorname{int} D - y$,

β) $\displaystyle\lim_{r\to 0} \int_{S_r(y)} \langle \operatorname{grad} f_y, N\rangle dA = 1,$

where $S_r(y)$ denotes the boundary of the metric ball $B_r(y)$ of radius r about y, and N the unit normal pointing into $B_r(y)$,

γ) $\displaystyle\lim_{r\to 0} \int_{S_r(y)} f_y \cdot dA = 0,$

δ) $f|\operatorname{int} D - y > 0$.

For fixed t, g_t is differentiable on $D - C_t$ where C_t denotes the cut locus of $\gamma(t)$ which is known to be a closed set of measure zero. Using the Fubini Theorem, one may easily construct sequences of smooth compact regions $B_{r,i,t} \subset \operatorname{int} D_{i,t} \subset D$ such that $\lim_{i\to\infty} \partial B_{r,i,t} = S_r(y)$, $\lim_{i\to\infty} \partial D_{i,t} = \partial D$, and the $(n-1)$-dimensional measures of $C_t \cap \partial B_{r,i,t}$ and $C_t \cap \partial D_{i,t}$ are both equal to zoro. For fixed t, i we may choose a sequence of regions $R_{i,j,t}$ such that

a′) $\bigcap_{j=1}^{\infty} R_{i,j,t} = C_t \subset \operatorname{int} R_{i,j,t}$,

b′) $\partial R_{i,j,t}$ is smooth and transversal to $\partial B_{r,i,t}$ and $\partial D_{i,t}$ for all j,

c′) for all j, $\operatorname{grad} g_t$ points into $R_{i,j,t}$.

a′) and b′) are straightforward. c′) may be seen by noting that $\operatorname{grad} g_t$ is the image under the exponential map of the radial vector field in $M_{\gamma(t)}$ and hence points inward towards C_t. Property c′) is a key point in the argument.

Now consider the region $D_{i,r,j,t} = D_{i,t} - B_{r,i,t} - R_{i,j,t}$. Its boundary, except for $(n-2)$-dimensional sets, is the disjoint union of

$$A = \partial D_{i,t} \cap D_{i,r,j,t}\,, \quad B = \partial B_{r,i,t} \cap D_{i,r,j,t}\,, \quad C = \partial R_{i,t,j} \cap D_{i,r,j,t}\,.$$

Let g_t^k be a sequence of functions as in Lemma 3. Assuming that A, B, C are oriented properly, by Stokes Theorem we have

$$\int_A g_t^k * df_y + \int_B g_t^k * df_y + \int_C g_t^k * df_y = \int_{D_{i,r,j,t}} dg_t^k \wedge * df_y + \int_{D_{i,r,j,t}} g_t^k \wedge d * df_y\,, \tag{12}$$

$$\int_A f_y * dg_t + \int_B f_y * dg_t + \int_C f_y * dg_t = \int_{D_{i,r,j,t}} df_y \wedge * dg_t + \int_{D_{i,r,j,t}} f_y \wedge d * dg_t \ . \tag{13}$$

Since on functions $*d*d = \Delta$, by α) the second term on the right hand side of (12) vanishes. By Stokes Theorem the third term on the left hand side of (12) may be rewritten as

$$\int_C g_t^k * df_y = \int_{R_{i,j,t} \cap D_{i,t}} dg_t^k \wedge * df_y + g_t^k \wedge d * df_y - \int_{R_{i,j,t} \cap \partial D_{i,t}} g_t^k * df_y - \int_{R_{i,j,t} \cap \partial B_{r,i,t}} g_t^k * df_y \ . \tag{14}$$

Then by α) and Lemma 3,

$$\left| \int_C g_t^k * df_y \right| \leq L \cdot L' \cdot V(R_{i,j,t} \cap D_{i,t}) + \int_{R_{i,j,t} \cap \partial D_{i,t}} \|g_t^k * df_y\| \, dA + \int_{R_{i,j,t} \cap \partial B_{r,i,t}} \|g_t^k * df_y\| \, dA \ , \tag{15}$$

where $L' = \max_{D_{i,r,j,t}} \|* df_y\|$, and $V(\quad)$ denotes volume. Letting $k \to \infty$ and then $j \to \infty$ yields

$$\lim_{j \to \infty} \int_C g_t * df_y = 0 \ . \tag{16}$$

Then by letting $j \to \infty$, (12) becomes

$$\int_{\partial D_{i,t}} g_t * df_y + \int_{\partial B_{r,i,t}} g_t * df_y = \int_{D_{i,t} - B_{r,i,t}} \langle dg_t, df_y \rangle dV \ . \tag{17}$$

