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ISOMETRIC IMMERSIONS OF RIEMANNIAN PRODUCTS

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Introduction

For each integer i, $1 \leq i \leq p$, let M_i be a compact connected riemannian manifold of dimension $n_i \geq 2$, and M the riemannian product $M_1 \times M_2 \times \cdots \times M_p$. In this paper we will prove that any codimension p isometric immersion of M in euclidean space is a product of hypersurface immersions. This means that if $f: M \to E^N$ is an isometric immersion into euclidean space E^N of dimension $N = \left(\sum_{i=1}^{p} n_i\right) + p$, then there exist isometric immersions $f_i: M_i \to E^{n_i+1}$ $(1 \leq i \leq p)$ and a decomposition of E^N into a riemannian product

$$E^N = E^{n_1+1} \times \cdots \times E^{n_p+1}$$

so that $f(m_1, m_2, \cdots, m_p) = (f_1(m_1), f_2(m_2), \cdots, f_p(m_p))$ when $m_i \in M_i$ for $1 \leq i \leq p$. This generalizes a result of S. B. Alexander [1] which dealt with codimension two isometric immersions.

As an application, we mention that if S^2 is the two-dimensional sphere of constant curvature one, then it follows from Liebmann's theorem that the riemannian product $S^2 \times S^2 \times \cdots \times S^2$ (p times) is globally rigid in $3p$-dimensional euclidean space E^{3p}; it is clearly not locally rigid. Very few global rigidity theorems in codimensions higher than one are known, and this example is perhaps the simplest.

Unless otherwise stated all riemannian manifolds are C^∞ and connected, and we use [5] as a reference for the basic theorems of riemannian geometry. The author sincerely thanks Professors M. P. do Carmo and S. Kobayashi for their encouragement, and the referee for several valuable suggestions. The main results in this paper were included in the author's thesis written under the direction of Professor Kobayashi at the University of California, Berkeley.

1. Statement of results

If M is a riemannian manifold, let $F(M)$ denote the interior of the set of points of M at which all sectional curvatures vanish. Our first result is proven by local methods:

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Theorem 1. *For $1 \leq i \leq p$, let M_i be a connected riemannian manifold of dimension n_i such that $F(M_i) = \emptyset$, and let M_0 be a connected flat riemannian manifold of dimension n_0. Let $M_0 \times M_1 \times \cdots \times M_p$ be the riemannian product, and E^N a euclidean space of dimension $N = \left(\sum_{i=0}^{p} n_i\right) + p$. Then any isometric immersion*

$$f: M_0 \times M_1 \times \cdots \times M_p \to E^N$$

is a product immersion.

In the terminology of O'Neill [6], f is n_0-cylindrical; if $n_0 = 0$, f is the product of hypersurface immersions. Theorem 1 allows us to construct examples of complete noncompact riemannian manifolds which can be locally but not globally isometrically immersed in a euclidean space of a given dimension. For example, the following corollary follows from Theorem 1 and a theorem of Efimov [3]:

Corollary 1. *For $1 \leq i \leq p$, let M_i be a complete two-dimensional riemannian manifold whose curvature is bounded above by a negative constant, and let $N = 3p$. Then the riemannian product $M_1 \times M_2 \times \cdots \times M_p$ cannot be globally isometrically immersed in E^N.*

On the other hand, it follows from [4] that M can be locally isometrically immersed in E^N.

Our second result (with no assumptiom on the curvature) is proven by global methods:

Theorem 2. *For $1 \leq i \leq p$, let M_i be a complete connected riemannian manifold of dimension $n_i \geq 2, M = M_1 \times M_2 \times \cdots \times M_p$ the riemannian product, and E^N a euclidean space of dimension $N = \left(\sum_{i=1}^{p} n_i\right) + p$. Then any isometric immersion $f: M \to E^N$ satisfies at least one of the following conditions:* (a) *It is a product of hypersurface immersions.* (b) *It carries a complete geodesic onto a straight line in E^N.*

In particular, if M is compact, then any isometric immersion of M in E^N is a product immersion, which proves the assertion we made in the introduction. A direct consequence of Theorem 2 and the rigidity theorems of Cohn-Vossen and Sacksteder [7] is the following corollary:

