J. DIFFERENTIAL GEOMETRY
4 (1969) 425-451

CURVATURE STRUCTURES AND CONFORMAL TRANSFORMATIONS

RAVINDRA S. KULKARNI

By a curvature structure on a Riemann manifold (M, g) we mean any (1,3) tensorfield which has the algebraic properties of the Riemann curvature tensor. Some examples are given in § 1.

Let $G_2(M)$ denote the Grassmann bundle of 2-planes on M. A curvature structure naturally defines the corresponding sectional curvature, which is a real-valued function on $G_2(M)$. In this memoir we shall show that these sectional curvature functions are of considerable geometrical interest.

Let (M, g), $(\bar{M}, \bar{g})$ be two Riemann manifolds, and let K(resp. $\bar{K}$) be the usual sectional curvature functions canonically defined by g(resp. $\bar{g}$). Call (M, g), $(\bar{M}, \bar{g})$ *isocurved* if there exists a 1-1 onto sectional-curvature-perserving diffeomorphism $f: M \to \bar{M}$, i.e., for every $p \in M$, $\sigma \in G_2(M)_p$, $K(\sigma) = \bar{K}(f_*\sigma)$. The "theorema egregium" or what is essentially the "fundamental theorem of Riemann geometry" asserts that isometric manifolds are isocurved. The basic result of [8] is the converse.

Call (M, g) *nowhere of constant curvature* if there does not exist a nonempty open subset on which $K \equiv$ constant. We have

Theorem A. *Let (M, g), $(\bar{M}, \bar{g})$ be isocurved. Suppose that (M, g) is nowhere of constant curvature and of dimension ≥ 4. Then (M, g), $(\bar{M}, \bar{g})$ are isometric.*

In the following we use the techniques developed in the proof of this theorem. All manifolds in this paper are assumed to be connected; and all manifolds, metrics and maps are assumed to be C^4.

PART I. CURVATURE STRUCTURES

Introduction

In this part, we first develop some generalities on curvature structure. These are applied to two cases: conformal curvature structure which is defined by the conformal curvature tensor, and the Ricci curvature structure which is defined by a certain combination of the Ricci tensor.

Communicated by S. Sternberg, September 26, 1969.

It turns out that the spaces of constant conformal curvature of dimension ≥ 4 are precisely the conformally flat spaces. We also obtain a criterion for conformal flatness purely in terms of (Riemannian sectional) curvature.

Theorem A is partially generalized to: isoconformally curved, nowhere conformlly flat manifolds of dimension ≥ 4 are isometric.

In the case of Ricci curvature structure it turns out that the spaces of constant Ricci curvature and of dimension ≥ 3 are precisely the Einstein spaces. Analogous results for iso-Ricci-curved nowhere Einstein spaces are so far only partly established.

For considering the variation of these curvatures, we have found it useful to find the critical point structure of curvature regarded as a function on the Grassmannian. Later it is shown that in certain cases these are natural Morse functions on the Grassmannian.

1. Definition of a curvature structure

Let (M, g) be a Riemann manifold. Whenever convenient, we shall denote by $\langle\ ,\ \rangle$ the inner product defined by g.

Definition. A curvature structure on (M, g) is a $(1, 3)$ tensor field T such that for any vector fields X, Y, Z, W, we have

1) $T(X, Y) = -T(Y, X)$,
2) $\langle T(X, Y)Z, W\rangle = \langle T(Z, W)X, Y\rangle$,
3) $T(X, Y)Z + T(Y, Z)X + T(Z, X)Y = 0$.

Let $G_2(M)$ be the Grassman bundle of 2-planes on M. A curvature structure T defines the corresponding sectional curvature

$$K_T\colon G_2(M) \to R$$

via: if $p \in M$ and $\sigma = \{X, Y\}$, a 2-plane at p, then

$$K_T(\sigma) = \frac{\langle T(X, Y)X, Y\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}\ .$$

(It is easy to check that the definition of $K_T(\sigma)$ does not depend on the choice of the basis of σ.)

Examples. (a) *Trivial curvature structure*: Consider the curvature structure I defined by

$$I(X, Y)Z = \{\langle X, Z\rangle Y - \langle Y, Z\rangle X\}\ .$$

Clearly $K_I \equiv 1$.

(b) *Riemann curvature structure*: This is the canonical curvature structure defined by the metric g, namely, if ∇ denotes the corresponding covariant derivative then

$$R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z \ .$$

We shall denote the corresponding K_R simply by K. (Our choice of sign for R ensures that the unit sphere has $K \equiv 1$.)

(c) *Ricci curvature structure*: Recall that the Riemann curvature tensor R defines the Ricci tensor by $\mathrm{Ric}(X, Y) = \mathrm{trace}\colon Z \to R(X, Z)Y$, for $p \in M$ and $X, Y \in T_p(M)$, the tangent space at p. We shall denote by Ric. the corresponding linear transformation of $T(M)$, the tangent bundle of M, defined by $\langle \mathrm{Ric}.\ X, Y\rangle = \mathrm{Ric}.\ (X, Y)$. We recall that the scalar curvature $\mathrm{Sc}\colon M \to R$ is defined as $Sc = \mathrm{trace\ Ric}$.

Consider the following (1, 3) tensor

$$\mathscr{R}\mathrm{ic}\,(X, Y)Z = \{\mathrm{Ric}\,(X, Z)Y - \mathrm{Ric}\,(Y, Z)X + \langle X, Z\rangle\, \mathrm{Ric}.\ Y - \langle Y, Z\rangle\, \mathrm{Ric}.\ X\} \ .$$

It is easily seen that this is a curvature structure on M. We shall call it the Ricci curvature structure. It is plain that for a 2-plane σ,

$$K_{\mathscr{R}\mathrm{ic}}(\sigma) = \mathrm{trace\ Ric}|_\sigma \ .$$

(d) *Conformal curvature structure*: Suppose that M is n-dimensional, $n > 2$, andlet R, Ric, I, Sc be as defined above. It is obvious that the following tensor

$$C = R - \frac{1}{n-2}\mathscr{R}\mathrm{ic} + \frac{Sc}{(n-1)(n-2)} I$$

is a curvature structure on M. This tensor is Weyl's well known conformal curvature tensor. We shall call K_C the "conformal curvature", and denote it by K_{con}.

More generally it is clear that every curvature structure T gives rise to the corresponding Ricci-curvature structure Ric_T and the conformal curvature structure C_T.

We finally remark that the notion of a curvature structure may be developed in a more general set-up by replacing the tangent bundle of M by an arbitrary vector bundle equipped with an inner product. However, the set-up which we have adopted is sufficient for our purposes.

2. A general theorem

Let (M, g, T) be a Riemann manifold with a curvature structure T. Let $\pi\colon G_2(M) \to M$ be the canonical projection of the Grassmann bundle of 2-planes, and $K_T\colon G_2(M) \to R$ the corresponding curvature function. Call a point $p \in M$ *isotropic* (resp. nonisotropic) *with respect to* T if $K_T|_{\pi^{-1}(p)} \neq$ constant (resp. $K_T|_{\pi^{-1}(p)} \neq$ constant).

The following theorem is essentially Theorem 1 of [8]:

Theorem 2.1. *Let (M, g, T), $(\bar{M}, \bar{g}, \bar{T})$ be Riemann manifolds with curvature structures, and $f\colon M \to \bar{M}$ be a curvature-preserving diffeomorphism. Suppose that dimension $M \geq 3$, and that the set*

$$\{p \in M \mid p \text{ is nonisotropic}\}$$

is dense in M. Then f is conformal, i.e., *$f^*\bar{g} = \lambda \cdot g$ where $\lambda\colon M \to R$ is a positive real valued function.*

For the proof see [8]. It is this theorem which generically reduces considerations about curvature-preserving maps to those about conformal maps.

3. Conformal curvature structure

Let (M, g) be a Riemann manifold, and equip it with the conformal curvature structure (cf. §1, Example (d)). Let $\{e_1 \cdots e_n\}$ be an orthonormal frame at a point $p \in M$, and σ the 2-plane $\{e_1, e_2\}$. Denote by K_{ij} the (Riemannian) sectional curvature corresponding to the plane $\{e_i, e_j\}_{i \neq j}$. Then

$$\begin{aligned} K_{\text{con}}(\sigma) &= K(\sigma) - \frac{1}{n-2}\{\text{Ric}\,(e_1, e_1) + \text{Ric}\,(e_2, e_2)\} + \frac{Sc}{(n-1)(n-2)} \\ &= K(\sigma) - \frac{1}{n-2}\{\sum_{i \neq 1} K_{1i} + \sum_{i \neq 2} K_{2i}\} + \frac{Sc}{(n-1)(n-2)} \,. \end{aligned} \tag{3.1}$$

We define the corresponding Ricci tensor by

$$\text{Ric-Kon}\,(X, Z) = \text{trace}\colon\, Y \to C(X, Y)Z \;,$$

and the corresponding scalar curvature by

$$Sc\text{-Kon} = \text{trace Ric-Kon} \;.$$

Proposition 3.1. Ric-Kon $\equiv 0$.

Proof. Typically it suffices to show that Ric-Kon$(e_1, e_1) = 0$. Now,

$$\begin{aligned} \text{Ric-Kon}(e_1, e_1) &= \text{Ric}(e_1, e_1) - \frac{1}{n-2}[\sum_{i>1} \{\text{Ric}(e_1, e_1) + \text{Ric}(e_i, e_i)\}] \\ &\quad + \frac{Sc}{(n-1)(n-2)} \times (n-1) \\ &= \text{Ric}(e_1, e_1) - \frac{1}{n-2}[(n-2)\,\text{Ric}(e_1, e_1) + Sc] + \frac{Sc}{(n-2)} \\ &= 0 \,. \end{aligned}$$

Corollary. Sc-Kon $\equiv 0$.

