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CONFORMAL TRANSFORMATIONS OF RIEMANNIAN MANIFOLDS

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Introduction

Let (M, g), or simply M, be a Riemannian n-manifold with Riemannian metric g; throughout this paper manifolds are always assumed to be connected and C^∞. For any C^∞-function ρ the Riemannian metric $g^* = e^{2\rho}g$ is said to be *conformal* or *conformally related* to g, and for constant ρ it is said to be *homothetic to* g.

Let h be a C^∞-mapping of (M, g) into another Riemannian manifold (M^*, g^*). If the Riemannian metric h^*g^* induced on M by h is conformal (homothetic) to the original metric g, then h is called a *conformal* (*homothetic*) *mapping* of (M, g) into (M^*, g^*). (Under a conformal mapping the angle between two vectors is preserved.) h remains to be conformal under any conformal changes of metrics on M and M^*. If h is a diffeomorphism, then h is called a *conformal diffeomorphism* or briefly a *conformorphism*, and (M, g) is said to be *conformally diffeomorphic* or briefly *conformorphic* to (M, g^*) through h. If h is a conformorphism of (M, g) onto itself, then h is called a *conformal transformation* of (M, g).

For a group G of conformal transformations of (M, g), if there exists a conformally related metric $g^* = e^{2\rho}g$ with respect to which G is a group of isometries, then G is said to be *inessential*, otherwise *essential*.

Let $C(M, g)$, or simply $C(M)$, be the group of the conformal transformations of (M, g), and $I(M, g)$, or simply $I(M)$, the group of the isometries of (M, g); they both are known to be Lie groups with respect to the compact-open topology. It is known [4] that any compact subgroup of $C(M)$ is inessential. Therefore a maximal compact subgroup of $C(M)$ may be considered as a subgroup of $I(M)$ by a suitable conformal change of metric. In particular, if M is compact, so is $I(M)$. Hence there is a conformally related Riemannian metric g^* such that the group $I(M, g^*)$ is a maximal compact subgroup of $C(M, g) = C(M, g^*)$. It follows that on a compact Riemannian manifold M, $C(M)$ is essential if and only if it is not compact.

In this paper, we are mainly concerned with a 1-parameter group of conformal transformations of a Riemannian manifold M, and so we may

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assume that the connected component $C_0(M)$ of $C(M)$ is essential. As was mentioned above, in case M is compact, this assumption is equivalent to that $C_0(M)$ is not compact. However for a non-compact M, the condition for $C_0(M)$ to be essential is not known.

Let f_t be a 1-parameter group of conformal transformations of M. The vector field on M induced by f_t is called a *conformal vector field*, and is denoted by u. If f_t is essential, then u is said to be *essential*. It is known [16] that a vector field u is conformal if and only if

$$\mathscr{L}_u g_{ji} = \nabla_j u_i + \nabla_i u_j = 2\phi g_{ji}$$

for some function ϕ, where $\mathscr{L}_u$ denoted the Lie derivation with respect to u. u is homothetic or isometric according as ϕ is constant or identically zero. Since we have $n\phi = \nabla_i u^i$, we call ϕ the *divergence* of u. A fixed point of f_t, or a zero of the vector field u, is called a *fixed point* or a *singular point* of u. As we shall see in § 2, the value of the divergence of a vector field u at a singular point, if exists, is unchanged by any conformal change of metric and is therefore not zero only when u is an essential conformal vector field.

If u is induced by a global 1-parameter group f_t, $-\infty < t < \infty$, of global transformations of M, u is said to be *complete*. On a compact manifold M, every vector field is complete.

The main purpose of the present paper is to prove the following theorem:

If a Riemannian n-manifold, $n > 2$, admits a complete conformal vector field u with singular points at each of which its divergence does not vanish, then the manifold is conformorphic to either a Euclidean n-sphere S^n or a Euclidean space E^n (or a punctured Euclidean sphere $S^n - \{p_\infty\}$). In the latter case the vector field u is homothetic with respect to the Euclidean metric conformally related to the original Riemannian metric.

