J. DIFFERENTIAL GEOMETRY
4 (1970) 457-467

SUBMANIFOLDS OF CODIMENSION 2 WITH CERTAIN PROPERTIES

MASAFUMI OKUMURA

Introduction

H. Liebmann [5] has proved that the only ovaloid with constant mean curvature in a 3-dimensional Euclidean space is a sphere. Various generalizations of this theorem have been obtained recently. Y. Katsurada [1], [2] and K. Yano [9] have generalized this theorem to a hypersurface of a Riemannian manifold admitting an infinitesimal conformal or homothetic transformation. A generalization of the theorem to the case of codimension greater than 1 was first tried by the present author [8] when the enveloping Riemannian manifold is an odd dimensional sphere. In [8], the present author made full use of the natural contact structure on the sphere.

On the other hand Y. Katsurada [3], [4], H. Kōjyo [3], T. Nagai [4] and K. Yano [10] studied this problem when the enveloping manifold admits an infinitesimal conformal transformation. They made full use of the existence of an infinitesimal conformal transformation, and proved that under some conditions the submanifold in consideration is umbilical only with respect to the mean curvature normal. In the present paper the author studies the same problem as that in [3], [4], [10] and proves that under certain conditions the submanifold in consideration is not only umbilical with respect to the mean curvature normal but also is totally umbilical.

In § 1 we recall formulas for the submanifolds of codimension 2 in a Riemannian manifold which will be used in the sequel.

In § 2 we define a certain intrinsic normal vector field and consider some properties of the normal bundle. In § 3 we derive some integral formulas for a compact submanifold of codimension 2 in a Riemannian manifold admitting an infinitesimal conformal transformation. Using these formulas, we establish, in § 4, a certain generalization to the most general form of Liebmann's theorem above stated. In the last § 5 we study submanifolds of codimension 2 of a sphere or a Euclidean space.

1. Submanifolds of codimension 2 in a Riemannian manifold

Let M^n be an n-dimensional orientable differentiable manifold, and ι be an

Received June 3, 1969.

immersion of M^n into an $(n+2)$-dimensional Riemannian manifold $\tilde{M}^{n+2}$. Then the Riemannian metric $\tilde{g}$ of $\tilde{M}^{n+2}$ induces naturally a Riemannian metric g on M^n by the immersion ι in such a way that

$$g(X, Y) = \tilde{g}(d\iota(X), d\iota(Y)) ,$$

where we denote by $d\iota$ the differential map of ι, and by X, Y vector fields in M^n. In order to simplify the presentation we identify, for each point $p \in M^n$, the tangent space $T_p(M)$ with $d\iota(T_p(M)) \subset T_{\iota(p)}(M)$ by means of the immersion.

Since M^n is orientable, if we assume that $\tilde{M}^{n+2}$ is also orientable, in a certain neighborhood U of $p \in M^n$ we can choose two fields of mutually orthogonal unit normal vectors C and D of M^n at each point of U in such a way that, if $(B_1, \cdots, B_n)_p$ is a positively oriented frame of tangent vectors at p then the frame $(d\iota(B_1), \cdots, d\iota(B_n), C, D)_{\iota(p)}$ is also positively oriented.

Let X, Y be tangent to M^n. Then the covariant derivative of $d\iota(X)$ in the direction of $d\iota(Y)$ is expressed as

$$\tilde{\nabla}_{d\iota(Y)} d\iota(X) = \nabla_Y X + h(X, Y)\, C + k(X, Y) D . \tag{1.1}$$

Although $\nabla_Y X$ denotes the tangential components of $\tilde{\nabla}_{d\iota(Y)} d\iota(X)$, it is easily verified that $\nabla_Y X$ is identical with the covariant derivative of X in the direction of Y with respect to the induced Riemannian metric g.

The tensors h and k of type (0.2) over M^n are called the second fundamental tensors of M^n in $\tilde{M}^{n+2}$ with respect to the normal vectors C and D respectively. Since the Riemannian connections $\tilde{\nabla}$ and ∇ are both torsionless we easily see that h and k are symmetric.