Recall that a Riemann manifold (M, g) is called *conformally flat* if for every point $p \in M$, there exists a neighbourhood U such that $g|_U = \phi \cdot g_0$ where g_0 is a flat (i.e., Euclidean) metric, and $\phi\colon U \to R$ a positive real-valued function on U. Based on Weyl's well known result, we enumerate below several characterizations of a conformally flat space of dimension ≥ 4. Note that the last one is purely sectional-curvature-theoretic, which does not seem to have been noticed before.

Theorem 3.2. *Let (M, g) be a Riemann manifold of dimension $n \geq 4$. Then the following are equivalent:*

(1) *(M, g) is conformally flat.*

(2) $C = 0$.

(3) $K_{\mathrm{con}} = 0$.

(4) *At every point $p \in M$, $K_{\mathrm{con}}|_{\pi^{-1}(p)} =$ constant.*

(5) *At every point $p \in M$, for every 4-dimensional subspace $W \subseteq T_p(M)$, there exists a constant $c = c(W)$ such that for any two mutually perpendicular 2-plane sections σ_1, σ_2 spanning W,*

$$K(\sigma_1) + K(\sigma_2) = c\ .$$

(6) *At every point $p \in M$, for every quadruple of orthogonal vectors $\{e_1, e_2, e_3, e_4\}$,*

$$K_{12} + K_{34} = K_{14} + K_{23}\ ,$$

where $K_{ij} =$ Riemannian sectional curvature corresponding to the plane spanned by $\{e_i, e_j\}$.

Proof. (1) $\Longleftrightarrow$ (2) is a famous theorem of Weyl (cf. [4]) which requires $n \geq 4$.

(2) $\Longleftrightarrow$ (3) is a consequence of a well-known generality about curvature structures, which is sometimes expressed as 'sectional curvature determines the curvature tensor' (cf. [3]).

Clearly (3) $\Rightarrow$ (4). Conversely, if at $p \in M$, $K_{\mathrm{con}} =$ a constant α, then *Sc*-Kon $= n(n-1)\alpha$. As we showed above, *Sc*-Kon $= 0$. So $\alpha = 0$.

(1) $\Rightarrow$ (5): Assume (M, g) conformally flat, so that given a point $p \in M$, there exists a neighbourhood U of p such that $g|_U = e^{2\phi}g_0$, where g_0 is a flat metric and $\phi\colon U \to R$.

In § 7 we have collected together the formulas arising from a conformal change of a metric. In particular, using the bilinear form Q defined by ϕ and writing $\langle X, Y\rangle$ for $g(X, Y)$ and $\langle X, Y\rangle_0$ for $g_0(X, Y)$ and $G = \mathrm{grad}_0\,\phi$ (the gradient of ϕ defined with respect to g_0), we have

$$R(X, Y)Z = \{Q(Y, Z) + \langle Y, Z\rangle_0\|G\|_0^2\}X - \{Q(X, Z) + \langle X, Z\rangle_0\|G\|_0^2\}Y + \langle Y, Z\rangle_0 Q.X - \langle X, Z\rangle_0 Q.Y\ .$$

Consequently, if $\{e_1, e_2, e_3, e_4\}$ is an orthonormal quadruple spanning W and $\sigma_1 = \{e_1, e_2\}$ and $\sigma_2 = \{e_3, e_4\}$, then

$$K(\sigma_1) + K(\sigma_2) = -\{Q(e_1, e_1) + Q(e_2, e_2) + Q(e_3, e_3) + Q(e_4, e_4) + 2\|G\|^2\}$$
$$= -\{\text{trace } Q|_W + 2\|G\|^2\},$$

which is a constant depending on W alone. This proves 5).

Clearly 5) $\Rightarrow$ 6).

6) $\Rightarrow$ 4): Let $\{e_1, \cdots, e_n\}$ be an orthonormal frame at a point $p \in M$. Typically it suffices to show that $K_{\text{con}}(e_1, e_2) = K_{\text{con}}(e_1, e_3)$. Using the formula (3.1) for K_{con}, and writing $K_{ij} = K(e_i, e_j)$ we have

$$K_{\text{con}}(e_1, e_2) - K_{\text{con}}(e_1, e_3) = K_{12} - K_{13} - \frac{1}{(n-2)}\{\sum_{i \neq 2,3} (K_{i2} - K_{i3})\}\ .$$

Using (6), $K_{12} - K_{13} = K_{42} - K_{43} = \cdots = K_{i2} - K_{i3}$, $i > 3$. So the right hand side of the above inequality clearly vanishes and the proof is finished.

q.e.d.

We shall call a Riemann manifold (M, g) *nowhere conformally flat*, if there is no nonempty open subset $\subseteq M$ which is conformally flat (in the inherited Riemann metric).

The condition (4) of the above theorem easily shows the following

Proposition 3.3. *Suppose that a Riemann manifold* (M, g), $\dim M \geq 4$, *is nowhere conformally flat. Consider the conformal curvature structure* (M, g, C). *Then the set*

$$\{p \in M \mid p \text{ is non-isotropic}\} \quad (\text{cf. } \S\, 2)$$

is dense in M.

We call two Riemann manifolds (M, g), $(\bar{M}, \bar{g})$ *isoconformally curved* if there exists a 1–1, onto conformal-curvature-preserving diffeomorphism F: $M \to \bar{M}$ i.e., denoting the quantities related to $\bar{M}$ by a bar overhead we have, for every $p \in M$ and every 2-plane σ at p,

$$K_{\text{con}}(\sigma) = \bar{K}_{\text{con}}(F_*\sigma)\ .$$

The following theorem generalizes Theorem A (cf. the introduction) in the case of nowhere conformally flat manifolds:

Theorem 3.4. *Let* (M, g), $(\bar{M}, \bar{g})$ *be isoconformally curved Riemann manifolds of* $\dim \geq 4$, *and suppose that* (M, g) *is nowhere conformally flat. Then* (M, g), $(\bar{M}, \bar{g})$ *are isometric.*

Proof. Let $F: M \to \bar{M}$ be a conformal-curvature-preserving diffeomorphism. By Proposition 3.3, the set of nonisotropic points of M (with respect to the conformal curvature structure) is dense. So by the general theorem of § 2, F is conformal. We identify M with $\bar{M}$ via F, and again write $\bar{g}$ for $F^*\bar{g}$, so that $\bar{g} = \phi \cdot g$ where ϕ: $M \to R$ is some positive real-valued function on M.

As is well-known, the conformal curvature tensor is invariant under a conformal change of the metric. So $\bar{C} = C$, and hence $K_{\text{con}} = \phi \bar{K}_{\text{con}}$. By hypothesis, $K_{\text{con}} = \bar{K}_{\text{con}}$ and also $K_{\text{con}} \neq 0$ on an open dense subset, so $\phi \equiv 1$. It follows that F is an isometry.

Remark. In the above theorem, in case g is analytic, the condition "nowhere conformally flat" is equivalent to "$C \not\equiv 0$" or $K_{\text{con}} \not\equiv 0$", or (in view of Theorem 3.1) "$K_{\text{con}} \not\equiv$ constant". This is immediate, since analyticity of g implies analyticity of C and K_{con}. If C(or K_{con}) $= 0$ on a nonempty open subset, C(or K_{con}) $\equiv 0$ everywhere by analyticity.

Remark. Call (M, g), $(\bar{M}, \bar{g})$ homoconformally curved if there exist a one-one, onto diffeomorphism $F\colon M \to \bar{M}$, and a function $\phi\colon M \to R$ such that for every $p \in M$, every σ, and a 2-plane at p, we have

$$K_{\text{con}}(\sigma) = \phi(p)\bar{K}_{\text{con}}(\sigma) \ .$$

The general theorem of § 2, and the argument of the above theorem easily imply the following result:

Two Riemann manifolds (M, g), $(\bar{M}, \bar{g})$ of dimension ≥ 4 are homoconformally curved if and only if they are conformal.

4. Ricci curvature structure

Let (M, g) be a Riemann manifold. Equip it with the Ricci curvature structure (cf. § 1, Example (c)). The corresponding curvature is given by: if $\sigma = \{X, Y\}$, X, Y orthonormal, then

$$K_{\mathscr{R}\text{ic}}(\sigma) = \text{Ric}(X, X) + \text{Ric}(Y, Y) = \text{trace Ric}|_\sigma \ . \tag{4.1}$$

Notice that $\text{Ric}\left(\frac{X}{\|X\|}, \frac{X}{\|X\|}\right)$, for $X \neq 0$, is the quantity which is classically called *the Ricci curvature in the direction X*. (M, g) is called an *Einstein manifold* if there exists a constant α such that for any vector fields X, Y, $\text{Ric}(X, Y) = \alpha\langle X, Y\rangle$. (For a detailed discussion see § 8.) The following proposition shows that for the Ricci curvature structure, the Einstein manifolds play a role of constant curvature spaces.

Proposition 4.1. *Let (M, g) be a Riemann manifold of dimension ≥ 3. Then $K_{\mathscr{R}\text{ic}} =$ constant if and only if (M, g) is an Einstein manifold.*

Proof. The 'if' part is obvious. Conversely, if $K_{\mathscr{R}\text{ic}} = 2\alpha$, it is easy to see from (4.1) and the hypothesis 'dimension ≥ 3' that $\text{Ric}(X, X) = \alpha$ for every unit vector X. From this, it easily follows that (M, g) is an Einstein manifold.

q.e.d.

We shall call (M, g) *nowhere Einsteinian* if there does not exist a nonempty open subset of M, which is an Einstein manifold (in the inherited Riemann metric).

Proposition 4.2. *Let (M, g) be a Riemann manifold of dimension ≥ 3 which is nowhere Einsteinian. Then the set of points*

$$\{p \in M \mid p \text{ non-isotropic with respect to Ricci curvature structure}\}$$

is dense in M.

