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NONDEGENERATE HOMOTOPIES OF CURVES ON THE UNIT 2-SPHERE

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The purpose of this paper is to prove

Theorem 1. *There are 6 second order nondegenerate regular homotopy classes of closed curves on the unit 2-sphere.*

Throughout this paper S^2 refers to the unit 2-sphere in E^3. A second order nondegenerate curve in S^2 is an immersion of S^1 in S^2 such that the geodesic curvature is continuous and nonzero. A regular homotopy of curves on S^2, $h\colon S^1 \times I \to S^2$, is called nondegenerate if each curve $h_t\colon S^1 \to S^2$ is nondegenerate and if the geodesic curvature is continuous on $S^1 \times I$. The homotopies we consider are free, or without base point, and the curves are oriented curves.

Proposition 2. *The following 6 curves, when projected via central projection into a hemisphere of S^2, are in different nondegenerate homotopy classes.*

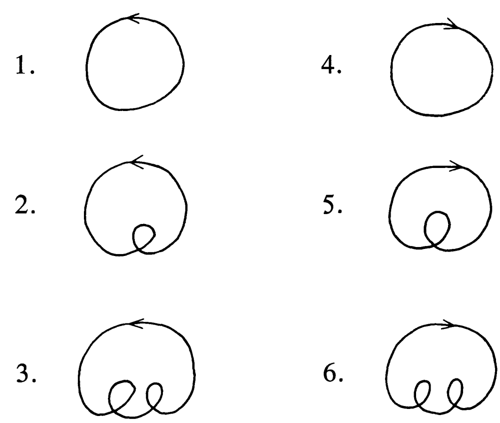

Fig. 1

This proposition is an observation of William F. Pohl.

Proof. We fix an orientation of S^2 by saying that a tangent frame e_1e_2 to

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S^2 is positive if $e_1e_2e_3$ is right-handed where e_3 is the outward normal. If $h\colon S^1 \to S^2$ is an immersion we define e_1 to be the unit tangent vector to the curve and e_2 to be normal to the curve so that e_1e_2 is a tangent frame agreeing with the orientation of S^2. The geodesic curvature may be defined by the equation

$$\frac{de_1}{ds}\cdot e_2 = k_g\ ,$$

where s is the arc length. The geodesic curvature is greater than zero for curves 1, 2, 3, and less than zero for curves 4, 5, 6. Since the sign of the geodesic curvature is preserved under nondegenerate homotopy 1, 2, 3 are distinct from 4, 5, 6. By symmetry, it is enough to show that 1, 2, 3 are in distinct nondegenerate homotopy classes. 1 and 2 are not regularly homotopic; neither are 2 and 3. (See Smale [4] and Feldman [1].) Thus it is enough to show that 1 and 3 are in different nondegenerate homotopy classes. This follows from

Proposition 3. *A nondegenerate homotopy of a simple curve on S^2 can introduce no double points.*

Proof. Let $h_t\colon S^1 \to S^2$ be a nondegenerate homotopy such that h_0 is simple. If there are double points there is a first time when they appear, say t_0. We now use

Theorem 4 (*Fenchel* [2]). *A nondegenerate simple arc, simple closed curve, or closed curve with one double point which lies in S^2 must lie in an open hemisphere.*