The normal vectors C and D are unit vectors, and so we can put

$$\tilde{\nabla}_X C = - A(X) + l(X) D , \qquad \tilde{\nabla}_X D = - A'(X) - l(X) C , \tag{1.2}$$

where $A(X)$ and $A'(X)$ denote the tangential components of $\tilde{\nabla}_X C$ and $\tilde{\nabla}_X D$ on M^n respectively, and l is the third fundamental form of M^n in $\tilde{M}^{n+2}$.

Let $X, Y \in T_p(M)$. Then we have the equations of Weingarten:

$$\tilde{g}(\tilde{\nabla}_X C, Y) = - h(X, Y) , \qquad g(\tilde{\nabla}_X D, Y) = - k(X, Y) . \tag{1.3}$$

Let $\{x^i\}$, $i = 1, \cdots, n$, be local coordinates in an open neighborhood U' of $p \in M^n$. The set of vector fields $(\partial/\partial x^1, \cdots, \partial/\partial x^n)$ is called the natural frame of M^n, and spans the tangent plane of M^n at each point of U'. We choose a positively oriented frame $(B_1, \cdots, B_n, C, D)$, where $B_i = d\iota(\partial/\partial x^i)$, $i = 1, \cdots, n$, at each point of the neighborhood $U \cap \iota(U')$ of $\iota(p) \in \tilde{M}^{n+2}$. Then $A(X)$ and $A'(X)$ are represented as linear combinations of B_i, $i = 1, \cdots, n$, and consequently we get, by (1.2),

$$\tilde{\nabla}_{B_j} C = - \sum_{i=1}^{n} H_j{}^i B_i + L_j D , \qquad \tilde{\nabla}_{B_j} D = - \sum_{i=1}^{n} K_j{}^i B_i - L_j C , \tag{1.4}$$

where L_j denotes $l(B_j)$. Thus by virtue of (1.3) we have

$$
\begin{aligned}
H_{ji} &\overset{\text{def}}{=} h(\partial/\partial x^j, \partial/\partial x^i) = -\tilde{g}(\tilde{\nabla}_{B_j}C, B_i) \\
&= \sum_{i=1}^{n} H_j{}^k \tilde{g}(B_k, B_i) = H_j{}^k g_{ki} ,
\end{aligned} \tag{1.5}
$$

$$
\begin{aligned}
K_{ji} &\overset{\text{def}}{=} k(\partial/\partial x^j, \partial/\partial x^i) = -\tilde{g}(\tilde{\nabla}_{B_j}D, B_i) \\
&= \sum_{i=1}^{n} K_j{}^i \tilde{g}(B_k, B_i) = K_j{}^k g_{ki} ,
\end{aligned} \tag{1.6}
$$

where g_{ji} denotes $g(\partial/\partial x^j, \partial/\partial x^i)$, and we use Einstein's summation convention for simplicity.

Let $\tilde{R}$ and R be curvature tensors of $\tilde{M}^{n+2}$ and M respectively. Then the equations of Gauss, Mainardi-Codazzi and Ricci-Kühne are respectively given by

$$
\begin{aligned}
\tilde{g}(\tilde{R}(B_k, B_j)B_i, B_h) = R_{kjih} &- H_{ki}H_{kh} + H_{ki}H_{jh} \\
&- K_{ji}K_{kh} + K_{ki}K_{jh} ,
\end{aligned} \tag{1.7}
$$

$$
\begin{aligned}
\tilde{g}(\tilde{R}(B_k, B_j)B_i, C) &= \nabla_k H_{ji} - \nabla_j H_{ki} - L_k K_{ji} + L_j K_{ki} , \\
\tilde{g}(\tilde{R}(B_k, B_j)B_i, D) &= \nabla_k K_{ji} - \nabla_j K_{ki} + L_k H_{ji} - L_j H_{ki} ,
\end{aligned} \tag{1.8}
$$

and

$$
\tilde{g}(\tilde{R}(B_k, B_j)C, D) = \nabla_k L_j - \nabla_j L_k - K_{ki}H_j{}^i + K_{ji}H_k{}^i , \tag{1.9}
$$

where

$$
R_{kjih} = g(R(\partial/\partial x^k, \partial/\partial x^j)\partial/\partial x^i, \partial/\partial x^h) ,
$$

and ∇_j denotes the operation of covariant differentiation in classical tensor calculus.