Corollary 2. *For $1 \leq i \leq p$, let M_i be a compact connected riemannian manifold of nonnegative curvature and dimension $n_i \geq 2, M = M_1 \times M_2 \times \cdots \times M_p$ the riemannian product, and E^N a euclidean space of dimension $N = \left(\sum_{i=1}^{p} n_i\right) + p$. Then any isometric immersion of M in E^N is rigid.*

2. The main lemma

Consider two riemannian manifolds M_1 and M_2 of dimensions n_1 and n_2 respectively, and suppose that the riemannian product $M = M_1 \times M_2$ is isometrically immersed in an N-dimensional euclidean space E^N. Let $n = n_1 + n_2$ and adopt the following conventions on ranges of indices:

$$1 \leq A, B, C \leq N ;$$
$$1 \leq i, j, k \leq n ; n + 1 \leq \lambda, \mu, \nu \leq N ;$$
$$1 \leq a, b, c \leq n_1 ; n_1 + 1 \leq r, s, t \leq n .$$

In an open neighborhood U in E^N of a point in M choose a moving orthonormal frame $e_1, e_2, \cdots, e_N$ so that at every point (m_1, m_2) in $M \cap U = (M_1 \times M_2) \cap U$, the first n_1 frame vectors $e_1, e_2, \cdots, e_{n_1}$ are tangent to $M_1 \times \{m_2\}$ and the next n_2 vectors $e_{n_1+1}, \cdots, e_n$ are tangent to $\{m_1\} \times M_2$. It follows that the remaining $N-n$ vectors are normal to $M_1 \times M_2$. Let $\theta^1, \theta^2, \cdots, \theta^N$ be the dual coframe, and θ^A_B the corresponding connection forms defined by the equations

$$de_A = \sum_B e_B \theta^B_A . \tag{1}$$

Then $\theta^A_B = -\theta^B_A$, and one can easily derive the following structure equations of Cartan:

$$d\theta^A = - \sum_B \theta^A_B \wedge \theta^B ,$$
$$d\theta^A_B = - \sum_C \theta^A_C \wedge \theta^C_B .$$

We restrict all the differential forms to the submanifold $M \cap U$, and denote the restrictions by the same letters, so that $\theta^\lambda = 0$. From the structure equations it follows in the usual fashion that

$$\theta^\lambda_i = \sum_j b^\lambda_{ij} \theta^j ,$$

where b^λ_{ij} is a differentiable function on $M \cap U$ for each i, j, λ, and $b^\lambda_{ij} = b^\lambda_{ji}$. The symmetric bilinear forms

$$\Phi^\lambda = \sum_{i,j} b^\lambda_{ij} \theta^i \otimes \theta^j$$

are called the second fundamental forms.

We will say that a moving orthonormal frame constructed as above is *compatible with the product structure* of M, if $\theta^a_b = 0$ when restricted to $(\{m_1\} \times M_2) \cap U$ and if $\theta^r_s = 0$ when restricted to $(M_1 \times \{m_2\}) \cap U$ for all $(m_1, m_2) \in U$. Such a moving frame can always be constructed if U is a sufficiently small open neighborhood of any point in $M_1 \times M_2$.

We will now prove the following assertion: If any moving frame choosen in the above manner satisfies the equations $b^\lambda_{ar} = 0$, then the isometric immersion of $M = M_1 \times M_2$ in E^N is a product immersion.

The first step in the proof of the assertion goes as follows: we take a vector X tangent to $M_1 \times \{m_2\}$ at $(m_1, m_2) \in M$ and a vector Y tangent to $\{m_1'\} \times M_2$ at $(m_1', m_2') \in M$, and show that they are perpendicular in E^N. Let σ be a minimizing geodesic in $\{m_1\} \times M_2$ connecting (m_1, m_2) to (m_1, m_2'), and τ a minimizing geodesic in $M_1 \times \{m_2'\}$ connecting (m_1', m_2') to (m_1, m_2'). We can then construct a moving frame $e_1, e_2, \cdots, e_N$ compatible with the product structure of M and defined on an open set in E^N, which contains both σ and τ. We claim that $e_1, e_2, \cdots, e_{n_1}$ are constant along σ, and $e_{n_1+1}, \cdots, e_n$ are constant along τ. Indeed equation (1) implies that