Proof. To say that $p \in M$ is isotropic with respect to the Ricci curvature structure amounts to, for $X, Y \in T_p(M)$, $\mathrm{Ric}(X, Y) = \alpha\langle X, Y\rangle$ for some $\alpha = \alpha(p)$ depending on p. However, a well-known Schur's theorem-type argument (cf. [4]) shows that if every point of an open subset of M is isotropic (with respect to the Ricci curvature structure) then $\alpha(p)$ is constant over the connected components of the open subset. q.e.d.

Call two manifolds (M, g), $(\bar{M}, \bar{g})$ *iso-Ricci-curved* if there exists a $K_{\mathscr{R}\mathrm{ic}}$-preserving diffeomorphism $F\colon M \to \bar{M}$. The above proposition and the general theorem of § 2 imply

Theorem 4.3. *Let (M, g), $(\bar{M}, \bar{g})$ be iso-Ricci-curved manifolds of dimension ≥ 3. Suppose that (M, g) is nowhere Einsteinian. Then (M, g), $(\bar{M}, \bar{g})$ are conformal.*

How does the notion "a diffeomorphism F is $K_{\mathscr{R}\mathrm{ic}}$-preserving" compare with the notion "a diffeomorphism F preserves $\mathrm{Ric}\left(\frac{X}{\|X\|}, \frac{X}{\|X\|}\right)$, i.e., the classical Ricci curvature in the direction $X \neq 0$"? For short, in the latter case we shall say that F is *Ric-preserving*. In case the manifolds are Einsteinian, every diffeomorphism is both $K_{\mathscr{R}\mathrm{ic}}$-preserving and Ric-preserving. In case the diffeomorphism F is conformal and the dimension of $M \geq 3$ it is easily seen that F is $K_{\mathscr{R}\mathrm{ic}}$-preserving if and only if it is Ric-preserving. So in view of the Theorem 4.3, $K_{\mathscr{R}\mathrm{ic}}$-preserving always implies Ric-preserving. The converse also holds *generically*, as is shown by the following theorem (recall that as in § 1, Ric. denotes the linear transformation induced by Ric).

Theorem 4.4. *Let (M, g) be a Riemann manifold of dimension $n \geq 3$ such that the set*

$$\{p \in M \mid \mathrm{Ric}. \text{ at } p \text{ has } n \text{ distinct eigenvalues}\}$$

is dense in M. Then every Ric-preserving diffeomorphism $F\colon M \to \bar{M}$ of Riemann manifolds (M, g), $(\bar{M}, \bar{g})$ is conformal.

Proof. This proof follows the pattern of the proof of the general theorem of § 2 (cf. [8]).

We denote the quantities corresponding to $(\bar{M}, \bar{g})$ by a bar overhead. Choose an orthonormal frame $e_1, \cdots, e_n$ at $p \in M$, consisting of eigenvectors of Ric. and assume that Ric. at p has distinct eigenvalues. Write $\mathrm{Ric}(e_i, e_j) = R_{ij}$. Then $R_{ij} = 0$ if $i \neq j$, and $R_{ii} \neq R_{jj}$ if $i \neq j$. Let

$$F_* e_i = \bar{e}_i, \overline{\langle \bar{e}_i, \bar{e}_j \rangle} = a_{ij} , \quad \bar{\mathrm{Ric}}(\bar{e}_i, \bar{e}_j) = \bar{R}_{ij} .$$

For $x, y \in R$, $(x, y) \neq (0,0)$, $i \neq j$,

$$\overline{\mathrm{Ric}}\left(\frac{x\bar{e}_i + y\bar{e}_j}{\|x\bar{e}_i + y\bar{e}_j\|}, \frac{x\bar{e}_i + y\bar{e}_j}{\|x\bar{e}_i + y\bar{e}_j\|}\right) = \mathrm{Ric}\left(\frac{xe_i + ye_j}{\|xe_i + ye_j\|}, \frac{xe_i + ye_j}{\|xe_i + ye_j\|}\right)$$

implies

$$\frac{x^2\bar{R}_{ii} + 2xy\bar{R}_{ij} + y^2\bar{R}_{jj}}{a_{ii}x^2 + 2a_{ij}xy + a_{jj}y^2} = \frac{x^2R_{ii} + y^2R_{jj}}{x^2 + y^2} .$$

Cross multiplying and equating coefficients of like powers of x, y we have

$$\bar{R}_{ii} = a_{ii}R_{ii} , \tag{4.1}$$

$$\bar{R}_{ij} = a_{ij}R_{ii} = a_{ij}R_{jj} , \tag{4.2}$$

$$\bar{R}_{ii} + \bar{R}_{jj} = a_{ii}R_{jj} + a_{jj}R_{ii} . \tag{4.3}$$

Since $R_{ii} \neq R_{jj}$ for $i \neq j$, (4.2) implies $a_{ij} = 0$. (4.1) and (4.3) together imply $(a_{ii} - a_{jj})(R_{ii} - R_{jj}) = 0$. Since $R_{ii} \neq R_{jj}$, we have $a_{ii} = a_{jj}$.

It follows that $\{\bar{e}_i\}$ are mutually orthogonal, and have the same length. So F is a homothety at p. Since the set $\{p \mid \text{Ric. at } p \text{ has } n \text{ distinct eigenvalues}\}$ is dense in M, it follows by continuity considerations that F is conformal.

q.e.d.

It now follows that under the hypotheses of Theorem 4.4, the concepts "a diffeomorphism is $K_{\mathscr{R}\mathrm{ic}}$-preserving" and "a diffeomorphism is Ric-preserving" are equivalent.

Now we consider the question: when is a $K_{\mathscr{R}\mathrm{ic}}$-preserving diffeomorphism necessarily an isometry? In view of Theorem A and Theorem 3.4, we would like to settle this question for all nowhere Einstein manifolds. Unfortunately we have been able to settle it only under further hypotheses.

An important case of conformally flat manifolds (which are more susceptible to admit nonisometric conformal maps) is settled in the following

Theorem 4.5. *Let (M, g), $(\bar{M}, \bar{g})$ be iso-Ricci-curved, and (M, g) be a conformally flat, nowhere Einsteinian manifold of dimension ≥ 4. Then (M, g), $(\bar{M}, \bar{g})$ are isometric.*

Proof. Let $F: M \to \bar{M}$ be a $K_{\mathscr{R}\mathrm{ic}}$-preserving diffeomorphism. Then it is conformal by Theorem 4.3, and equation (3.1) reads

$$K_{\mathrm{con}} = K - \frac{1}{n-2}K_{\mathscr{R}\mathrm{ic}} + \frac{Sc}{(n-1)(n-2)} .$$

Now $K_{\mathscr{R}\mathrm{ic}}$-preserving $\Rightarrow$ Ric-preserving $\Rightarrow$ Sc-preserving. So a $K_{\mathscr{R}\mathrm{ic}}$-preserving diffeomorphism is K-preserving if and only if it is K_{con}-preserving. But (M, g) is conformally flat, so F is automatically K_{con}-preserving; hence F is K-preserving.

By Theorem A, F is an isometry if (M, g) is nowhere of constant (Riemannian sectional) curvature—which is indeed the case, since (M, g) is nowhere Einsteinian. q.e.d.

The results of § 12 will show that the conclusion of Theorem 4.5 is valid also under the hypothesis: (M, g) is locally homogeneous, nowhere Einsteinian manifold of dimension ≥ 4. Some cases when (M, g) is assumed to be compact will also be settled later.

5. Critical points of curvature on a fiber

We follow the notation of § 2. Let $p \in (M, g, T)$, and write K for $K_T|_{\pi^{-1}(p)}$. In this section we shall evaluate the gradient and hessian of K at a point σ in the fiber $\pi^{-1}(p)$. This calculation is important in itself. We shall use it in Part II.

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis at p, and σ be the 2-plane spanned by $\{e_1, e_2\}$. Denote $\langle T(e_i, e_j)e_k, e_l\rangle$ by T_{ijkl}, and T_{ijij} by K_{ij}. It is easy to see that

$$(x, y) \equiv (x_3, x_4, \cdots, x_n, y_3, y_4, \cdots, y_n)$$
$$\to \text{the 2-plane spanned by } \left\{e_1 + \sum_{i=3}^n x_i e_i,\ e_2 + \sum_{i=3}^n y_i e_i\right\}$$

gives a coordinatization of a neighbourhood of σ such that σ is given by $(0, 0)$. Par abus, we shall denote a 2-plane in this neighbourhood by (x, y), and also write x for $\sum_{i=3}^n x_i e_i$.

Now, by the definition of K, for $x \neq 0$, $y \neq 0$,

$$\begin{aligned}
&\left\{\left(1 + \sum_{i=3}^n x_i^2\right)\left(1 + \sum_{i=3}^n y_1^2\right) - \left(\sum_{i=3}^n x_i y_i\right)^2\right\} K(x, y) \\
&\quad = \left\langle T\left(e_1 + \sum_{i=3}^n x_i e_i,\ e_2 + \sum_{i=3}^n y_i e_i\right) e_1 + \sum_{i=3}^n x_i e_i,\ e_2 + \sum_{i=3}^n y_i e_i \right\rangle \\
&\quad = T_{1212} + 2\sum_{i=3}^n T_{12i2} x_i + 2\sum_{i=3}^n T_{121i} y_i + \sum_{i=3}^n T_{i2i2} x_i^2 \\
&\qquad + \sum_{i=3}^n T_{1i1i} y_i^2 + 2\sum_{i<j} T_{i2j2} x_i x_j + 2\sum_{i<j} T_{1i1j} y_i y_j \\
&\qquad + 2\sum_{i,j} (T_{12ij} + T_{1ji2}) x_i y_j + \text{third and higher order terms.}
\end{aligned} \tag{5.1}$$

Proposition 5.1. *The gradient of K at σ is given by*

$$\partial K / \partial x_i|_\sigma = T_{12i2}\ , \qquad \partial K / \partial y_i|_\sigma = T_{121j}\ .$$

Proof. Differentiate the above identity (5.1) with respect to x_i to get

$$(5.2)\qquad \begin{aligned} &2\{x_i(1 + \sum y_i^2) - (\sum x_j y_j)y_i\}K(x, y) + \{\cdots\}\, \partial K/\partial x_i \\ &\quad = 2T_{12i2} + \text{first and higher order terms.} \end{aligned}$$

This and similar identities prove the proposition.