2. Submanifolds and some vector fields

Let g^{ji} be the inverse matrix of g_{ji}, and put $H_r{}^r = g^{jr}H_{rj}$, $K_r{}^r = g^{jr}K_{rj}$. Then the vector H defined by

$$
H = \frac{1}{n}(H_r{}^r C + K_r{}^r D) \tag{2.1}
$$

is independent of the choice of mutually orthogonal unit normal vectors of M^n, and so defines a vector field along M^n. We call this vector field the mean curvature vector field along M^n with respect to $\tilde{M}^{n+2}$. Next putting $H^{ji} = g^{jr}H_r{}^i, K^{ji} = g^{jr}K_r{}^i$, we consider a vector

$$(2.2)\quad W = (K_r{}^r H_{ji}K^{ji} - H_r{}^r K_{ji}K^{ji})C + (H_r{}^r H_{ji}K^{ji} - K_r{}^r H_{ji}H^{ji})D \;.$$

It can be easily verified that W is also independent of the choice of mutually orthogonal unit normal vectors of M^n, and so defines a vector field along M^n.

When at each point of M^n there exist functions h' and k' such that $h(X, Y) = h'g(X, Y), k(X, Y) = k'g(X, Y)$ or equivalently

$$(2.3)\qquad H_{ji} = h'g_{ji} \;, \qquad K_{ji} = k'g_{ji} \;,$$

we call M^n a totally umbilical submanifold. From this definition, if M^n is totally umbilical we have

$$(2.4)\qquad h' = \frac{1}{n}H_r{}^r \;, \qquad k' = \frac{1}{n}K_r{}^r \;.$$

Proposition 2.1. *A necessary and sufficient condition for a submanifold of codimension* 2 *to be umbilical is that the following equations are satisfied*:

$$(2.5)\qquad H_{ji}H^{ji} = \frac{1}{n}(H_i{}^i)^2 \;, \qquad K_{ji}K^{ji} = \frac{1}{n}(K_i{}^i)^2 \;.$$

Proof. This follows from the identities

$$\left(H_{ji} - \frac{H_r{}^r}{n}g_{ji}\right)\left(H^{ji} - \frac{H_r{}^r}{n}g^{ji}\right) = H_{ji}H^{ji} - \frac{1}{n}(H_r{}^r)^2 \;,$$

$$\left(K_{ji} - \frac{K_r{}^r}{n}g_{ji}\right)\left(K^{ji} - \frac{K_r{}^r}{n}g^{ji}\right) = K_{ji}K^{ji} - \frac{1}{n}(K_i{}^i)^2 \;,$$

and the positive definiteness of the Riemannian metric g_{ji}.

Proposition 2.2. *The vector field W vanishes identically if the submanifold is totally umbilical.*

Proof. Substituting (2.3) and (2.4) into (2.2), we get $W = 0$. This completes the proof.

Next we consider the normal bundle $N(M^n)$ of M^n. For $X \in T(M^n)$, $N \in N(M^n)$, a connection $'\nabla$ on $N(M^n)$ is defined by

$$(2.6)\qquad {}'\nabla_X N = (\tilde{\nabla}_X N)^N \;,$$

where $(\tilde{\nabla}_X N)^N$ denotes the normal part of $\tilde{\nabla}_X N$. When $'\nabla_X N$ vanishes identically along M^n we say that N is parallel with respect to the connection of the normal bundle $N(M^n)$.