$$de_a = \sum_b e_b \theta_a^b + \sum_\lambda e_\lambda \theta_a^\lambda ,$$

because $\theta_a^r = 0$ for a product manifold. But θ_a^b vanishes when restricted to $\{m_1\} \times M_2$, and our assumption that $b_{ar}^\lambda = 0$ implies that $\theta_a^\lambda = \sum_b b_{ab}^\lambda \theta^b$ so that θ_a^λ also vanishes on $\{m_1\} \times M_2$. It follows that de_a vanishes on $\{m_1\} \times M_2$ and consequently e_a is constant along σ; in a similar fashion one shows that e_r is constant along τ. Now choose constants x^a, y^r so that $X = \sum_a x^a e_a(m_1, m_2)$ and $Y = \sum_r y^r e_r(m_1', m_2')$. Then $X = \sum_a x^a e_a(m_1, m_2')$ and $Y = \sum_r y^r e_r(m_1, m_2')$ which proves that X and Y are perpendicular in E^N.

If we choose an origin in E^N we can regard E^N as a vector space. Let E_1 be the subspace generated by all vectors tangent to $M_1 \times \{m_2\}$ for all $m_2 \in M_2$, and E_2 the subspace generated by all vectors tangent to $\{m_1\} \times M_2$ for all $m_1 \in M_1$. The preceding paragraph proves that E_1 and E_2 are orthogonal, and consequently we can choose a subspace E_0 of E^N so that

$$E^N = E_0 \oplus E_1 \oplus E_2$$

is an orthogonal direct sum decomposition. If E_0, E_1, E_2 are regarded as riemannian manifolds, then E^N is their riemannian product. Let p_0, p_1, and p_2 be the natural projections of E^N on E_0, E_1, and E_2 respectively.

If $m_2 \in M_2$, then the isometric immersion $f\colon M_1 \times M_2 \to E^N$ defines an isometric immersion $f_{m_2} : M_1 \to E^N$ by $f_{m_2}(m_1) = f(m_1, m_2)$. Let f_1 be the composition $p_1 \circ f_{m_2}$. We now show that f_1 is independent of the choice of $m_2 \in M_2$. Let m_1 be an arbitrary fixed point of M_1, let m_2, m_2' be two points of M_2, and let σ be a path in $\{m_1\} \times M_2$ joining (m_1, m_2) to (m_1, m_2'). The tangent vector of σ always lies in E_2, and therefore its endpoints have the same projection on E_1. This proves that f_1 is well-defined. Define $f_2\colon M_2 \to E_2$ in a similar fashion. Since $p_0 \circ f$ is constant, $f(m_1, m_2) = (\text{constant}, f_1(m_1), f_2(m_2))$ and hence f is a product immersion, which proves our assertion. In all our applications, E_0 will be the zero subspace.

By a straightforward induction on the number of manifolds considered, the above argument yields the following lemma:

Lemma. *Suppose that $M_1, M_2, \cdots, M_p$ are connected riemannian manifolds and that*

$$f\colon M_1 \times M_2 \times \cdots \times M_p \to E^N$$

is an isometric immersion of the riemannian product. If the second fundamental forms Φ^λ have the property that

$$\Phi^\lambda(X, Y) = 0\ , \quad \text{when } X \text{ is tangent to } M_i\ , \quad Y \text{ is tangent to } M_j, \text{ and } i \neq j\ , \tag{2}$$

then f is a product immersion.

3. Proof of Theorem 1

For reference, we state an algebraic theorem due to E. Cartan, a proof of which can be found in [2]. If $B^1, B^2, \cdots, B^n$ are n symmetric bilinear forms on a real vector space V, they are said to be *exteriorly orthogonal* if

$$\sum_{i=1}^{n} [B^i(X, Y)B^i(Z, W) - B^i(X, W)B^i(Z, Y)] = 0$$

for all vectors $X, Y, Z, W \in V$. Cartan's theorem states that if $B^1, B^2, \cdots, B^n$ are n exteriorly orthogonal symmetric bilinear forms and the codimension of the subspace $\{X \in V \mid B^i(X, Y) = 0 \text{ for all } Y \in V, 1 \leq i \leq n\}$ is at least n, then $B^1, B^2, \cdots, B^n$ can be diagonalized simultaneously. The theorem implies that n exteriorly orthogonal symmetric bilinear forms depend upon at most n variables.