Since $h_t(S^1)$ for $t < t_0$ are nondegenerate simple closed curves, they lie in hemispheres. Let $G_t(\theta)$ with θ varying over S^1 be the great circle which is tangent to $h_t(S^1)$ at $h_t(\theta)$. Thus we see that $h_t(S^1)$ lies in the closed hemisphere $H_t(\theta)$ bounded by $G_t(\theta)$ for $t < t_0$. In the limit $h_{t_0}(S^1)$ is contained in $H_{t_0}(\theta)$ for all $\theta \in S^1$. Take $\theta_1 \in S^1$, $h_{t_0}(S^1) \subset H_{t_0}(\theta_1)$. Since h_{t_0} is nondegenerate, $h_{t_0}(S^1)$ must have a point in the interior of $H_{t_0}(\theta_1)$, say $h_{t_0}(\theta_2)$. Thus $h_{t_0}(S^1)$ lies in the closed sector between the two great circles $G_{t_0}(\theta_1)$, $G_{t_0}(\theta_2)$ and meets each of them tangentially. Again because h_{t_0} is nondegenerate, $h_{t_0}(S^1)$ must have a point in the interior of this sector, say $h_{t_0}(\theta_3)$. $G_{t_0}(\theta_3)$ cannot pass through $G_{t_0}(\theta_1) \cap G_{t_0}(\theta_2)$ because $h_{t_0}(S^1)$ meets $G_{t_0}(\theta_1)$ and $G_{t_0}(\theta_2)$ tangentially. Thus $h_{t_0}(S^1)$ lies in a proper spherical triangle bounded by the great circles $G_{t_0}(\theta_i)$, $i = 1, 2, 3$. Hence it lies in an open hemisphere. Thus h_t, $t_0 - \varepsilon \leq t \leq t_0$ for some $\varepsilon > 0$ provides a nondegenerate homotopy between simple curves and a curve with doubles points which lies in an open hemisphere. But this is impossible for plane curves and so, by central projection, for hemispherical curves.

The total turning of an arc $h\colon [0, 1] \to E^2$ is defined to be $\theta(1) - \theta(0)$ where θ is the argument of the tangent vector, and is a continuous function on $[0, 1]$. If k_g is the geodesic curvature, the total turning is given by $\int k_g ds$, where the integral is over the arc, and s is the arc length. For a closed curve the total turning is 2π times the index of rotation.

We shall need the following result (unpublished) of William F. Pohl.

Theorem 5. *Two nondegenerate plane curves are nondegenerately homotopic if and only if they have the same total turning. Two nondegenerate plane arcs, which agree on neighbourhoods of their endpoints and have the same total turning, are nondegenerately homotopic by a homotopy which is constant on the neighbourhoods of the endpoints.*

We shall prove the second statement concerning arcs. The proof of the first statement involves the same idea.

Proof (William F. Pohl). Let $h_i\colon [0,3] \to E^2$, $i = 1, 2$, be two nondegenerate plane arcs. Suppose that $h_1 = h_2$ on $[0,1] \cup [2,3]$ and that

$$\int_1^2 k_1 ds = \int_1^2 k_2 ds ,$$

where k_i is the curvature of h_i, and ds is the element of arc length. Let $e_i(t)$ be the unit tangent vector of h_i at t, taking into account the orientation, $i = 1, 2$. Let $\theta_i(t) = \angle(e_i(t), e_i(0))$, $i = 1, 2$, be continuous functions such that $\theta_i(0) = 0$, $i = 1, 2$. Assume that $k_i > 0$, $i = 1, 2$ (if $k_i < 0$ the proof is similar). Then θ_1 and θ_2 are monotonically increasing functions of t. Hence we may use θ_1 and θ_2 as parameters for the arcs h_1 and h_2 respectively. Now $\theta_1(t) = \theta_2(t)$ for $0 \le t \le 1$. Let $2 \le t \le 3$. Then

$$\begin{aligned}\theta_1(t) - \theta_1(1) = \int_1^t k_1 ds = \int_1^2 k_1 ds + \int_2^t k_1 ds &= \int_1^2 k_2 ds + \int_2^t k_2 ds \\ &= \theta_2(t) - \theta_2(1) .\end{aligned}$$

Thus $\theta_1(t) = \theta_2(t)$ for $2 \le t \le 3$. So regarding h_i as parametrized by θ_i we see that h_1 and h_2 both map $[0, \theta_1(3)]$ into E^2 and agree on $[0, \theta_1(1)] \cup [\theta_1(2), \theta_1(3)]$. Let y vary over $[0, \theta_1(3)]$. Define

$$h_t(y) = t h_2(y) + (1 - t) h_1(y) ;$$

h_t is the required homotopy.