Corollary. *σ is a critical point of K if and only if $T_{12i2} = 0 = T_{121j}$ for $i, j \geq 3$.*

Proposition 5.2. *The Hessian of K at σ is given by*

$$\begin{aligned} &\partial^2K/\partial x_i^2|_\sigma = 2(K_{i2} - K_{12}); \qquad &&\partial^2K/\partial y_i^2|_\sigma = 2(K_{1i} - K_{12}); \\ &\partial^2K/\partial x_i \partial x_j|_\sigma = 2T_{i2j2}\,, \quad i \neq j; \qquad &&\partial^2K/\partial y_i \partial y_j|_\sigma = 2T_{1i1j}\,, \quad i \neq j; \\ &\partial^2K/\partial x_i \partial y_j|_\sigma = 2(T_{12ij} + T_{1ji2})\,, \quad 3 \leq i,\, j \leq n\,. \end{aligned}$$

Proof. Differentiate (5.2) with respect to x_i and evaluate at σ to get

$$2K(x, y) + \partial^2K/\partial x_i^2|_\sigma = 2T_{i2i2}\,,$$

i.e., $\partial^2K/\partial x_i^2|_\sigma = 2(K_{i2} - K_{12})$. The rest follows similarly.

6. Curvature as a Morse function on the Grassmannian

It is natural to ask whether some of these curvature functions are canonical Morse functions on the Grassmannian of 2-planes in n-space. We shall show that such indeed is the case; and in fact, one can construct curvature-like Morse functions on the Grassmannian of k-planes in $n + k$ space over reals, complexes or quaternions.

The author wishes to thank Professor R. Bott for asking this question. The calculations presented here are essentially those in [8, Theorem 7], and they generalize the well-known Morse functions on projection spaces. Recently J. C. Alexander [1] constructed some different curvature like Morse functions on Grassmannians which have other novel features.

We shall exhibit the Morse functions on the Grassmannian of k-planes in R^{n+k}, the real vector space of dimension $n + k$. By putting appropriate bars for complex conjugation and quaternionic conjugation, one gets Morse functions on the complex or quaternionic Grassmannians.

Let σ be a k-plane in R^{n+k}, and τ its any complementary subspace. Fix a basis $\{e_1, \cdots, e_k\}$ of σ, and consider

$$\underbrace{\tau \times \cdots \times \tau}_{k} \xrightarrow{F} G_{k,n}$$

given by $(w_1, \cdots, w_k) \to (e_1 + w_1, \cdots, e_k + w_k)$. It is easily seen that $\{e_1 + w_1, \cdots, e_k + w_k\}$ indeed span a k-plane, and the k-planes spanned by

$\{e_1 + w_1, \cdots, e_k + w_k\}$ and $\{e_1 + w_1^1, \cdots, e_k + w_k^1\}$ are identical if and only if $w_i = w_i^1$, $i = 1, \cdots, k$. Hence the above correspondence F is indeed a well defined map, and in fact gives a coordinate neighbourhood at σ with σ given by $(0, \cdots, 0)$.

Let $A: R^{n+k} \to R^{n+k}$ be a symmetric operator. Consider the corresponding bilinear map $g_A: R^{n+k} \times R^{n+k} \to R$ given by $g_A(v, w) = \langle Av, w\rangle$ where $\langle\ ,\ \rangle$ denotes the Euclidean inner product in R^{n+k}, and the function $K: G_{n,k} \to R$ defined by $K(\sigma) = \text{trace } g_A|_\sigma$.

Note. A will be taken to be a Hermitian operator in the complex case, and to be a Hermitian quaternionic operator in the quaternionic case. (A matrix $[a_{ij}]$ of such an operator with respect to a quaternionic orthonormal basis has the property $a_{ij} = \bar{a}_{ji}$.) In every case, A has real eigenvalues (cf. [12, Theorem 4.4] for the quaternionic case), so we get a real-valued function K on Grassmannians.

Proposition 6.1. *σ is a critical point of K iff $A(\sigma) \subseteq \sigma$.*

Proof. By $[v, w, \cdots]$ we shall denote the subspace spanned by the vectors $v, w, \cdots$. Let $\{e_1, \cdots, e_{n+k}\}$ be an orthonormal basis of R^{n+k} such that $\sigma = [e_1, \cdots, e_k]$. Let $\tau = [e_{k+1}, \cdots, e_{k+n}]$, and consider the coordinate neighbourhood of σ described by the map F defined above.

Let $w_i = \sum_{r=k+1}^{k+n} x_{ri} e_r$, $i = 1, \cdots, k$; $Ae_q = \sum_{p=1}^{k+n} a_{pq} e_p$. We wish to show that σ is a critical point of K iff $a_{pq} = 0$ for $q \leq k$, $p > k$. Consider the k-plane $\sigma' = [e_1 + w_1, \cdots, e_k + w_k]$, and let us calculate $K(\sigma')$. Consider the Gram-Schmidt orthogonalization:

$$f_1 = e_1 + w_1, \cdots, f_i = e_i + w_i - \sum_{j=1}^{i-1} \frac{\langle e_i + w_i, f_j\rangle}{\|f_j\|^2} f_j\ , \quad i = 2, \cdots, k\ .$$

Let $A_i = \dfrac{\langle Af_i, f_i\rangle}{\|f_i\|^2}$, $i = 1, \cdots, k$. Clearly $K(\sigma') = \sum_{j=1}^{k} A_i$.

We are interested in small values of x_{ri}'s which describe the k-planes in a neighbourhood of σ. An easy induction shows that

$$\|f_i\|^2 = 1 + \|w_i\|^2 + 0(x^4)\ ,$$
$$\langle e_i + w_i, f_j\rangle = \langle w_i, w_j\rangle + 0(x^4)\ , \qquad 1 \leq j < i \leq k\ .$$

So

$$f_i = e_i + w_i - \sum_{j=1}^{i-1} \langle w_i, w_j\rangle f_j + 0(x^4)\ .$$

Hence

$$A_i = a_{ii} + 2 \sum_{r=k+1}^{k+n} a_{ir} x_{ri} + 0(x^2)\ .$$

It follows that

$$\left.\frac{\partial A_i}{\partial x_{rl}}\right|_\sigma = \begin{cases} 2\,a_{ir}\ , & \text{if}\quad i = l\ ,\quad r > k\ , \\ 0\quad , & \text{if}\quad i \neq l\ ,\quad r > k\ , \end{cases}$$

and so also

$$\partial K/\partial x_{ri}|_\sigma = 2\,a_{ir}\ ,\quad \text{if}\quad i \leq k\ ,\quad r > k\ .$$

The proposition is now clear.

Proposition 6.2. *K has only nondegenerate critical values iff A has $n + k$ distinct eigenvalues.*

Proof. By Proposition 6.1 it follows that the k-planes spanned by k linearly independent eigenvectors of A are precisely the critical points of K. Let $\{e_1, \cdots, e_{n+k}\}$ be an orthonormal frame of R^{n+k} such that $Ae_i = \lambda_i e_i$.

Introduce coordinates and notations as in Proposition 6.1. Then

$$\langle Af_i, f_i\rangle = \lambda_i + \sum_{r=k+1}^{k+n} \lambda_r x_{ri}^2 + 0(x^3)\ ,$$

$$\|f_i\|^2 = 1 + \sum_{r=k+1}^{k+n} x_{ri}^2 + 0(x^4)\ .$$

So

$$A_i = \left\{\lambda_i + \sum_{r=k+1}^{k+n} \lambda_r x_{ri}^2 + 0(x^3)\right\}\left\{1 - \sum_{r=k+1}^{k+n} x_{ri}^2 + 0(x^4)\right\}\ .$$

Hence

$$\frac{\partial A_i}{\partial x_{rl}} = \begin{cases} 2(\lambda_r - \lambda_i)x_{ri} + 0(x^2)\ , & \text{if}\quad i = l\ ,\quad r > k\ . \\ 0(x^2)\ , & \text{otherwise.} \end{cases}$$

It follows that

$$\left.\frac{\partial^2 A_i}{\partial x_{rl}\partial x_{rm}}\right|_\sigma = \begin{cases} 2(\lambda_r - \lambda_i)\ , & \text{if}\quad i = l = m\ ,\quad r > k\ , \\ 0\ , & \text{otherwise,} \end{cases}$$

and so

$$\left.\frac{\partial^2 K}{\partial x_{rl}\partial x_{rm}}\right|_\sigma = \begin{cases} 2(\lambda_r - \lambda_i)\ , & \text{if}\quad i = l = m\ ,\quad r > k\ . \\ 0\ , & \text{otherwise.} \end{cases}$$

From these formulas the assertion of the proposition is obvious. q.e.d.

It thus follows for instance that if Ric. has distinct eigenvalues (at a point p of a Riemann manifold M) then $K_{\mathscr{R}\mathrm{ic}}$, restricted to the fiber of $G_2(M)$ at p, is a Morse function on $G_{2,n}$.

PART II. CONFORMAL TRANSFORMATIONS

In this part we shall deal with a classical problem: when is a conformal map of a Riemann manifold into itself necessarily an isometry (or a homothety)? We cannot expect an affirmative answer without some hypothesis about uniformity of curvature. Indeed, if (M, g) admits an isometry group $I \neq \{1\}$, and $\bar{g}$ is a conformal deformation of g, then $(M, \bar{g})$ admits a conformal group $\supseteq I$, but its isometry group may be smaller than I.