To prove Theorem 1, it clearly suffices to establish (2) on a dense subset of $M = M_0 \times M_1 \times \cdots \times M_p$, and we will establish it at every point m in M at which no $M_i, i > 0$, is flat. The point m will be fixed throughout the proof.

We first consider the special case where M_0 is a point, and each M_i is two-dimensional for $i > 0$. Then we can choose an orthonormal basis $\{e_1, e_2, \cdots, e_{2p}\}$ for the tangent space of M at m so that e_1, e_2 are tangent to M_1, e_3 and e_4 are tangent to $M_2, \cdots$, and e_{2p-1}, e_{2p} are tangent to M_p. Let $\{\theta^1, \theta^2, \cdots, \theta^{2p}\}$ be the dual basis for the cotangent space, and k_i the curvature of M_i. In addition to the second fundamental forms $\Phi^\lambda, 2p + 1 \leq \lambda \leq 3p$, we will need to consider the symmetric bilinear forms $\Psi^i, 1 \leq i \leq p$, defined as follows:

$$\Psi^i = \begin{cases} \sqrt{-k_i}(\theta^{2i-1} \otimes \theta^{2i-1} + \theta^{2i} \otimes \theta^{2i}) & \text{if} \quad k_i < 0\ , \\ \sqrt{k_i}(\theta^{2i-1} \otimes \theta^{2i-1} - \theta^{2i} \otimes \theta^{2i}) & \text{if} \quad k_i > 0\ . \end{cases}$$

Then the $2p$ symmetric bilinear forms $\{\Phi^\lambda, \Psi^i\}$ are exteriorly orthogonal. Therefore Cartan's theorem implies that there exists a vector space basis $\{v_1, v_2, \cdots, v_{2p}\}$ for the tangent space of M at m, which diagonalizes all of these symmetric bilinear forms simultaneously. It follows that the null space of any of these symmetric bilinear forms is the vector space spanned by a subset of $\{v_1, v_2, \cdots, v_{2p}\}$. Since the space of vectors tangent to M_i is the intersection of the null spaces of $\Psi^j, j \neq i$, it is also spanned by a subset of this basis. Hence the basis $\{v_1, v_2, \cdots, v_{2p}\}$ is consistent with the product structure of M, and after a permutation if necessary, we can arrange that v_1, v_2 are tangent to M_1, v_3 and v_4 are tangent to M_2, etc. If X is a vector tangent to M_i at m and Y is a vector tangent to $M_j, i \neq j$, then X is a linear combination of v_{2i-1} and v_{2i}, and Y is a linear combination of v_{2j-1} and v_{2j}. Since the basis $\{v_1, v_2, \cdots, v_{2p}\}$ diagonalizes the second fundamental forms Φ^λ, it follows that $\Phi^\lambda(X, Y) = 0$, which proves that (2) holds at m in this special case.

Now we can consider the general case, and choose a set of orthonormal vectors $\{e_1, e_2, \cdots, e_{2p}\}$ at m so that e_{2l-1} and e_{2l} form a basis for a two-plane of nonzero sectional curvature tangent to M_l, $1 \leq l \leq p$. We can assume without loss of generality that $\sum_\lambda \Phi^\lambda(e_{2i}, e_{2i})e_\lambda$ is nonzero for $1 \leq i \leq p$. If we restrict the second fundamental forms Φ^λ to the vector space generated by $\{e_1, e_2, \cdots, e_{2p}\}$ and apply the algebraic argument of the preceding paragraph, we find that $\Phi^\lambda(e_{2i}, e_{2j}) = 0$, for $i \neq j$. Since the sectional curvature of the two-plane spanned by e_{2i} and e_{2j} is zero when $i \neq j$, it follows that

$$\sum_\lambda \Phi^\lambda(e_{2i}, e_{2i})\Phi^\lambda(e_{2j}, e_{2j}) = 0 , \qquad i \neq j . \tag{3}$$