Proposition 2.3. *The mean curvature vector field H is parallel with respect to the connection of the normal bundle if and only if the following two equations are both valid.*

$$(2.7)\qquad \nabla_j H_r{}^r = K_r{}^r L_j \;, \qquad \nabla_j K_r{}^r = -H_r{}^r L_j \;.$$

Proof. Since

$$\tilde{\nabla}_{B_j}H = \frac{1}{n}(\nabla_j H_r{}^r C + H_r{}^r \tilde{\nabla}_{B_j} C + \nabla_j K_r{}^r D + K_r{}^r \tilde{\nabla}_{B_j} D) ,$$

we have

$$\tilde{\nabla}_{B_j}H = \frac{-1}{n}\{(H_r{}^r H_j{}^i + K_r{}^r K_j{}^i)B_i - (\nabla_j H_r{}^r - K_r{}^r L_j)C - (\nabla_j K_r{}^r + H_r{}^r L_j)D\}$$

because of (1.4). Thus we get

$$('\nabla\partial/\partial x^j)H = \frac{1}{n}\{(\nabla_j H_r{}^r - K_r{}^r L_j)C + (\nabla_j K_r{}^r + H_r{}^r L_j)D\} , \tag{2.8}$$

from which we have Proposition 2.3.

Proposition 2.4. *If the mean curvature vector field is parallel with respect to the induced connection of the normal bundle, then* $(H_i{}^i)^2 + (K_i{}^i)^2$ *is constant.*

Proof. From Proposition 2.3 this is easily verified.

Proposition 2.5. *Let* M^n *be a totally umbilical submanifold of* $\tilde{M}^{n+2}$ *such that at each point of* M^n *the tangent space is invariant under the curvature transformation of* $\tilde{M}^{n+2}$*. Then the mean curvature vector field H is parallel with respect to the induced connection of the normal bundle.*

Proof. Since at each point of M^n the tangent space is invariant under the curvature transformation of $\tilde{M}^{n+2}$, equation (1.8) reduces to

$$\begin{aligned} \nabla_k H_{ji} - \nabla_j H_{ki} &= L_k K_{ji} - L_j K_{ki} , \\ \nabla_k K_{ji} - \nabla_j K_{ki} &= -L_k H_{ji} + L_j H_{ki} , \end{aligned} \tag{2.9}$$

from which we get equations (2.7), which, together with Proposition 2.3, thus imply the assertion of Proposition 2.5.

Proposition 2.6. *Let* M^n *be a totally umbilical submanifold of an* $(n+2)$*-dimensional Riemannian manifold of constant curvature. Then the mean curvature vector field H is parallel with respect to the induced connection of the normal bundle.*

Proof. Since the enveloping manifold is of constant curvature, for a constant c, the curvature tensor of the enveloping manifold has the form $R(B_k, B_j)B_i = C\{g(B_j, B_i)B_k - g(B_k, B_i)B_j\}$. This shows that the tangent space of M^n is invariant under the curvature transformation of the enveloping manifold. Thus, by Proposition 2.5, we have Proposition 2.6.

3. Integral formulas

Let M^n be a compact submanifold of $\tilde{M}^{n+2}$ in which there exists an infinitesimal conformal transformation $\tilde{X}$, that is, in which $\tilde{X}$ is a field of $\tilde{M}^{n+2}$ and satisfies for any vector fields $\tilde{Y}, \tilde{Z} \in T(\tilde{M}^{n+2})$,

$$(\mathscr{L}(\tilde{X})\tilde{g})(\tilde{Y}, \tilde{Z}) = \tilde{g}(\tilde{\nabla}_{\tilde{Y}}\tilde{X}, \tilde{Z}) + \tilde{g}(\tilde{Y}, \tilde{\nabla}_{\tilde{Z}}\tilde{X}) = 2\,\rho\tilde{g}(\tilde{Y}, \tilde{Z}) \,, \tag{3.1}$$

where $\mathscr{L}(\tilde{X})$ is the operator of Lie derivative with respect to $\tilde{X}$ and ρ is a function on $\tilde{M}^{n+2}$. The vector $\tilde{X}$ being a tangent field of $\tilde{M}^{n+2}$, it is represented as a linear combination of B_i, C and D. Hence we put

$$\tilde{X} = X + \alpha C + \beta D \,, \qquad X = v^i B_i \,, \tag{3.2}$$

from which we get

$$\alpha = \tilde{g}(\tilde{X}, C) \,, \qquad \beta = \tilde{g}(\tilde{X}, D) \,. \tag{3.3}$$