Lemma 6. *Every nondegenerately immersed curve in S^2 is nondegenerately homotopic to a curve lying in a hemisphere.*

In order to prove this lemma it will be necessary to have some information about nondegenerate plane arcs. The proof of Lemma 6 is postponed until after the proof of Lemma 9.

Lemma 7. *Suppose that f is an oriented planar arc, and let θ be the angle between the tangent vector and a fixed vector. Suppose that θ is a monotone increasing function differentiable except for a jump discontinuity at one point*

$f(p)$, and that the jump is less than π. Then f may be approximated by an arc f' which agrees with f outside any chosen neighbourhood of $f(p)$, f' has positive continuous curvature, and the total turning of f' equals the total turning of f plus the jump. If f has no double point in the chosen neighbourhood of $f(p)$ neither will f'.

We do not prove this lemma but remark only that a proof may be obtained using spiral arcs, see for example Guggenheimer [3, pp. 48–52].

Lemma 8. *Suppose that f: $[0, 1] \to E^2$ is an oriented arc with positive curvature such that $f(0)$, $f(1)$ lie in the lower half plane and such that the curve crosses the x-axis transversally twice. Suppose that either a) the second crossing (in the sense of the orientation of the arc) is to the left of the first crossing or b) the second crossing is to the right and the total turning of the arc in the upper half plane is greater than or equal to 2π. Then the arc is nondegenerately homotopic to an arc lying entirely in the lower half plane by a homotopy which leaves a neighbourhood of the endpoints fixed.*

Proof. Note first that if two oriented arcs agree on neighbourhoods of their endpoints, then their total turnings differ by a multiple of 2π. Let l be a line parallel to the x-axis and a little below chosen so that l meets the arc transversally at two points just as the x-axis does. One constructs using Lemma 7 the following arcs.

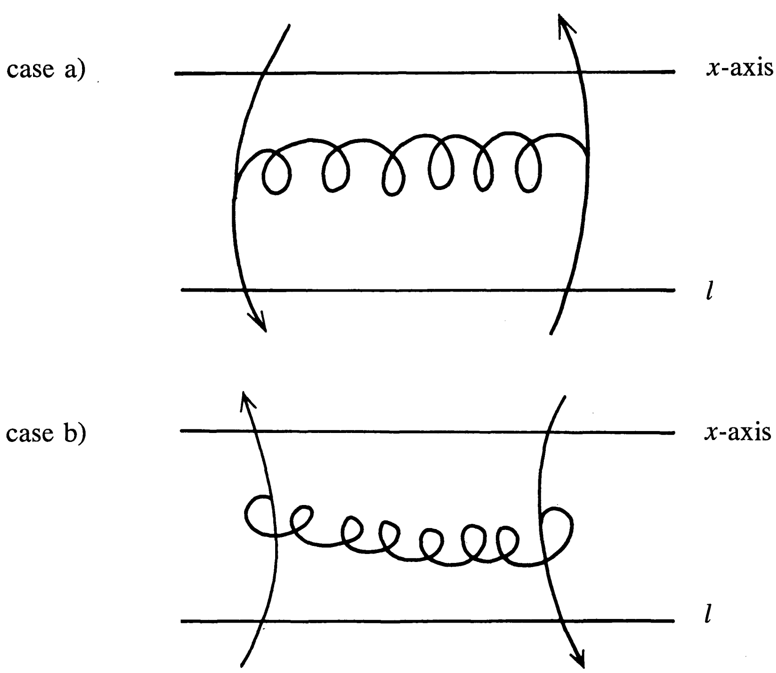

Fig. 2

Each loop adds 2π to the turning. By putting in the proper number of loops one sees that the turning of the constructed arc is equal to that of the original arc. Thus using Lemma 5 the conclusion is reached.