Now letting N denote the outward normal to the boundary of $D - B_r(y)$ and letting $i \to \infty$, (17) implies

$$\int_{S_r(y)} g_t \langle \operatorname{grad} f_y, N \rangle dA + \int_{\partial D} g_t \langle \operatorname{grad} f_y, N \rangle dA = \int_{D - B_r(y)} \langle dg_t, df_y \rangle dV \ . \tag{18}$$

Letting N denote the outward normal to the region $D_{i,r,j,t}$, (13) may be rewritten as

$$\int_A f_y\langle \operatorname{grad} g_t, N\rangle dA + \int_B f_y\langle \operatorname{grad} g_t, N\rangle dA + \int_C f_y\langle \operatorname{grad} g_t, N\rangle dA \tag{19}$$
$$= \int_{D_{i,r,j,t}} \langle df_y, dg_t\rangle dV + \int_{D_{i,r,j,t}} f_y\cdot \Delta g_t\cdot dV\ .$$

By c′), we see that the third term on the left hand side of (19) is *positive*. Thus

$$\int_A f_y\langle \operatorname{grad} g_t, N\rangle dA + \int_B f_y\langle \operatorname{grad} g_t, N\rangle dA \tag{20}$$
$$\leq \int_{D_{i,r,j,t}} \langle df_y, dg_t\rangle dV + \int_{D_{i,r,j,t}} f_y\cdot \Delta g_t\cdot dV\ .$$

Since $\|\operatorname{grad} g_t\| = 1$ wherever $\operatorname{grad} g_t$ is defined, the second term on the left hand side of (20) is $\geq -\int_B f_y\cdot dA$. Now letting $i\to\infty$, the first term on the left hand side of (20) approaches 0 ($f_y\,|\,\partial D\equiv 0$) giving

$$-\int_{S_r(y)} f_y dA \leq \int_{D-B_r(y)} \langle df_y, dg_t\rangle dV + \int_{D-B_r(y)} f_y\cdot \Delta g_t\cdot dV\ . \tag{21}$$

Substituting (21) in (18) we have

$$\int_{S_r(y)} g_t\langle N, \operatorname{grad} f_y\rangle\, dA + \int_{\partial D} g_t\langle N, \operatorname{grad} f_y\rangle dA \tag{22}$$
$$\geq -\int_{S_r(y)} f_y dA - \int_{D-B_r(y)} f_y\cdot \Delta g_t\cdot dV\ .$$

Letting $t\to\infty$ and using Lemma 2 we obtain

$$\int_{S_r(y)} g_\gamma\langle N, \operatorname{grad} f_y\rangle dA + \int_{\partial D} g_\gamma\langle N, \operatorname{grad} f_y\rangle dA \geq -\int_{S_r(y)} f_y\cdot dA\ . \tag{23}$$

Letting $r\to 0$ and using γ) yield

$$g_\gamma(y) + \int_{\partial D} g_\gamma\langle N, \operatorname{grad} f_y\rangle dA \geq 0\ . \tag{24}$$

Now for any harmonic function h, a simpler more standard version of the above yields

$$h(y) + \int_{\partial D} h\langle N, \operatorname{grad} f_y\rangle dA = 0 \;. \tag{25}$$

In particular, if $h = h_\gamma$ has the same boundary values as g_γ, then (24), (25) imply

$$h_\gamma(y) \leq g_\gamma(y) \;, \tag{26}$$

which completes the proof of Theorem 1.

As in the case of nonnegative sectional curvature, we now have the following structure theorem for the fundamental group in the compact case.

Theorem 3. *Let M be a compact manifold of nonnegative Ricci curvature. Then $\pi_1(M)$ contains a finite normal subgroup ψ such that $\pi_1(M)/\psi$ is a finite group extended by $\overbrace{Z \oplus \cdots \oplus Z}^{k}$, and $\tilde{M}$, the universal covering of M, splits isometrically as $\bar{M} \times R^k$ where $\bar{M}$ is compact*[1].