Since (1.1) and (3.2) yield that

$$\begin{aligned} g(\nabla_Y X, Z) &= \tilde{g}(\tilde{\nabla}_Y X - h(X, Y)C - k(X, Y)D, Z) \\ &= \tilde{g}(\tilde{\nabla}_Y X, Z) = \tilde{g}(\tilde{\nabla}_Y(\tilde{X} - \alpha C - \beta D), Z) \\ &= \tilde{g}(\tilde{\nabla}_Y \tilde{X}, Z) - \alpha\tilde{g}(\tilde{\nabla}_Y C, Z) - \beta\tilde{g}(\tilde{\nabla}_Y D, Z) \\ &= \tilde{g}(\tilde{\nabla}_Y \tilde{X}, Z) + \alpha h(Y, Z) + \beta k(Y, Z) \,, \end{aligned} \tag{3.4}$$

we have

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 2\,\{\rho g(Y, Z) + \alpha h(Y, Z) + \beta k(Y, Z)\} \tag{3.5}$$

because of (3.1). Substituting $v^i\partial/\partial x^i, \partial/\partial x^j$ and $\partial/\partial x^h$ for X, Y and Z respectively in (3.5), we get

$$\nabla_j v_h + \nabla_h v_j = 2(\rho g_{jh} + \alpha H_{jh} + \beta K_{jh}) \,, \tag{3.6}$$

which implies that

$$\operatorname{div} X = \nabla_i v^i = n\rho + \alpha H_i{}^i + \beta K_i{}^i = n\{\rho + \tilde{g}(H, \tilde{X})\} \,. \tag{3.7}$$

Since M^n is compact we have

$$\int_{M^n} \rho \, dM = -\int_{M^n} \tilde{g}(H, \tilde{X}) dM \tag{3.8}$$

because of (3.3).

Now we put $F = H_r{}^r A + K_r{}^r A'$. Then it is easily verified that F is independent of the choice of mutually orthogonal normal vectors C and D and consequently that F defines a linear transformation on $T(M^n)$. Let $Y = FX$, that is,

$$Y = H_r{}^r A(v^i \partial/\partial x^i) + K_r{}^r A'(v^i \partial/\partial x^i) . \tag{3.9}$$

Putting $Y = u^j \partial/\partial x^j$, we have

$$u_h = g(Y, \partial/\partial x^i) = H_r{}^r H_{ih} v^i + K_r{}^r K_{ih} v^i , \tag{3.10}$$

from which

$$\begin{aligned} \nabla_j u_h = {} & H_r{}^r (H_{ih} \nabla_j v^i + v^i \nabla_j H_{ih}) + K_r{}^r (K_{ih} \nabla_j v^i + v^i \nabla_i K_{ih}) \\ & + \nabla_j H_r{}^r v^i H_{ih} + \nabla_j K_r{}^r v^i K_{ih} . \end{aligned}$$

Thus we get

$$\begin{aligned} \operatorname{div} Y = \nabla_j u^j = {} & \tfrac{1}{2}(H_r{}^r H^{ji} + K_r{}^r K^{ji})(\nabla_j v_i + \nabla_i v_j) \\ & + H_r{}^r \nabla^j H_{ij} v^i + K_r{}^r \nabla^j K_{ij} v^i + \nabla_j H_r{}^r H_{ih} v^i \\ & + \nabla_j K_r{}^r K_{ih} v^i \end{aligned} \tag{3.11}$$

because of the fact that H is symmetric.