To the other extreme, if curvature is very degenerate (e.g. the unit sphere) the manifold is more likely to admit nonisometric (or nonhomothetic) conformal transformations. The case of two-dimensional manifolds has again to be excluded since, as is well known, every complex analytic diffeomorphism of a Riemann surface is conformal.

A general feeling is that these are typical situations. A well known classical result in this direction is due to Liouville, which says that every conformal map of R^n, $n \geq 3$, is a homothety. A significant partial generalization of this result was obtained by Yano and Nagano [13], who proved: a complete Einstein manifold admitting a 1-parameter group of nonhomothetic conformal transformations is compact, simply connected and in fact isometric to a standard sphere. One of our main observations is that this result is essentially local; one need not invoke completeness. Moreover "a 1-parameter group $\cdots$" may be replaced by "a *single*, nonhomothetic, conformal transformation".

Various results which were based on Yano-Nagano's result, e.g., an important result on conformal transformations of a homogeneous manifold due to Goldberg and Kobayashi [5], and other results depending on signs of certain curvatures, e.g., Lichnerowicz [9, § 83], Obata [10], are also improved in a similar fashion.

Some particular cases of these results have been obtained previously. Professors M. Berger and Alfred Gray kindly referred me to the papers of C. Barbance [2] and C. C. Hsiung [6]. The treatment given here is in the spirit of curvature structures, and the arguments are essentially geometric.

7. Conformal change of a metric

Our basic interest is in the following situation: let (M, g) be a Riemann manifold, and $F: M \to M$ be a conformal diffeomorphism into itself. A more functorial situation of a conformal correspondence between two Riemann manifolds can essentially be reduced to this situation. We identify two copies of M via F and consider the conformal deformation $g \to \bar{g} = F{*}g \equiv f \cdot g$ where f is a positive real valued function on M. In several cases we shall conclude $f \equiv$ constant (or $f \equiv 1$) which will imply that F is a homothety (or isometry).

We collect together relevant formulas arising from the conformal deformation $g \to \bar{g} = f \cdot g$. We shall denote the quantities related to $\bar{g}$ by a bar overhead, and use the notation of § 1. ∇(resp. $\bar{\nabla}$) denotes the covariant derivatives with respect to g (resp. $\bar{g}$), and $g(X, Y)$ (resp. $\bar{g}(X, Y)$) will be abbreviated to $\langle X, Y\rangle$ (resp. $\overline{\langle X, Y\rangle}$).

Write $f = e^{2\phi}$ (which is valid since f is positive real-valued), and let $G =$ grad $\phi =$ the gradient of ϕ with respect to g. The function ϕ defines the symmetric bilinear form Q on the tangent bundle $T(M)$, viz.,

$$Q(X, Y) = XY\phi - (\nabla_X Y)\phi - X\phi Y\phi \equiv \text{Hess}\,\phi(X, Y) - X\phi Y\phi$$

(cf. [8, § 2]). Let $Q.$ denote the corresponding linear transformation defined by $\langle Q.\, X, Y\rangle = Q(X, Y)$.

Proposition 7.1. $\bar{\nabla}_X Y = \nabla_X Y + S(X, Y)$,

where

$$S(X, Y) = X\phi Y + Y\phi X - \langle X, Y\rangle G .$$

Proposition 7.2. $\bar{R}(X, Y)Z = R(X, Y)Z + T(X, Y)Z$,

where

$$T(X, Y)Z = \{Q(Y, Z) + \langle Y, Z\rangle \|G\|^2\}X - \{Q(X, Z) + \langle X, Z\rangle \|G\|^2\}Y + \langle Y, Z\rangle Q.X - \langle X, Z\rangle Q.Y .$$

Proposition 7.3. $\overline{\text{Ric}}(X, Z) = \text{Ric}(X, Z) + \mathcal{T}(X, Z)$,

where

$$\mathcal{T}(X, Z) = -\{(n-2)Q(X, Z) + (n-1)\langle X, Z\rangle \|G\|^2 + \langle X, Z\rangle \text{ trace } Q.\} .$$

Proposition 7.4. $e^{2\phi}\, \overline{\text{Ric}}.\, X = \text{Ric}.\, X - (n-2)Q.X - \{(n-1)\|G\|^2 + \text{trace}\, Q.\}X$.

Proposition 7.5. $e^{2\phi}\overline{Sc} = Sc - \{n(n-1)\|G\|^2 + (2n-2) \text{ trace}\, Q.\}$.

These formulas are classical; see, e.g., Eisenhart [4], where calculations are made in local coordinates.

Proposition 7.6.

$$\begin{aligned}(\bar{\nabla}_X \bar{R})(Y, Z)W = {} & (\nabla_X R)(Y, Z)W + (\nabla_X T)(Y, Z)W \\ & + \{-2X\phi\bar{R}(Y, Z)W + Y\phi\bar{R}(Z, X)W + Z\phi\bar{R}(X, Y)W\} \\ & - \{W\phi\bar{R}(Y, Z)X\} + \{\langle\bar{R}(Y, Z)X, W\rangle G\} \\ & + \{\langle X, Y\rangle\bar{R}(G, Z)W - \langle X, Z\rangle\bar{R}(G, Y)W\} \\ & + \langle\bar{R}(Y, Z)W, G\rangle X + \langle X, W\rangle\bar{R}(Y, Z)G .\end{aligned}$$

Proof. Observe that both sides of the equation are tensors. So to prove it, fix a point $p \in M$, choose a normal coordinate system at p. Next, extend by linearity the vectors $X_p \in T_p(M)$, the tangent space at p, to vector fields X in a neighbourhood at p so that

$$\text{i)}\quad [X, Y] = 0\,, \qquad \text{ii)}\quad (\nabla_X Y)(p) = 0\,.$$

Now,

$$\begin{aligned}&(\bar{\nabla}_X \bar{R})(Y, Z)W \\ &\quad = \bar{\nabla}_X(\bar{R}(Y, Z)W) - \bar{R}(\bar{\nabla}_X Y, Z)W - \bar{R}(Y, \bar{\nabla}_X Z)W - \bar{R}(Y, Z)\bar{\nabla}_X W\,.\end{aligned}$$

Using Proposition 7.1, the first term on the right hand side becomes

$$\begin{aligned}\bar{\nabla}_X(\bar{R}(Y, Z)W) &= \nabla_X(\bar{R}(Y, Z)W) + S(X, \bar{R}(Y, Z)W) \\ &= \nabla_X(R(Y, Z)W) + \nabla_X(T(Y, Z)W) + S(X, \bar{R}(Y, Z)W) \\ &= (\nabla_X R)(Y, Z)W + (\nabla_X T)(Y, Z)W + S(X, \bar{R}(Y, Z)W) \text{ by ii)}\,.\end{aligned}$$

Similarly, $\bar{R}(\bar{\nabla}_X Y, Z)W = \bar{R}(S(X, Y), Z)W$, etc., and finally substituting the formula for S in Proposition 7.1, we get the required result.

Proposition 7.7. *Let $\sum_{\text{cycl}}$ denote a cyclic sum over X, Y, Z. Then*

$$\textstyle\sum_{\text{cycl}}(\nabla_X T)(Y, Z)W = -\sum_{\text{cycl}}\langle\bar{R}(Y, Z)W, G\rangle X - \sum_{\text{cycl}}\langle X, W\rangle\bar{R}(Y, Z)G\,.$$

Proof. Take a cyclic sum over X, Y, Z in the formula in the above proposition. The terms involving $\bar{\nabla}\bar{R}$ and ∇R, and the terms in { } vanish by the Bianchi identity and the usual properties of the curvature tensor.

8. **Einstein manifolds** (Generalities)

As in § 4, we call a Riemann manifold (M, g) an *Einstein manifold* if there exists a constant α such that $\mathrm{Ric}(X, Y) = \alpha\langle X, Y\rangle$, which are the gravitational field equations for the empty universe with the Lorentz metric.

We may note: $Sc = \operatorname{trace}\mathrm{Ric} = n\alpha = $ constant for an n-dimensional Einstien manifold. From this, it easily follows that if $\alpha \neq 0$, homothety of M into itself is necessarily an isometry.

If $\dim M = 2$(resp. 3), an elementary calculation shows that M is of constant curvature α (resp. $\alpha/2$).

Proposition 8.1. *A conformally flat Einstein manifold is of constant curvature.*

Proof. Let K, K_{con}, K_{Ric} denote the Riemannian, conformal and Ricci sectional curvature as in Part I, § 1. For an Einstein manifold with $\mathrm{Ric}(X, Y) = \alpha\langle X, Y\rangle$ and dimension n, $K_{\text{Ric}} = 2\alpha$. So

$$K_{\text{con}} = K - \frac{1}{n-2} K_{\mathscr{R}\text{ic}} + \frac{Sc}{(n-1)(n-2)} \equiv K - \frac{\alpha}{n-1}.$$

For $n \leq 3$, the manifold is already of constant curvature. For $n \geq 4$ conformal flatness amounts to $K_{\text{con}} \equiv 0$, so $K \equiv \alpha/(n-1)$. q.e.d.

The local situation of conformal maps of constant curvature spaces is well known. So in view of the above observations our interest is the case where (M, g) is an Einstein manifold of dimension ≥ 4 which is nowhere of constant curvature (i.e., there does not exist a nonempty open subset of constant curvature in the induced metric). In case dimension $= 4$, we shall show that the Einstein manifold which is nowhere of constant curvature does not admit a nonhomothetic conformal map. In case dimension ≥ 5, we have been able to assert the validity of the same result only if the curvature is not too degenerate.