Proof. By Theorem 2, we may write the universal covering space $\tilde{M}$ isometrically as $\bar{M} \times R^k$ where $\bar{M}$ contains no line. The covering transformations $\pi \simeq \pi_1(M)$ are of the form $(f, g)(x, y) = (f(x), g(y))$ where $f\colon \bar{M} \to \bar{M}$ and $g\colon R^k \to R^k$ are isometric. Let ρ be the projection of $\tilde{M}$ on the first factor in $\bar{M} \times R^k$, and K be a compact fundamental domain for π which exists by compactness of M. Then the orbit $\rho(K)$ under $\rho(\pi)$ is all of $\bar{M}$. We claim $\bar{M}$ must be compact, otherwise there exist a ray γ and a sequence $g_n \in \rho(\pi)$ such that $g_n^{-1}(\gamma(n)) \in \rho(K)$. By compactness we find a subsequence g_{n_i} such that $dg_{n_i}^{-1}(\gamma'(n_i)) \to v$, a tangent vector at $p \in \rho(K)$. If $\sigma\colon (-\infty, \infty) \to M$ is the geodesic with $\sigma'(0) = v$, then σ is easily seen to be a line. The rest of the argument follows word for word from the proof of Theorems 9.1 and 9.2 of [4].

Theorem 3 generalizes classical results of Bochner [2] and Myers [6] as well as a recent result of Milnor [5] in the compact case. Although the noncompact case is still open, we have the following sharpening of Milnor's theorem.

Theorem 4. *Let M be complete and $\mathrm{Ric}_M \geq 0$. Then every finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree $\leq n$, and there exists a subgroup for which equality holds if and only if M is compact and flat.*

Proof. $\tilde{M} = \bar{M} \times R^k \ni (\bar{m}, 0)$ where $\bar{M}$ does not contain a line. Let C_d, $d > 0$, denote the closed set of points x in $\bar{M}$ with the property that for every geodesic $\gamma\colon [0, \infty) \to \bar{M}$ such that $\gamma(0) = \bar{m}$, $C_d \ni x = \gamma(t)$ implies either $t \leq d$ or $\gamma\,|\,[0, s]$ is not minimal for $s > t + d$. We claim there exists d such that

$$\pi(\bar{m}, 0) \subset C_d \times R^k \;, \tag{27}$$

which implies for all r

[1] For more detailed results which also remain valid in case $\mathrm{Ric}_M \geq 0$, see [4].

$$\pi(\bar{m},0) \cap B_r(\bar{m},0) \subset \pi(\bar{m},0) \cap [B_{2r}(\bar{m}) \times B_{2r}(0)] \subset \pi(\bar{m},0) \cap [(B_{2r}(\bar{m}) \cap C_d) \times B_{2r}(0)] \,. \tag{28}$$

In addition,

$$V([B_{2r}(\bar{m}) \cap C_d] \times B_{2r}(0)) < Kr^{n-1} \,. \tag{29}$$

In fact, $B_{2r} \cap C_d$ is easily seen to be the image under $\exp_{\bar{m}}$ of the union of a piece of an annular region and a ball whose volume in $\bar{M}_{\bar{m}}$ is $\leq K_1(d\cdot(2r)^{\dim \bar{M}-1} + d^{\dim \bar{M}})$ where K_1 is the volume of the unit sphere in Euclidean space of dimension $\dim \bar{M}$. Then (29) follows from the fact that the exponential map does not increase volume for manifolds of nonnegative Ricci curvature. To see (27), note that as in Theorem 3, any element of π can be written as (f, g). Now, if there exists a sequence (f_i, g_i) such that $f_i(\bar{m})$ lies on a segment γ_i with $\gamma_i(t_i) = f_i(\bar{m})$ such that $\gamma_i | [t_i - i, t_i + i]$ is minimal, then $f_i^{-1} \circ \gamma_i | [-i, i]$ is minimal with $(f_i^{-1} \circ \gamma_i)(0) = \bar{m}$. Taking an accumulation point of the tangent directions of these segments would produce a line in $\bar{M}$ and hence a contradiction. Given (28) and (29), the proof now follows as in [5].