Substituting (3.6) into (3.11) and making use of (2.1) and (2.2), we obtain

$$\begin{aligned} \operatorname{div} Y = {} & n\bar{g}(H, \bar{X})(H_{ji} H^{ji} + K_{ji} K^{ji}) + \rho\{(H_i{}^i)^2 + (K_i{}^i)^2\} + \bar{g}(W, \bar{X}) \\ & + H_r{}^r \nabla^j H_{ij} v^i + K_r{}^r \nabla^j K_{ij} v^i + \nabla_j H_r{}^r H_{ih} v^i + \nabla_j K_r{}^r K_{ih} v^i . \end{aligned}$$

Since M^n is compact it follows that

$$\begin{aligned} - \int_{M^n} \rho\{(H_i{}^i)^2 + (K_i{}^i)^2\} dM = {} & \int_{M^n} \{ n\bar{g}(H, \bar{X})(H_{ji} H^{ji} + K_{ji} K^{ji}) + \bar{g}(W, \bar{X}) \\ & + H_r{}^r \nabla^j H_{ij} v^i + K_r{}^r \nabla^j K_{ij} v^i + \nabla_j H_r{}^r H_{ih} v^i + \nabla_j K_r{}^r K_{ih} v^i \} dM . \end{aligned} \tag{3.12}$$

4. Compact submanifolds with certain properties

In this section we assume that M^n is a compact submanifold of $\bar{M}^{n+2}$ in which there exists an infinitesimal conformal transformation $\bar{X}$ of $\bar{M}^{n+2}$ and that M^n satisfies the following conditions:

1) The tangent space at each point of M^n is invariant under the curvature transformation of $\bar{M}^{n+2}$.

2) The mean curvature vector of M^n in $\bar{M}^{n+2}$ is parallel with respect to the connection of the normal bundle and is non-vanishing at almost everywhere.

Then the condition 1), together with (1.8), implies that

$$\begin{aligned} \nabla_k H_{ji} - \nabla_j H_{ki} &= L_k K_{ji} - L_j K_{ki} , \\ \nabla_k K_{ji} - \nabla_j K_{ki} &= -L_k H_{ji} + L_j H_{ki} . \end{aligned} \tag{4.1}$$

Furthermore, from the condition 2), Proposition 2.3 and (4.1) it follows that

$$\nabla_j H_i{}^j = L_j K_i{}^j \,, \qquad \nabla_j K_i{}^j = -L_j H_i{}^j \,. \tag{4.2}$$

Substituting (2.7) and (4.2) into (3.12), we get

$$\begin{aligned} &-\{(H_i{}^i)^2 + (K_i{}^i)^2\} \int_{M^n} \rho\, dM \\ &= \int_{M^n} [n\bar{g}(H, \tilde{X})(H_{ji}H^{ji} + K_{ji}K^{ji}) + \bar{g}(W, \tilde{X})] dM \end{aligned} \tag{4.3}$$

because of Proposition 2.4. Substituting (3.8) into (4.3), we have

$$\begin{aligned} \int_{M^n} \Big[n\bar{g}(H, \tilde{X}) \Big\{ \Big(H_{ji}H^{ji} - \frac{1}{n} (H_i{}^i)^2 \Big) \\ + \Big(K_{ji}K^{ji} - \frac{1}{n} (K_i{}^i)^2 \Big) + \bar{g}(W, \tilde{X}) \Big\} \Big] dM = 0 \,. \end{aligned} \tag{4.4}$$

Thus, if the vector field W is on the same side as H with respect to the normal parts of $\tilde{X}$ in the normal bundle, that is, if H and W satisfy the inequality $\bar{g}(H, \tilde{X})\bar{g}(W, \tilde{X}) \geq 0$, then the integrand of (4.4) has a definite sign. In this case we have $H_{ji}H^{ji} = (1/n)(H_i{}^i)^2$ and $K_{ji}K^{ji} = (1/n)(K_i{}^i)^2$. Consequently we have, from Proposition 2.1,

Theorem 4.1. *Let M^n be a compact submanifold of $\tilde{M}^{n+2}$ whose tangent space at each point is invariant under the curvature transformation of $\tilde{M}^{n+2}$. Suppose that $\tilde{M}^{n+2}$ admits an infinitesimal conformal transformation $\tilde{X}$ and that the mean curvature vector field of M^n in $\tilde{M}^{n+2}$ is parallel with respect to the connection of the normal bundle and $\bar{g}(H, \tilde{X})$ is non-vanishing at almost everywhere on M^n. If, with respect to the normal part of $\tilde{X}$, the vector field W defined by* (2.2) *is on the same side as the mean curvature vector field in the normal bundle, then M^n is a totally umbilical submanifold of $\tilde{M}^{n+2}$.*