Lemma 9. *Suppose that* $f\colon [0, 1] \to E^2$ *is an oriented arc with the following properties*:

a) *f has positive curvature.*

b) *$f(0)$, $f(1)$ are in the lower half plane.*

c) *f meets the x-axis transversally at two points.*

d) *The point where the arc enters the upper half plane is to the left of the point where it returns to the lower half plane.*

e) *The total turning of f in the upper half plane is less than 2π.*

Then given any line L parallel to the x-axis and lying in the upper half plane, f is nondegenerately homotopic to a curve $g\colon [0, 1] \to E^2$ which meets L and meets L transversally. Furthermore the portion of g between the x-axis and L consists of two nondegenerate arcs each without double point, though the two may cross. The homotopy is constant on a neighbourhood of the endpoints.

Proof. Suppose that f enters the upper half plane at p_1 and returns at p_2. Let l_1, l_2 be the tangent lines of f at p_1 and p_2 respectively. l_1, l_2 are not the x-axis because f crosses transversally. Let $f[p_1, p_2]$ be the arc in the upper half plane. If l_1, l_2 meet in the lower half plane, then we may complete $f[p_1, p_2]$ to a nondegenerate closed curve by adding an arc in the lower half plane. We use Lemma 7 to patch the arcs together. Furthermore we may construct the arc in the lower half plane so that its turning is less than 2π and so that it has

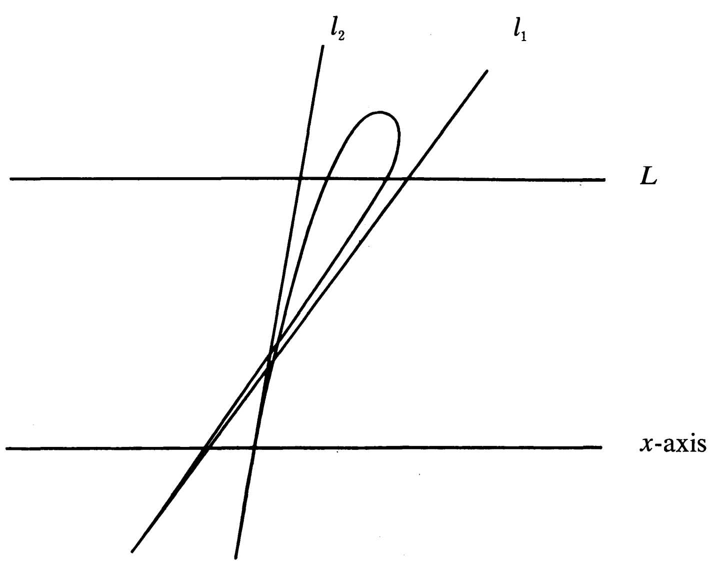

Fig 3.

exactly one double point. (Here we use the fact that f has positive curvature and that f enters the upper half plane to the left of where it returns.) Hence we obtain a closed nondegenerate curve of index of rotation less that 2 and hence 1 which has a double point. This is a contradiction. If l_1, l_2 are parallel, let l be a line parallel to the x-axis and slightly above it. Then the same argument may be applied to that portion of f lying above l. Thus l_1, l_2 must meet in the upper half plane.

Now l_1 and l_2 will eventually meet the given line L. We may construct an arc (using Lemma 7 to be sure that it has positive curvature) as shown in Figure 3. Again by Lemma 7 we may patch it in with the portion of f below the x-axis so that the resulting arc has positive curvature and agrees with f on a neighbourhood of 0 and 1. It is not difficult to check that f and the new arc have the same total turning. Hence by Lemma 5 the conclusion is reached.