Hence the normal vectors $\{\sum_\lambda \Phi^\lambda(e_{2i}, e_{2i})e_\lambda = \Phi(e_{2i}, e_{2i}) \mid 1 \leq i \leq p\}$ are orthogonal to each other and form a basis for the normal space. Similarly we can conclude that

$$\sum_\lambda \Phi^\lambda(e_{2i-1}, e_{2i})\Phi^\lambda(e_{2j}, e_{2j}) = 0 , \qquad \sum_\lambda \Phi^\lambda(e_{2i-1}, e_{2i-1})\Phi^\lambda(e_{2j}, e_{2j}) = 0$$

for $i \neq j$, from which it follows that the normal vectors $\Phi(e_{2i-1}, e_{2i})$ and $\Phi(e_{2i-1}, e_{2i-1})$ are scalar multiples of $\Phi(e_{2i}, e_{2i})$. After a rotation of e_{2i-1} and e_{2i}, we can arrange that $\Phi(e_{2i-1}, e_{2i}) = 0$, so that the normal vectors $\{\Phi(e_{2i-1}, e_{2i-1}) \mid 1 \leq i \leq p\}$ also form a basis for the normal space.

Suppose that X is a vector tangent to M_i, where $0 \leq i \leq p$. If $j \neq i$, the curvature relations imply that

$$\sum_\lambda \Phi^\lambda(X, e_{2j})\Phi^\lambda(e_{2k-1}, e_{2k-1}) = 0 , \qquad \text{for } 1 \leq k \leq p ,$$

so that $\Phi^\lambda(X, e_{2j}) = 0$. Since the two-plane splanned by X and e_{2j} has zero curvature, it follows that

$$\sum_\lambda \Phi^\lambda(X, X)\Phi^\lambda(e_{2j}, e_{2j}) = \sum_\lambda \Phi^\lambda(X, e_{2j})\Phi^\lambda(X, e_{2j}) = 0 , \qquad \text{for } j \neq i ,$$

so that the normal vector $\Phi(X, X)$ is zero if $i = 0$, or a multiple of $\Phi(e_{2i}, e_{2i})$ if $i > 0$. Similarly, if Y is a vector tangent to M_j at m, where $j \neq i$ and $j > 0$, then $\Phi(Y, Y)$ is a multiple of $\Phi(e_{2j}, e_{2j})$. Using equation (3), we see that

$$\sum_\lambda \Phi^\lambda(X, Y)\Phi^\lambda(X, Y) = \sum_\lambda \Phi^\lambda(X, X)\Phi^\lambda(Y, Y) = 0 ,$$

so that $\Phi^\lambda(X, Y) = 0$. This proves that (2) holds at m and concludes the proof of Theorem 1.

4. Proof of Theorem 2

We begin by mentioning some facts about the index of relative nullity which will be needed in the proof. Let M be an n-dimensional riemannian manifold, and $f\colon M \to E^N$ an isometric immersion, which defines $N - n$ second fundamental forms Φ^λ, $\lambda = n + 1, \cdots, N$. A vector X tangent to M at a point $m \in M$ is said to be a *relative nullity vector* for f or a vector in the relative nullity space $N(f, m)$ if $\Phi^\lambda(X, Y) = 0, n + 1 \leq \lambda \leq N$, for every vector Y tangent to M at m. The dimension of the vector space $N(f, m)$ is called the *index of relative nullity* of f at m. We will need to use the following two lemmas:

Lemma 1. *If U is an open subset of M on which the index of relative nullity is constant, then the distribution $N(f)$ of relative nullity spaces is completely integrable on U, and its integral submanifolds are totally geodesic.*

Lemma 2. *Suppose that the U is an open subset of M on which the index of relative nullity is equal to the constant ρ, $\sigma\colon [a, b] \to M$ is a geodesic segment such that $\sigma(s) \in U$ for $s \in (a, b)$, and the tangent vector $\sigma'(s)$ lies in the relative nullity space $N(f, \sigma(s))$ for $s \in (a, b)$. Then the index of relative nullity of f at $\sigma(a)$ and $\sigma(b)$ is equal to ρ, and the distribution of relative nullity spaces is parallel along $\sigma | [a, b]$.*

These lemmas are proven in [1] as Lemma 5.1 and Theorem 6.2 respectively. The second lemma states that the index of relative nullity cannot increase as one moves along a geodesic whose tangent vector is a relative nullity vector.