5. Applications

Let M^n be a submanifold of a Riemannian manifold of constant curvature. As we have seen in the proof of Proposition 2.3, the tangent space of M^n is invariant under the curvature transformation of the enveloping manifold. Thus Theorem 4.1 can be applied to submanifolds of a sphere or of a Euclidean space, and therefore in this section we consider submanifolds of such manifolds.

In order to get further results we use the following theorem due to M. Obata [6], [7].

Theorem. *Let M^n be a complete, connected Riemannian manifold of dimension $n(\geq 2)$. In order for M^n to admit a non-constant function f with*

(5.1) $$g(\nabla_X \operatorname{grad} f, Y) = -c^2 f g(X, Y) ,$$

for any X and $Y \in T(M^n)$, it is necessary and sufficient that M^n be isometric to a sphere S^n of radius $1/c$.

Let S^{n+2} be a sphere of radius $1/c$. Then on S^{n+2} there exists a function f satisfying (5.1) [7]. It is easily seen that for such f, grad f is an infinitesimal conformal transformation.

Theorem 5.1. *Let f be the function on a sphere S^{n+2} satisfying* (5.1), *and M^n a compact submanifold of S^{n+2}. Suppose that the mean curvature vector field H of M^n in S^{n+2} is parallel with respect to the induced connection in the normal bundle and $\tilde{g}(H, \operatorname{grad} f)$ is non-vanishing at almost everywhere on M^n. If, with respect to the normal part of* grad f, *the vector field W is on the same side as H in the normal bundle, M^n is a totally umbilical submanifold and consequently a sphere.*

Proof. Applying Theorem 4.1, we know that M^n is totally umbilical. The last part of Theorem 5.1 is proved in the following way. Let X be the tangential part of grad f to M^n, and put grad $f = X + \alpha C + \beta D$ as (3.2). Then

(5.2) $$\alpha = df(C) , \qquad \beta = df(D) ,$$

and, by (3.4),

$$g(\nabla_{\partial/\partial x^i} X, \partial/\partial x^i) = \tilde{g}(\tilde{\nabla}_{B_j} \operatorname{grad} f, B_i) + \alpha H_{ji} + \beta K_{ji} .$$

Since

$$g(\nabla_{\partial/\partial x^i} X, \partial/\partial x^i) = (\partial/\partial x^j)\, \tilde{g}(\operatorname{grad} f, B_i) - g(X, \nabla_{\partial/\partial x^j} \partial/\partial x^i) ,$$

the above equation can be written as

$$\nabla_j \nabla_i f = \tilde{g}(\tilde{\nabla}_{B_j} \operatorname{grad} f, B_i) + \alpha H_{ji} + \beta K_{ji} ,$$

from which we have

(5.3) $$\nabla_j \nabla_i f = \left\{ - c^2 f + \frac{1}{n}(\alpha H_r{}^r + \beta K_r{}^r) \right\} g_{ji}$$

because of (5.1). On the other hand, using (1.4), (5.1) and (5.2), we get

$$\begin{aligned} \nabla_j \alpha &= B_j(df(C)) = B_j(\tilde{g}(\operatorname{grad} f, C)) \\ &= \tilde{g}(\tilde{\nabla}_{B_j} \operatorname{grad} f, C) + \tilde{g}(\operatorname{grad} f, \tilde{\nabla}_{B_j} C) = -\frac{1}{n} H_r{}^r \nabla_j f + \beta L_j , \\ \nabla_j \beta &= -\frac{1}{n} K_r{}^r \nabla_j f - \alpha L_j . \end{aligned}$$