9. Conformal transformations of an Einstein manifold (Local case)

We consider a conformal diffeomorphism F of an n-dimensional Einstein manifold (M, g) into itself, and shall assume $n \geq 4$ and $\text{Ric}(X, Y) = \alpha\langle X, Y\rangle$. As explained at the beginning of § 7, we shall consider the corresponding conformal deformation $g \to \bar{g} = e^{2\phi}g$. Notice that both (M, g), $(M, \bar{g})$ are Einstein manifolds with the same constant α. We shall use the notation introduced in § 7; in particular, $G = \text{grad}\,\phi$, Q is the bilinear form: $Q(X, Y) = XY\phi - (\nabla_X Y)\phi - X\phi Y\phi$, and $Q.$ is the corresponding linear transformation.

Proposition 9.1. *$Q. = \beta E$ where E is the identity transformation and β: $M \to R$ given by* $\beta = -\dfrac{1}{2}\left\{\dfrac{(e^{2\phi}-1)\alpha}{n-1} + \|G\|^2\right\}$.

Proof. By Proposition 7.4, we have

$$(9.1) \qquad e^{2\phi}\,\overline{\text{Ric}}. = \text{Ric}. - (n-2)Q. - \{n-1)\|G\|^2 + \text{trace}\, Q.\}E.$$

In our case, $\overline{\text{Ric}}. = \text{Ric}. = \alpha E$, so $Q. = \beta E$ for some β: $M \to R$. Taking trace of both sides of (9.1) we have the formula for β.

Proposition 9.2. $T = \dfrac{\alpha(e^{2\phi}-1)}{n-1} I$ *where* $T = \bar{R} - R$ (*cf.* Proposition 7.2) *and I is the trivial curvature structure* (*cf.* § 1).

Proof. Plug the formula in Proposition 9.1, in the expression for T.

Proposition 9.3. *If Y, Z are two vector fields such that $\langle Y, G\rangle = 0 = \langle Z, G\rangle$, then $\bar{R}(Y, Z)G = 0 = T(Y, Z)G = R(Y, Z)G$.*

Proof. $T(Y, Z)G = 0$ is clear from the above proposition. On the other hand, if $\sum_{\text{cycl}}$ denotes a cyclic sum over X, Y, Z, then by the above proposition and Proposition 7.7 we have

$$\sum_{\text{cycl}}(\nabla_X T)(Y,Z)W = \frac{2\alpha e^{2\phi}}{n-1}\sum_{\text{cycl}} X\phi\{\langle Y,W\rangle Z - \langle Z,W\rangle Y\} \tag{9.2}$$
$$= -\sum_{\text{cycl}}\{\langle \bar{R}(Y,Z)W,G\rangle X + \langle X,W\rangle \bar{R}(Y,Z)G\}.$$

Let $\langle Y,G\rangle = 0 = \langle Z,G\rangle$ and put $X = W = G$, then (9.2) reduces to $\bar{R}(Y,Z)G = 0$. Since $\bar{R} = R + T$, the proof is complete.

Proposition 9.4. *Let $p \in M$ and suppose that $G_p \neq 0$. Then for every 2-plane σ containing G_p, $\bar{K}(\sigma) = K(\sigma) = \alpha/(n-1)$.*

Proof. In (9.2) of the above proposition, assume X, Y, Z to be linearly independent and all perpendicular to W, then we get $\langle \bar{R}(Y,Z)W,G\rangle = 0$. In particular, if Y, W, G are mutually orthogonal, then $\langle \bar{R}(Y,G)W,G\rangle = 0$. It follows that the bilinear form $(Y,W) \to \langle \bar{R}(Y,G)W,G\rangle$ defined on the orthogonal complement of G is a scalar multiple of $\langle\ ,\ \rangle$; so if X is perpendicular to G, then $\langle \bar{R}(X,G)X,G\rangle = \text{constant} \cdot \|X\|^2$. It follows that if $G_p \neq 0$, then for every 2-plane σ containing G_p, $\bar{K}(\sigma) = \text{constant} = c$ say. However $\overline{\text{Ric}}\,(G,G) = \alpha\|G\|^2 = (n-1)\,c\|G\|^2$. So $c = \alpha/(n-1)$. Since clearly by using $\bar{R} = R + T$ and Proposition 7.2, we have

$$e^{2\phi}\bar{K}(\sigma) = K(\sigma) + \frac{(e^{2\phi}-1)}{n-1}\alpha,$$

and also $K(\sigma) = \alpha/(n-1)$. q.e.d.

This proposition shows that if (M,g) does admit a nonhomothetic conformal map (so that $G \not\equiv 0$), then at points p where $G_p \neq 0$, the sectional curvature K is very degenerate. To see how exactly it is degenerate we apply the results of § 5.

Let $p \in M$ such that $G_p \neq 0$. Choose an orthonormal frame $\{e_1 = G_p/\|G_p\|, e_2, \cdots, e_n\}$ at p, and say $\sigma =$ the 2-plane spanned by $\{e_1, e_2\}$. Then in the notation of § 5 Proposition 9.3 says that $R_{12i2} = 0 = R_{121i}$, $i \geq 3$. So σ is a critical point of $K|_{\pi^{-1}(p)}$. Also using Proposition 9.4 and Proposition 5.2, its hessian at σ looks like

$$\left|\begin{array}{c|c} \dfrac{\partial^2 K}{\partial x_i \partial x_j} & 0 \\ \hline 0 & 0 \end{array}\right|.$$

So nullity $\geq n-2$. Moreover,

$$\Delta K(\sigma) = \text{trace of the hessian} = \sum_{i<2}(K_{2i} - K_{12})$$
$$= \sum_{i\neq 2} K_{2i} - (n-1)K_{12} = \alpha - \alpha = 0.$$

In view of this discussion, we have the following

Theorem 9.5. *Let (M, g) be an Einstein manifold such that the set*

$$S = \{p \in M \mid K|_{\pi^{-1}(p)} \text{ has only nondegenerate critical points}\} \tag{9.3}$$

is dense in M. Then every conformal map of (M, g) into itself is a homothety.

Proof. By the above discussion, if $p \in S$, then $G_p = 0$. Since S is dense in M, $G \equiv 0$.

Remark. It is clear from the above discussion that the condition (9.3) may be replaced by a still weaker condition:

(9.4) The set $S' = \{p \in M \mid$ If σ is a critical point of $K|_{\pi^{-1}(p)}$, then either nullity of $\sigma \leq n - 3$ or else $\Delta K(\sigma) \neq 0$, where Δ is the laplacian determined by the canonical metric on the grassmannian$\}$

is dense in M.

We have been able to get rid of the condition (9.3) or (9.4) in case dimension $M = 4$. This is because of the following general lemma on curvature structures which is of some interest in itself. This was kindly pointed out to me by Professor M. Berger.

Lemma on curvature structures. *Let (M, g) be a 4-dimensional Riemann manifold equipped with a curvature structure T, and* Ric_T *be the corresponding Ricci tensor* (*cf.* § 1, Part I). *Suppose that* $\mathrm{Ric}_T = \alpha g$ *for some* $\alpha\colon M \to R$. *Then for every $p \in M$ and two mutually orthogonal 2-plane sections σ_1, σ_2 at p,*

$$K_T(\sigma_1) = K_T(\sigma_2).$$

Proof. Choose an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ at p, and write $K_{ij} = K_T(e_i, e_j\}$. We typically show $K_{12} = K_{34}$. We have

$$K_{12} + K_{13} + K_{14} = \alpha(p) = K_{21} + K_{23} + K_{24}, \tag{9.5}$$

which implies

$$K_{13} + K_{14} = K_{23} + K_{24}. \tag{9.6}$$

Similarly,

$$K_{12} + K_{14} = K_{32} + K_{34}, \tag{9.7}$$

$$K_{12} + K_{13} = K_{42} + K_{43}. \tag{9.8}$$

Subtracting (9.7) and (9.8) from (9.6) implies $K_{12} = K_{34}$.

Corollary. *Let (M, g) be a 4-dimensional Riemann manifold. Then for every $p \in M$ and two mutually orthogonal 2-planes σ_1, σ_2 at p, $K_{\mathrm{con}}(\sigma_1) = K_{\mathrm{con}}(\sigma_2)$.*

Proof. This is because Ric-Kon $= 0$ (cf. § 3).

Corollary. *Let (M, g) be a 4-dimensional Einstein manifold. Then for every $p \in M$ and two mutually orthogonal 2-planes σ_1, σ_2 at p, $K(\sigma_1) = K(\sigma_2)$.*

We return to the standard situation of this section.

Theorem 9.6. *Let (M, g) be a 4-dimensional Einstein manifold which is nowhere of constant curvature. Then every conformal map of (M, g) into itself is a homothety.*

Proof. We use the by now standard notation. Suppose there exists $p \in M$ such that $G_p \neq 0$. As shown in Proposition 9.4 for every 2-plane σ at p containing G_p, $K(\sigma) = \alpha/3$. By the above lemma, for the orthogonal complement σ' of σ we also have $K(\sigma') = \alpha/3$. From this it easily follows that $K \equiv \alpha/3$ at p. The set $\{p \mid G_p \neq 0\}$ is open. So $K \equiv \alpha/3$ on an open set contradicting the hypothesis that (M, g) is nowhere of constant curvature.

10. Conformal transformations of an Einstein manifold (complete case)

We continue to use the notation of last section.

Proposition 10.1. *Trajectories of G are (pointsetwise) geodesics.*

Proof. The formula in Proposition 9.1 written in full reads: for any vector fields X, Y,

$$XY\phi - (\nabla_X Y)\phi - X\phi Y\phi + \frac{1}{2}\langle X, Y\rangle \|G\|^2 + \frac{\alpha}{2(n-1)}\langle X, Y\rangle (e^{2\phi} - 1) = 0 . \tag{10.1}$$

The assertion of our proposition is trivial if $G \equiv 0$. So suppose there exists $p \in M$ such that $G_p \neq 0$, and let $X_p = G_p$. Consider a geodesic γ through p with the initial velocity X_p, and let X be the tangent vector field of γ. Let $Y_p \in T_p(M)$ such that $Y_p\phi = \langle Y_p, G\rangle = 0$, and extend Y_p to a vector field Y by parallel translation along γ. Then $\nabla_X Y = 0$, $\langle X, Y\rangle = 0$, and (10.1) becomes

$$XY\phi - X\phi Y\phi = 0 .$$

Treating this as an ordinary differential equation for $Y\phi$, we see that $Y\phi = 0$ is the unique solution with the initial condition $Y_p\phi = 0$. This means that a vector Y_p orthogonal to G at p remains orthogonal to G under parallel translation along γ. It follows that G is tangential to γ.