To prove Theorem 2, we assume that (b) does not hold, i.e., that M does not contain a complete geodesic which f takes onto a straight line in E^N, and we prove that (a) holds by means of an induction. Let $A_\alpha(M)$ denote the set of points in M at which the index of relative nullity is α. The proof of Theorem 1 shows that (2) holds on the closure of $A_0(M)$, and we will prove the following inductive step: if (2) holds on the closure of $\cup \{A_\beta(M) | \beta < \alpha\}$, then (2) holds on the closure of $\cup \{A_\beta(M) | \beta < \alpha + 1\}$.

Let $\alpha \geq 1$ and let U be the open set

$$A_\alpha(M) - \mathrm{Cl}\,[\cup \{A_\beta(M) | \beta < \alpha\}] ,$$

where Cl denotes closure, a set on which the index of relative nullity is equal to the constant α. Let m be a point of U, and σ: $(a, b) \to U$ a unit speed geodesic passing through m whose tangent vector $\sigma'(s)$ is a relative nullity vector for each $s \in (a, b)$. Assume that σ cannot be extended beyond the interval (a, b) without leaving U. Either $a > -\infty$ or $b < +\infty$ because otherwise f would take the complete geodesic σ onto a straight line in E^N, and we assume without loss of generality that $b < +\infty$. We notice that by Lemma 2, $\sigma(b)$ lies in the closure of $\cup \{A_\beta(M) \mid \beta < \alpha\}$, a set on which (2) holds by inductive hypothesis. At $\sigma(b)$ the relative nullity space is the direct sum of its projections on the M_i's, and Lemma 2 implies that this also holds at m. Since m is an arbitrary point of U, the distribution of relative nullity spaces on U is consistent with the product structure of M.

Again we choose a point m in U and let σ: $(a, b) \to U$ be a unit speed geodesic passing through m such that $\sigma'(s)$ is a relative nullity vector for each $s \in (a, b)$. This time, however, we require that σ be tangent to some M_i (say M_1). As before we can assume that σ cannot be extended beyond (a, b) without leaving U and that $b < +\infty$. We can choose a moving orthonormal frame $e_1, e_2, \cdots, e_N$ in a neighborhood V of $\sigma \mid (a, b)$ in M so that $e_1, e_2, \cdots, e_{n_1}$ are tangent to M_1 (where $n_1 = \dim M_1$), $e_{n_1+1}, \cdots, e_n$ are tangent to $M_2 \times M_3 \times \cdots \times M_p$ (where $n = \dim M$), and $e_{n+1}, \cdots, e_N$ are normal to M. Moreover, we can assume that $e_1, e_2, \cdots, e_n$ are parallel along σ with respect to the connection in the tangent bundle, that $e_{n+1}, \cdots, e_N$ are parallel along σ with respect to the connection in the normal bundle, and that at points of σ, e_1 is the tangent vector of σ. Finally by the argument of the preceding paragraph we can arrange that e_1 be a relative nullity vector at every point in V. We adopt the following index conventions:

$$1 \leq i, j, k \leq n \ ; \qquad n + 1 \leq \lambda, \mu, \nu \leq N \ ;$$
$$1 \leq a, b, c \leq n_1 \ ; \qquad n_1 + 1 \leq r, s, t \leq n \ .$$

As usual, we can construct a dual coframe $\theta^1, \cdots, \theta^n$, connection forms $\theta^i_j, \theta^\lambda_\mu$, and components b^λ_{ij} of the second fundamental forms. We notice that $\theta^r_a = 0$ and $b^\lambda_{1j} = 0$.

The covariant derivatives b^λ_{ijk} of the second fundamental forms are defined by the equation

$$db^\lambda_{ij} + \sum_\mu b^\mu_{ij}\theta^\lambda_\mu - \sum_k b^\lambda_{kj}\theta^k_i - \sum_k b^\lambda_{ik}\theta^k_j = \sum_k b^\lambda_{ijk}\theta^k \ , \tag{4}$$

and we recall that the b^λ_{ijk}'s are symmetric in their lower indices. Since $b^\lambda_{1j} = 0$ on V,

$$- \sum_b b^\lambda_{br}\theta^b_1(e_a) = b^\lambda_{1ra} = b^\lambda_{ar1} = e_1(b^\lambda_{ar})$$

at points of σ, so that we have the following system of ordinary differential equations along $\sigma | (a, b)$:

$$e_1(b^\lambda_{ar}) + \sum_b b^\lambda_{br}\theta^b_1(e_a) = 0 \ .$$

Since $b^\lambda_{ar} = 0$ at $\sigma(b)$, it follows that $b^\lambda_{ar} = 0$ at m. Therefore, if X is a vector tangent to M_1 at m and Y is a vector tangent to M_i at m, for some $i \neq 1$, then $\Phi^\lambda(X, Y) = 0$ for all λ, $n + 1 \leq \lambda \leq N$.