Proof of Lemma 6. Let $f\colon S^1 \to S^2$ be a closed curve with positive geodesic curvature (negative curvature is handled similarly). By a nondegenerate homotopy we may assume that the curve has only finitely many transversal double points and no triple points. We may choose a hemisphere H such that ∂H meets the curve transversally at a finite number of points and also such that no double point lies on ∂H. We may by a nondegenerate homotopy (which just flattens the curve a bit locally) assume that the curve does not meet ∂H in a pair of anitpodal points. Thus $H \cap f(S^1)$ consists of a finite number of arcs (connected components) each meeting ∂H transversally in two distinct nonantipodal points. Let $A = f[p_1, p_2]$ be such an arc, and give ∂H the orientation induced from H. Let us call the arc A *troublesome* if the total turning of the central projection from H is less than 2π and if the shorter arc from $f(p_1)$ to $f(p_2)$ agrees with the orientation of ∂H. (We assume the curve crosses into H at p_1 and out of H at p_2.)

We first take care of the troublesome arcs. Note that a simple hemispherical arc cannot be troublesome. This follows from the fact that if a simple plane arc of positive curvature with endpoints in the lower half plane meets the x-axis transversally at two points, then the point where it returns to the lower half plane must lie to the left of the point where it enters the upper half plane. Our procedure will be to convert a troublesome arc into two simple arcs using Lemma 9. To do this let $A = f[p_1, p_2]$ be a troublesome arc. Since $f(p_1)$ and $f(p_2)$ are not antipodal points, they lie in a half circle $C \subset \partial H$.

Let R_1, R_2 be rotations of S^2 about the axis through the endpoints of C, and assume that R_2 is a rotation with the same sense as R_1 but of greater magnitude than R_1. These rotations will carry the hemisphere H into new hemispheres H_1 and H_2, and the half circle C into new half circles (joining the same endpoints) C_1, C_2. If the sense of the rotation is correctly chosen we have

$$C_2 \subset H_1 , \qquad \partial H - C \subset H_1 .$$

We may choose the rotations R_1, R_2 small enough in magnitude so that the arc A meets ∂H_i transversally at two points contained in C_i for $i = 1, 2$. Let $\tau\colon H_1 \to E^2$ be central projection. Again by choosing the rotation R_1 small enough in magnitude we may assume that the total turning of the arc $\tau(H_1 \cap A)$ is less than 2π. Now C_2 and $\partial H - C$ are half great circles with the same endpoints. Thus $\tau(C_2)$ and $\tau(\partial H - C)$ are parallel straight lines.

We take $\tau(C_2)$ to be the x-axis and $\tau(\partial H - C)$ to be the line L in Lemma 9. Let the two points at which A meets C_2 be $f(p_3)$, $f(p_4)$ where p_3 is the "first point", i.e., $f(p_3, p_4) \subset H_2$, where (p_3, p_4) is the oriented arc from p_3 to p_4. Choose $\varepsilon > 0$ so that $f[p_3 - \varepsilon, p_4 + \varepsilon] \subset H_1$, where $[p_3 - \varepsilon, p_4 + \varepsilon]$ is an oriented arc containing (p_3, p_4). Applying Lemma 9 to the arc

$$\tau \circ f\colon [p_3 - \varepsilon, p_4 + \varepsilon] \to E^2 ,$$

we obtain a nondegenerate homotopy

$$h'_t\colon [p_3 - \varepsilon, p_4 + \varepsilon] \to E^2 .$$

Define a homotopy $h_t\colon S^1 \to S^2$ as follows:

$$h_t(p) = \begin{cases} f(p) , & p \in S^1 - [p_3 - \varepsilon, p_4 + \varepsilon] , \\ \tau^{-1} \circ h'_t(p) , & p \in [p_3 - \varepsilon, p_4 + \varepsilon] . \end{cases}$$

The curve h_1 is the same as f except that the troublesome arc $A = f(p_1, p_2)$ has been changed. $h_1(p_1, p_2)$ meets H in two simple arcs, which as we have previously noted cannot be troublesome. Thus $h_1(S^1)$ has one less troublesome arc than f had. We may repeat this argument to eliminate all troublesome arcs.