Corollary. *Let p be a critical point of ϕ. Then ϕ is constant on circles centered at p.*

Proof. Indeed by the above proposition, ϕ is a function of the radial distance from p in a normal coordinate neighbourhood of p.

Corollary. *Suppose $\phi \not\equiv$ constant. Then the critical points of ϕ are isolated.*

Proof. Suppose not. Then clearly the set

$$S = \{p \in M \mid p \text{ is a non-isolated critical point of } \phi\}$$

is a nonempty closed set. We show that it is also open. This would imply, since M is connected, that $S = M$, and so $\phi \equiv$ constant, contradicting the hypothesis.

Let $p \in S$, $B(p, \varepsilon)$ be a ball of radius $\varepsilon > 0$ centered at p on which normal coordinates are valid, and d denote the distance defined by the Riemannian metric. Since p is a nonisolated critical point, there exists a critical point $q \in B(p, \varepsilon)$. Let $d(p, q) = \varepsilon' \leq \varepsilon$. Since ϕ is a radial function in $B(p, \varepsilon)$, the circle $S(p, \varepsilon')$ of radius ε' centered at p consists of critical points of ϕ. Now we can apply the same argument to every point of $S(p, \varepsilon')$, which shows that $B(p, \varepsilon')$ consists of critical points of ϕ. So S is open and the proof is finished.

Proposition 10.2. *Suppose that $\phi \not\equiv$ constant, and let p be a critical point of ϕ. Then there exists a neighbourhood of p on which $K \equiv \alpha/(n-1)$.*

Proof. Let $\bar{p}$ be any point of the standard space $(S, \bar{g})$ of constant curvature $\alpha/(n-1)$. Choose any isometry $u\colon T_p(M) \to T_{\bar{p}}(S)$. Let $X_p \in T_p(M)$ and $X_{\bar{p}} = u(X_p)$. Let γ be the geodesic consisting of $p_t = \exp_p tX_p$, and $\bar{\gamma}$ be the correspoding geodesic consisting of $\bar{p}_t = \exp_{\bar{p}} tX_{\bar{p}}$. Let σ be a 2-plane containing X_p, and let σ_t denote the 2-plane at p_t obtained by parallel translation of σ along γ. Correspondingly, let $\bar{\sigma} = u(\sigma)$, and let $\bar{\sigma}_t$ denote the 2-plane at $\bar{p}_t$ obtained by parallelly translating $\bar{\sigma}$ along $\bar{\gamma}$.

σ_t clearly contains G_{p_t}, so using the previous proposition $K(\sigma_t) = \alpha/(n-1) = \bar{K}(\bar{\sigma}_t)$, i.e., sectional curvature is preserved under parallel translation along geodesics. It follows by a well known criterion of Eli Cartan (cf. [3]) that u extends to an isometry of a neighbourhood of p onto a neighbourhood of $\bar{p}$. On this neighbourhood of p, $K \equiv \alpha/(n-1)$. q.e.d.

After these preliminaries we state the principal result of this section which is a common generalization of the result of Liouville and that of Yano-Nagano.

Theorem 10.3. *Let (M, g) be a complete Einstein space which admits a nonhomothetic conformal diffeomorphism F onto itself. Then (M, g) is compact, simply connected and in fact isometric to a standard sphere.*

The proof is divided into several steps. Many thanks to Alan Weinstein for his help in the final global argument. First we prove

Proposition 10.4. *A complete Einstein (M, g) with $\alpha \leq 0$ does not admit a nonhomothetic conformal diffeomorphism onto itself.*

Proof. (i) *First consider the case $\alpha = 0$.* Using the standard notation of this and the last section, suppose there exists $p \in M$ such that $G_p \neq 0$. Let $X_p = G_p/\|G_p\|$, and γ be a geodesic through p with initial velocity X_p. By completeness, γ extends infinitely. Let X be the tangent vector field to γ.

Since $\alpha = 0$ and $\nabla_X X = 0$, by Proposition 9.1, ϕ satisfies the differential equation

$$XX\phi = (X\phi)^2 - \frac{1}{2}\|G\|^2 = \frac{1}{2}(X\phi)^2 . \tag{10.2}$$

By considering this as an ordinary differential equation for $X\phi$ with the initial condition $X_p\phi = \|G_p\| > 0$, obviously its solution has a singularity at finite positive time. Since this is not possible, G_p must be 0.

(ii) *Next suppose* $\alpha < 0$. With the above notation, ϕ satisfies the differential equation

$$XX\phi = \frac{1}{2}(X_\phi)^2 - \frac{\alpha}{2(n-1)}(e^{2\phi} - 1) . \tag{10.3}$$

$X_p \neq 0$ so $\sigma \not\equiv$ constant. Since we have supposed that the conformal diffeomorphism F (which gives rise to ϕ) is *onto*, the conformal deformation of metric with respect to F^{-1} is obviously $g \to e^{-2\phi}g$. Hence considering F or F^{-1} we may suppose $\phi(p) \geq 0$.

Since $\alpha < 0$, (10.3) shows that $X_p\phi = \|G_p\| > 0$ and $X\phi$ is monotonically increasing along γ, and therefore ϕ also monotonically increases along γ. It follows that the right hand side of (10.3) majorizes the right hand side of (10.2). Since (10.2) has a singularity at a finite positive time, so does (10.3). So again we must have $G \equiv 0$.

Remark. In case $\alpha < 0$, we have made essential use of the fact that F is a 1–1 onto diffeomorphism. Restriction to 1–1 onto diffeomorphism is clearly elegant and sufficient for group theoretical purposes. However, as a point of geometric interest we wish to remark that Theorem 10.3 as stated is not valid if we merely assume that F is a 1–1 *into* diffeomorphism. Indeed in the usual open disc model for the hyperbolic space the map $X \to \lambda X$, $0 < \lambda < 1$, which is a contracting homothety for the flat metric, is a 1–1 into, nonhomothetic conformal diffeomorphism with respect to the hyperbolic metric.

We return to the situation of the theorem. Because of the above proposition, α must be > 0. Consequently the Ricci tensor is positive definite. Hence by the well known argument of Meyers, (M, g) has finite diameter, and moreover it must be compact, since it is complete.

We have assumed $\phi \not\equiv$ constant. So ϕ has at least two critical points p and q say. Let $d(p, q) = l$.

Lemma. *Every geodesic from* p *and of length* l *has* q *as its endpoint.* Indeed, let

$$\mathcal{S} = \{X_p \in T_p(M) | \; \|X_p\| = 1 \text{ and the geodesic with initial velocity } X_p \text{ reach } q \text{ after time } l\} .$$

Since $d(p, q) = l$ and the manifold is complete, by the theorem of Hopf and

Rinow there is at least one geodesic of length l from p to q. So $\mathscr{S}$ is nonempty. It is manifestly closed. It now suffices to show that $\mathscr{S}$ is also open.

Let $X_p \in \mathscr{S}$, and γ be the corresponding geodesic. Let m be the midpoint of γ, and consider the arc A_p (resp. A_p) of the circle $S(p, l/2)$ (resp. $S(q, l/2)$). A_p and A_q must coincide in a neighbourhood of m; for, the normal coordinates at p (resp. q) are valid in a neighbourhood of m, and so by the first corollary of Proposition 10·1, ϕ is constant on $A_p \cup A_q$. If A_p and A_q do not coincide, m must be a critical point of ϕ, and so $A_p \cup A_q$ consists of critical points. This contradicts the fact that the critical points of ϕ are isolated (cf. the second corollary of Proposition 10.1). Thus $\mathscr{S}$ is also open, and the lemma is proved.

We can now set up a homeomorphism of M with the n-dimensional sphere in the obvious way: map $B(p, l/2)$ on the northern hemisphere, and $B(q, l/2)$ on the southern hemisphere. The lemma says that the equators match.

The normal coordinates at p (resp. g) are valid in $M - \{q\}$ (resp. $M - \{p\}$). So by Proposition 10.2, (M, g) is of constant curvature $K = \alpha/(n-1)$. By the fact about constant curvature spaces, (M, g) is isometric to the standard n-dimensional sphere of constant curvature $\alpha/(n-1)$. This finishes the proof of the theorem.

11. Conformal transformations of a compact manifold

Let (M, g) be a compact manifold admitting a conformal diffeomorphism $F: M \to M$, and $g \to \bar{g} = e^{2\phi}g$ be the corresponding conformal deformation. *Compactness guarantees that ϕ has critical points on M.* If a geometric condition of negativity of certain curvatures is satisfied, then looking at the behaviour of ϕ at its critical points we can conclude that $\phi \equiv 0$. A typical result is the following:

We recall that π: $G_2(M) \to M$ is the canonical projection and K: $G_2(M) \to \mathscr{R}$ is the sectional curvature function.

Proposition 11.1. *Let (M, g) be a compact manifold, denote* $\inf_{\sigma \in \pi^{-1}(p)} K(\sigma) = \lambda(p)$, *and suppose that $\lambda \equiv$ a negative constant $\equiv \lambda_0$. Then a conformal diffeomorphism of M onto itself is necessarily an isometry.*

Proof. Let $F: M \to M$ be a conformal diffeomorphism, and $g \to e^{2\phi}g = \bar{g}$ the corresponding conformal deformation. Note that both (M, g) and $(M, \bar{g})$ have the same $\lambda \equiv \lambda_0$.