Thus we have

$$\nabla_j(\alpha H_r{}^r + \beta K_r{}^r) = -\frac{1}{n}((H_r{}^r)^2 + (K_r{}^r)^2)\nabla_j f$$

because of Proposition (2.3). This, together with Proposition 2.5, implies that

$$\alpha H_r{}^r + \beta K_r{}^r + \frac{1}{n}\{(H_r{}^r)^2 + (K_r{}^r)^2\}f = c_0 (= \text{const.}) \,. \tag{5.4}$$

Substituting (5.4) into (5.3), we find that

$$\nabla_j\nabla_i f = \left\{-c^2 f - \frac{1}{n}((H_r{}^r)^2 + (K_r{}^r)^2)f + \frac{1}{n}c_0\right\} g_{ji} \,. \tag{5.5}$$

Since $(H_r{}^r)^2 + (K_r{}^r)^2$ and c_0 are both constants, from (4.9) we have

$$\begin{aligned} &\nabla_j\nabla_i\left[f - \frac{nc_0}{n^2c^2 + (H_r{}^r)^2 + (K_r{}^r)^2}\right] \\ &= -\frac{n^2c^2 + (H_r{}^r)^2 + (K_r{}^r)^2}{n^2} \times \left[f - \frac{nc_0}{n^2c^2 + (H_r{}^r)^2 + (K_r{}^r)^2}\right] g_{ji} \,, \end{aligned} \tag{5.6}$$

and consequently by Obata's theorem the submanifold is isometric to a sphere; this hence completes the proof.

Next let E^{n+2} be an $(n+2)$-dimensional Euclidean space. Then the so-called position vector field $\bar{X}$ satisfies (3.1) with constant coefficient $\rho = 1$. Furthermore $\bar{X}$ satisfies $\bar{\nabla}_{\bar{Y}}\bar{X} = \bar{g}(\bar{X}, \bar{Y})$ for any $\bar{Y}$. Thus we can apply the whole discussions in this section to the submanifold M^n of E^{n+2}. So we have

Theorem 5.2. *Let M^n be a compact submanifold of an $(n+2)$-dimen sional Euclidean space. Suppose that the mean curvature vector field of M^n is parallel with respect to the induced connection in the normal bundle and $\bar{g}(H, \bar{X})$ is non-vanishing at almost everywhere on M^n. If, with respect to the normal part of the position vector in the Euclidean space, the vector field W is on the same side as the mean curvature vector field in the normal bundle, M^n is a totally umbilical submanifold and consequently a sphere.*

Bibliography

[1] Y. Katsurada, *Generalized Minkowski formulas for closed hypersurfaces in Riemann space*, Ann. Mat. Pura Appl. **57** (1962) 283–293.

[2] ——, *On certain property of closed hypersurfaces in an Einstein space*, Comment. Math. Helv. **38** (1964) 165–171.

[3] Y. Katsurada & T. Nagai, *On some properties of a submanifold with constant mean curvature in a Riemann space*, J. Fac. Sci. Hokkaido Univ. Ser. I, **22** (1968) 79–89.

[4] Y. Katsurada & H. Kōjyo, *Some integral formulas for closed submanifolds in a Riemann space*, J. Fac. Sci. Hokkaido Univ. Ser. I, **22** (1968) 90–100.

[5] H. Liebmann, *Über die Verbiegung der geschlossen Flächen positiver Krümmung*, Math. Ann. **53** (1900) 91–112.

[6] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14** (1962) 333–340.
[7] ——, *Riemannian manifolds admitting a solution of a certain system of differential equations*, Proc. United States-Japan Sem. in Differential Geometry, Kyoto, Japan, 1965, 101–114.
[8] M. Okumura, *Compact orientable submanifold of codimension* 2 *in an odd dimensional sphere*, Tôhoku Math. J. **20** (1968) 8–20.
[9] K. Yano, *Closed hypersurfaces with constant mean curvature in a Riemannian manifold*, J. Math. Soc. Japan **17** (1965) 333–340.
[10] ——, *Integral formulas for submanifolds and their applications*, Canad. J. Math. **22** (1970) 376–388.

SAITAMA UNIVERSITY, JAPAN