Now suppose that $A = f[p_1, p_2]$ is an arc which is not troublesome. A is a connected component of $f(S^1) \cap H$. Let $\tau_H\colon H \to E^2$ be the central projection. In the case where the shorter arc from $f(p_1)$ to $f(p_2)$ agrees with the orientation of ∂H, the total turning of $\tau_H(A) \geq 2\pi$. If the total turning is equal to 2π, then $f(p_1)$ and $f(p_2)$ are antipodal points but f has no antipodal points on ∂H. Thus in this case the total turning of $\tau_H(A) > 2\pi$. Now let $C \subset \partial H$ be a half circle containing $f(p_1)$ and $f(p_2)$, and R_1 be a rotation of S^2 about the axis through the endpoints of C. R_1 carries the hemisphere H into a hemisphere H_1, and the circle C into a circle C_1. By choosing the correct sence of rotation we may suppose that

$$C \subset H_1 .$$

A is compact and so is a positive distance from the half circle $\partial H - C$. Thus, by choosing the magnitude of R_1 small enough, we may suppose that

$$A \subset H_1 .$$

Let $\tau\colon H_1 \to E^2$ be central projection. By choosing the magnitude of R_1 small enough we may suppose that the total turning of $\tau(A)$ is greater than 2π if the total turning of $\tau_H(A)$ is greater than 2π. Choose $\varepsilon > 0$ so that

$$f(p_1 - \varepsilon, p_2 + \varepsilon) \subset H_1 .$$

Let $\tau(C)$ be the x-axis in E^2. We may now apply Lemma 8 to the arc

$$\tau \circ f \colon [p_1 - \varepsilon, p_2 + \varepsilon] \to E^2$$

to obtain a nondegenerate homotopy

$$h'_t \colon [p_1 - \varepsilon, p_2 + \varepsilon] \to E^2 .$$

Define a homotopy $h_t \colon S^1 \to S^2$ by

$$h_t(p) = \begin{cases} f(p), & \text{for } f \in S^1 - [p_1 - \varepsilon, p_2 + \varepsilon] , \\ \tau^{-1} \circ h'_t(p), & \text{for } p \in [p_1 - \varepsilon, p_2 + \varepsilon] . \end{cases}$$

h_t is a nondegenerate homotopy which pulls the arc A out of H. By this process we may remove all arcs from H; thus concluding Lemma 6.

Lemma 10. *The following two curves, when projected via central projection onto the northern hemisphere, are nondegenerately homotopic.*

1. 2.

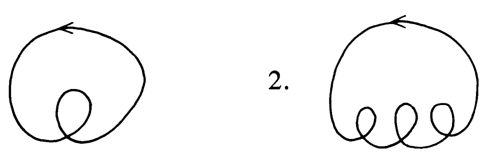

Fig. 4

Proof. We give a rather explicit construction of the homotopy.

Let C_i be three great circles through the north pole which meet at 120°, and D_i be three small circles of the same radius which form an equilateral spherical triangle containing the north pole and such that D_i is parallel to C_i, $i = 1, 2, 3$. C_i divides the sphere into two hemispheres H_i^+ and H_i^-. H_i^+ is the hemisphere which contains D_i. The three small circles D_i will also form an equilateral spherical triangle in the southern hemisphere about the south pole. Let E be the small circle through the verticies of that triangle.