We now treat the structure of locally homogeneous spaces.

Theorem 5. *Let M be complete and locally homogeneous with* $\mathrm{Ric}_M \geq 0$. *Then M is isometric to a flat vector bundle over a compact locally homogeneous space S. S and hence M admit locally homogeneous metrics of nonnegative sectional curvature.*

Proof. $\tilde{M}$ has a transitive group of isometries $I(\tilde{M})$. From the argument of Theorem 3 it follows that $\tilde{M} = \bar{M} \times R^k$ where $\bar{M}$ is compact homogeneous. Then $I(\tilde{M}) = I(\bar{M}) \times I(R^k)$, and the compact group $I(\bar{M})$ preserves a normal homogeneous metric of nonnegative sectional curvature. Since $\pi \subset I(\bar{M}) \times I(R^k)$, M also admits a locally homogeneous metric of nonnegative sectional curvature, and therefore, by [4], M is isometrically a flat vector bundle over a locally homogeneous space S whose inverse image in $\tilde{M}$ is of the from $\bar{M} \times R^l$, $l \leq k$. Hence M is also a flat vector bundle over S with respect to the original metric.

The following application arose during a conversation with L. Charlap.

Theorem 6. *Let M be compact and suppose $\tilde{M} = \hat{M} \times R^l$ isometrically. If either $\hat{M}$ is compact or $l \leq 1$, then the holonomy group Φ of M is compact.*

Proof. Write $\tilde{M}$ isometrically as $M_1 \times \cdots \times M_k \times M_R \times R^l$ where $M_1, \cdots, M_k$ are irreducible, M_R is maximal non-Euclidean Ricci-flat, and R^l is maximal Euclidean. Identify $\pi_1(M, m)$ with the group of isometric covering transformations. Then the projection of π on M_R is a group of isometries which is transitive modulo compact sets, and by the argument of the proof of Theorem 3 we conclude that M_R is compact. Now the identity component Φ^0 of the holonomy group at m may be naturally identified with the holonomy group of $\tilde{M}$ at $\tilde{m} \in \pi^{-1}(m)$. Φ^0 acts reducibly on the direct sum decomposition $T_1 \oplus \cdots T_k \oplus T_R \oplus T_l$ of $\tilde{M}_{\tilde{m}}$ corresponding to the product decomposition and irreducibly on each factor. Since there are only finitely many factors, it follows that Φ contains a normal subgroup $\bar{\Phi}$ of finite index which preserves the decomposition of $\tilde{M}_{\tilde{m}}$.

Since $\bar{\Phi}/\Phi^0$ is contained in the product of the component groups of the restrictions $\rho_i\bar{\Phi}$ to the various factors, it suffices to prove that these are finite. $\rho_i\Phi^0 \mid T_i$ $(i = 1, \cdots, k)$ is the holonomy group of M_i and, by a result of Berger [1], has finite index in its normalizer. Actually, Berger only checks this in case M_i is not a symmetric space, but the symmetric case is also known; see, for example, [8]. The elements of Φ restricted to T_R may be represented as $d\rho_R(h)^{-1} \circ P_{\rho_R(c)}$ where c is a curve from $\tilde{m}$ to $h(\tilde{m})$ for some $h \in \pi$, and P denotes parallel translation. Clearly, to prove the finiteness of $\rho_R(\bar{\Phi})/\rho_R(\Phi^0)$, it suffices to prove that the group $\rho_R(\pi)$ is finite. However, since M_R is compact, if $\rho_R(\pi)$ were infinite, then M_R would carry a non-trivial Killing field which would be harmonic (since $\mathrm{Ric}_{M_R} \equiv 0$) and hence would contradict the fact that M_R is simply connected. Now, if $l \leq 1$, clearly the component group of $\bar{\Phi}$ restricted to R^l is finite. If $M_1 \times \cdots M_k \times M_R$ is compact and l is arbitrary, then by the argument of Theorem 3, the restriction of π to R^k is seen to be a Bieberbach group. In either case the theorem follows.

It would be interesting to know whether there are actually examples of non-flat complete manifolds with $\mathrm{Ric}_M \equiv 0$.

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State University of New York, Stony Brook