The argument of the preceding paragraph can be used to show that $b^\lambda_{ar} = 0$ on all of V so that equation (4) implies that $b^\lambda_{ark} = 0$. Hence at points of σ,

$$e_1(b^\lambda_{rs}) = b^\lambda_{rs1} = 0 \ ,$$

so that b^λ_{rs} is constant along $\sigma | (a, b)$. Therefore, if X is a vector tangent to M_i at m and Y is a vector tangent to M_j at m, where $i, j, 1$ are distinct, and if $\bar{X}$, $\bar{Y}$ are the parallel translates of X, Y along σ to $\sigma(b)$, then $\Phi^\lambda(X, Y) = \Phi^\lambda(\bar{X}, \bar{Y})$, $n+1 \leq \lambda \leq N$. Since (2) holds at $\sigma(b)$, it follows that $\Phi^\lambda(X, Y) = 0$, $n + 1 \leq \lambda \leq N$. Together with the result of the preceding paragraph, this shows that (2) holds at m. Since m was an arbitrary point of U it follows that (2) holds on the closure of U and hence on the closure of $\cup \{A_\beta(M) | \beta < \alpha + 1\}$. The inductive step is completed and Theorem 2 is proven.

5. Concluding remarks

As a first step toward generalizations in higher codimensions, we mention the following result:

Proposition. *For $1 \leq i \leq p$, let M_i be a connected riemannian manifold of dimension $n_i \geq 2$ and constant negative curvature k_i, and E^N a euclidean space of dimension $N = 2\left(\sum_{i=1}^p n_i\right) - p$. Then any isometric immersion of the riemannian product*

$$f\colon M_1 \times M_2 \times \cdots \times M_p \to E^N$$

is a product immersion.

The dimension N is the lowest such that local isometric immersions of $M = M_1 \times M_2 \times \cdots \times M_p$ in E^N exist.

The proof proceeds along the following lines: Let $s(j) = \sum_{i=1}^j n_i$ and $n = \sum_{i=1}^p n_i$, and choose a moving orthonormal frame $e_1, e_2, \cdots, e_N$ so that the first n_1 frame vectors are tangent to M_1, the next n_2 frame vectors are tangent to $M_2, \cdots$, and the last $N - n$ frame vectors are normal to M. Let $\theta^1, \theta^2, \cdots, \theta^N$ be the dual coframe. Then the second fundamental forms Φ^λ and the symmetric bilinear forms

$$\Psi^i = \sqrt{-k_i}(\theta^{s(i-1)+1} \otimes \theta^{s(i-1)+1} + \cdots + \theta^{s(i)} \otimes \theta^{s(i)}) ,$$

$i \leq i \leq p$, are exteriorly orthogonal in the tangent space of M at any point $m \in M$. Therefore we can apply Cartan's theorem on exteriorly orthogonal forms to conclude that there is a vector space basis $\{v_1, v_2, \cdots, v_n\}$ for the tangent space of M at m, which diagonalizes the symmetric bilinear forms $\{\Phi^\lambda, \Psi^i\}$ simultaneously. A straightforward algebraic argument shows that the basis is orthonormal and consistent with the product structure of M. In other words, after a permutation if necessary, $v_1, v_2, \cdots, v_{n_1}$ are tangent to $M_1, v_{n_1+1}, \cdots,$ $v_{s(2)}$ are tangent to M_2, etc. One can then verify in the usual fashion that (2) holds at every point of M.

Theorem 1 and the above proposition suggest the following conjecture: If M_1 (resp. M_2) is a riemannian manifold which can be locally isometrically immersed in a euclidean space of dimension n_1 (resp. n_2) but in no lower-dimensional euclidean space, then every isometric immersion of $M_1 \times M_2$ in an $(n_1 + n_2)$-dimensional euclidean space is a product immersion.

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