We construct four curves γ_i, $i = 1, 2, 3, 4$ according to Figure 5. The figures are parallel projections of the northern and southern hemispheres as seen from above. The curves γ_i go along the small circles as indicated turning corners with loops as indicated. The loops may be constructed with the aid of Lemma 7. Since the curves are along small circles, their geodesic curvature is nonzero. γ_1 is seen to be nondegenerately homotopic to curve 1) in Figure 4 and γ_4 is homotopic to curve 2). We show that γ_1 and γ_2 are nondegenerately homotopic by observing that they are identical in H_1^- and that in H_1^+ they are arcs which

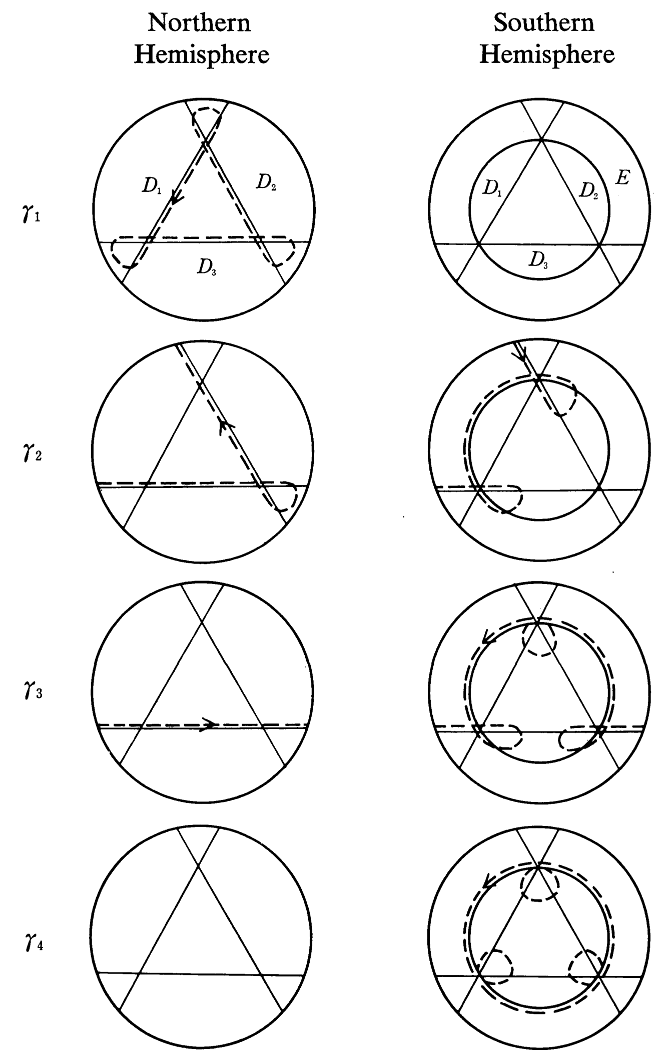

Fig. 5

agree near ∂H_1^+. Thus projecting H_1^+ centrally we may use Lemma 5. Similarly project centrally from H_2^+ to show that γ_2 and γ_3 are nondegenerately homotopic, and finally project centrally from H_3^+ to show that γ_3 and γ_4 are nondegenerately homotopic.

Conclusion of the proof of Theoem 1. In Lemma 6 we have seen that every nondegenerate curve is nodegenerately homotopic to a curve lying in a hemisphere. But nondegenerate plane curves and, vis central projection, curves in a hemisphere are characterized by their total turning. Assume that the geodesic curvature is positive. Hence the total turning is $2\pi n$ for some positive number n. If $n \leq 3$ we are finished by Proposition 2. If $n = 4$ we are finished by Lemma 10. If $n > 4$ we attach $n - 4$ loops to the curve γ_1 of Lemma 10. This curve will then have index n. We may nondegenerately homotope this curve to a hemispherical curve of index $n - 2$. Just use the same homotopy as in Lemma 10 and allow the extra loops to be carried along. By this procedure we eventually reach a curve with index 2 or 3. If the geodesic curvature is negative, reverse the orientation of the curve, apply the above argument, and then reverse the orientation of the homotopy. Thus we see that curves in Proposition 2 represent the only nondegenerate homotopy classes on S^2.

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