Assume $\phi \not\equiv 0$. Conformal deformation of the metric corresponding to F^{-1} is clearly $g \to e^{-2\phi}g$, so considering F of F^{-1} we may assume that ϕ attains its maximum at a point p and $\phi(p) > 0$.

Denoting the quantities related to $\bar{g}$ by a bar overhead, and using Proposition 7.2 we see that for a 2-plane σ at p spanned by $\{X_p, Y_p,$ orthonormal$\}$,

$$e^{2\phi(p)}\bar{K}(\sigma) = K(\sigma) - \{X_pX_p\phi + Y_pY_p\phi\} .$$

If $\bar{K}$ attains the minimum λ_0 at σ, we have

$$e^{2\phi(p)}\lambda_0 = K(\sigma) - \{X_pX_p\phi + Y_pY_p\phi\} . \tag{11.1}$$

The left hand side $< \lambda_0$, while on the right hand side $K(\sigma) \geq \lambda_0$, and $X_pX_p\phi \leq 0$, $Y_pY_p\phi \leq 0$ since p is a maximum. Thus (11.1) cannot hold. So $\phi \equiv 0$.

Remark. Similar arguments will also show the following:

Let (M, g) be compact. Then under any of the following conditions (M, g) does not admit a nonisometric conformal diffeomorphism onto itself.

$$\sup_{\sigma \in \pi^{-1}(p)} K(\sigma) \equiv \text{constant (independent of } p) < 0 . \tag{11.2}$$

$$\inf_{\substack{\|X_p\|=1 \\ X_p \in T_p(M)}} \operatorname{Ric}(X_p, X_p) \equiv \text{constant (independent of } p) < 0 . \tag{11.3}$$

$$\inf_{\sigma \in \pi^{-1}(p)} K_{\mathrm{Ric}}(\sigma) = \text{constant (independent of } p) < 0 . \tag{11.4}$$

We may replace inf by sup in (11.3) and (11.4).

(11.5) Let μ_p denote a density $\pi^{-1}(p)$ which depends differentiably on p, and assume

$$\int_{\sigma \in \pi^{-1}(p)} K(\sigma) d\mu(\sigma) = \text{constant (independent of } p) < 0 .$$

A particular case of (11.5) is the one when μ_p is defined canonically by the Riemann metric. In this case, the integral in (11.5) is the scalar curvature.

This result for scalar curvature was obtained by Lichnerowicz for an infinitesimal conformal diffeomorphism (cf. [9, p. 134]) and by Obata [10] for a (finite) conformal diffeomorphism.

Remark. A more functorial formulation of Proposition 11.1 would be:

Let (M, g) and $(\bar{M}, \bar{g})$ be compact manifolds, and $F: M \to \bar{M}$ a conformal diffeomorphism such that for every $p \in M$,

$$\inf_{\sigma \in \pi^{-1}(p)} K(\sigma) = \inf_{\sigma \in \pi^{-1}(F(p))} \bar{K}(\bar{\sigma}) < 0 .$$

Then F is an isometry.

Similar generalizations may be made in cases listed in the above remark. In particular, in view of the results of Part I, these generalizations apply to sectional curvature preserving, $K_{\mathscr{R}\mathrm{ic}}$-preserving or Ric-preserving diffeomorphisms of compact manifolds.

12. Conformal transformation of a locally homogeneous manifold

Recall that a Riemann manifold (M, g) is called *locally homogeneous* if for any two points $p, q \in M$, there exists an isometry of a neighbourhood of p onto a neighbourhood of q carrying p into q. It is called *homogeneous* if for any two points $p, q \in M$, there exists a (global) isometry carrying p into q.

Using the result of Yano-Nagano and a theorem of Kuiper, Goldberg and Kobayashi [5] prove the following beautiful result:

A compact homogeneous (M, g) of dimension ≥ 4 admitting a 1-parameter group of nonisometric conformal transformations is isometric to a standard sphere.

We shall replace "a 1-parameter group$\cdots$" by "a *single* nonisometric conformal transformation", and try to get rid of compactness.

Proposition 12.1. *Let* (M, g) *be a nowhere conformally flat* (*cf.* § 3) *Riemann manifold of* dim ≥ 4. *Suppose moreover that either*

$$\text{i)} \quad \inf_{\sigma \in \pi^{-1}(p)} K_{\text{con}}(\sigma) = \textit{constant (independent of } p) \equiv a \,,$$

or

$$\text{ii)} \quad \sup_{\sigma \in \pi^{-1}(p)} K_{\text{con}}(\sigma) = \textit{constant (independent of } p) \equiv b \,,$$

Then a conformal map $F: M \to M$ *into itself is an isometry.*

Proof. Consider the corresponding conformal deformation $g \to \bar{g} = e^{2\phi}g$. Then both (M, g) and $(M, \bar{g})$ together satisfy (i) (or (ii)) with the same constant a (or b). Since (M, g) is nowhere conformally flat, by Theorem 3.2 and the corollary to Proposition 3.1 we see that $a \neq 0$ in case i) or $b \neq 0$ in case ii).

On the other hand, using the fact that the conformal curvature tensor is a conformal invariant (and denoting the quantities related to $\bar{g}$ by a bar overhead) we see that

$$e^{2\phi}\bar{K}_{\text{con}} = K_{\text{con}} \,.$$

However, the inf of the left hand side in case i) is $e^{2\phi}a$ and that of the right side is a. So $\phi \equiv 0$. Similary the case ii) is treated. q.e.d.

Since both i) and ii) clearly hold for a locally homogeneous space, we have

Theorem 12.2. *A locally homogeneous nowhere conformally flat* (M, g) *of dimension* ≥ 4 *does not admit a nonisometric conformal map.*

Leaving aside the case of 3-dimensional manifolds, a natural question is:

Is a locally homogeneous conformally flat Riemann manifold admitting a nonisometric conformal map onto itself necessarily of constant curvature?

Unfortunately, the answer is *no* in general. Consider the following example: Let $v \in R^n$, $v \neq 0$, and consider $M = R^n - \{0, \pm v, \pm 2v, \cdots\}$. Equip M with the metric $g = g_0/r^2$ where r is the radial distance from 0, and g_0 is the flat metric. This is so arranged that on $R^n - \{0\}$, rotations and homothetics with respect to g_0 become isometries with respect to g. So $R^n - \{0\}$ is a homogeneous space, and consequently M is a locally homogeneous space. It may be checked that (M, g) is not of constant curvature, but the map $F: M \to M$ defined by $X \to X + v$ is a nonisometric conformal map of M onto itself.

The author does not know whether the answer to the above question is affirmative for complete (but non-compact) Riemann manifolds.

The answer is *affirmative for locally homogeneous Einstein* manifolds due to the fact that 3-dimensional Einstein manifolds and conformally flat Einstein manifolds are of constant curvature (cf. § 8). Similarly the answer is affirmative for $K_{\mathscr{R}\mathrm{ic}}$-preserving diffeomorphism (cf. § 4).

We shall end this section by the following generalization of the result of Goldberg-Kobayashi.

Theorem 12.3. *Let (M, g) be a compact locally homogeneous manifold of* dim ≥ 4 *admitting a nonisometric conformal diffeomorphism. Then it is isometric to a standard sphere.*

C. Barbance [2] has also obtained this result by a different method using a theorem of Obata [10]. We shall present here a more geometric argument which is closer to that of Goldberg and Kobayashi.

Lemma. *Under the hypothesis of the theorem, the Ricci tensor must be positive definite.*

Proof. By using Proposition 11.1 we see that $K \geq 0$. So the Ricci tensor is certainly positive semidefinite. Suppose the Ricci tensor is not positive definite. Then by local homogeneity, at each point p there exists a vector X_p such that Ric $(X_p, X_p) = 0$.

Let $F: M \to M$ be the nonisometric conformal diffeomorphism. Then as in Proposition 11.1, we can assume that the corresponding function ϕ takes a *positive* value at its maximum p.

Let $X_p \in T_p(M)$ be such that

$$0 = \overline{\mathrm{Ric}}\,(X_p, X_p) = \mathrm{Ric}\,(X_p, X_p) - \{n - 2)X_pX_p\phi + \text{trace } Q.\} .$$

Since Ric is positive semidefinite and $Q.$ is negative semidefinite, we see that trace $Q. = 0$, so $Q. = 0$. But then for any 2-plane σ at p, $\sigma = \{X_p, Y_p$ orthonormal$\}$,

$$e^{2\phi(p)}K(\sigma) = K(\sigma) - \{X_pX_p\phi + Y_pY_p\phi\} \equiv K(\sigma) .$$

Since K and $\bar{K}$ take the same range of values, $\phi(p) = 0$ unless $K \equiv 0$ (at p and hence everywhere by local homogeneity). But if $K \equiv 0$, then the space is Einsteinian, and so by Theorem 10.3, this cannot happen. q.e.d.

Let $(\tilde{M}, \tilde{g})$ be the covering space of (M, g). Since $(\tilde{M}, \tilde{g})$ also has positive definite Ricci curvature, $\tilde{M}$ is also compact. It is simply connected and locally homogeneous, so it is actually homogeneous (cf. [11]).

Because of Proposition 12.1 we can also assume that (M, g) and hence $(\tilde{M}, \tilde{g})$ to be conformally flat. Following [5], we invoke a theorem Kuiper [7] to assert that $(\tilde{M}, \tilde{g})$ is conformal to a standard sphere.

Let C be the group of all conformal diffeomorphisms of M, and I the transitive group of isometries of M. Let 0 be a maximal compact subgroup of C which contains I, and let $\bar{g}$ denote the constant curvature metric on the sphere which is invariant under 0. Since $\bar{g}$ is conformal to $\tilde{g}$, $\tilde{g} = f\bar{g}$ where f is a positive real-valued function on M. But I is transitive and $I \subseteq 0$. If easily follows that f must be constant. This finishes the proof of the theorem.

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JOHNS HOPKINS UNIVERSITY