It should be remarked that in a Euclidean space any global conformal transformation is automatically homothetic and therefore is an affine transformation preserving angles. Thus the latter case of the above theorem is reduced to a study of those affine transformations mentioned above, and therefore is not specially important for our general purpose of studying conformal transformations. Furthermore it is well-known that a homothetic transformation on a compact Riemannian manifold must be isometric.

Now we can state a well-known conjecture as follows:

Conjecture I. *A compact Riemannian n-manifold, $n > 2$, admitting an essential conformal vector field is conformorphic to a sphere.*

Our result is a step to this conjecture without assuming the compactness of the manifold. Indeed the assumption in the theorem is satisfied only by an essential conformal vector field (§ 2). On a sphere or rather on a Möbius space the vector field under consideration in our theorem is conformally equivalent to the one which leaves the origin and the point at infinity fixed. A conformal vector field corresponding to the one which does not leave fixed the point at infinity in the Möbius space is not considered in our theorem.

A. Avez [1] announced that Conjecture I was true by using a lemma, given in the first paper of [1], on the behavior of singular points of an essential conformal vector field. Unfortunately this lemma is not true [see MR **31** (1966), # 1635], and a counter example will be naturally shown by some discussions to be given in § 3. However the method of Avez employed in the second paper of [1] can be applied to prove our theorem here, and our proof is simpler and more direct than his given in [1] although both proofs have similar basic ideas.

It is remarked that the completeness of a Riemannian metric is not of conformal nature, and is not assumed in our theorem. However it is true that on any manifold there exists a complete metric conformal to a given metric [8].

Another result concerning Conjecture I is a theorem of Ishihara & Tashiro [5], [13] that the conjecture is true if the conformal vector field under consideration is a gradient field; their proof is based on an interesting geometrical consideration of the hypersurfaces defined by the gradient field. However, the assumption is not of conformal nature in the sense that a gradient field does not remain to be a gradient field under a conformal change of metric unless the change is homothetic.

The following is another well-known conjecture:

Conjecture II. *If a compact Riemannian n-manifold, $n > 2$, with constant scalar curvature admits a nonisometric conformal vector field, then it is isometric to a Euclidean n-sphere S^n.*

If Conjecture II is true, so is Conjecture I by means of a theorem of Yamabe [15] that any Riemannian metric on a compact n-manifold, $n > 2$, is conformally related to one with constant scalar curvature. However, the converse implication is not known to be true in general, even though it is true when the conformal vector field is a gradient field.

As for Conjecture II there are published many results, most of which reduce the conjecture with an extra condition to the case of a gradient conformal vector field; some remarks on this will be given in the last section, § 5.

In § 1 notation and terminology are given for later use. § 2 is devoted to the establishment of the main theorem and its related results. In § 3 the behavior of singular points of essential vector fields on a Euclidean sphere will be considered. It will be seen that such vector fields fall into two classes, one having exactly two singular points with non-vanishing divergence and one having exactly one singular point with vanishing divergence; such a phenomenon must be a model of Conjectures I and II. In § 4 several sufficient conditions will be given for a vector field to satisfy the assumption of our main theorem. § 5 will consist of remarks on the above two conjectures.

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1. Preliminaries

Let u be a conformal vector field on a Riemannian n-manifold (M, g), and $\mathscr{L}_u$ denote the Lie derivation with respect to u. Then we have

$$\mathscr{L}_u g_{ji} = \nabla_j u_i + \nabla_i u_j = 2\phi g_{ji} , \tag{1.1}$$

ϕ being a scalar. Since we have

$$n\phi = \nabla_i u^i , \tag{1.2}$$

we call ϕ the *divergence* of u. Denoting the curvature tensor, Ricci tensor and the scalar curvature of the manifold M respectively by $K_{kji}{}^h$, $K_{ji} = K_{hji}{}^h$ and $K = K_{ji}g^{ji}$, it is known (see, for example, [16]) that (1.1) implies

$$\nabla_j \nabla_i u^h = -u^k K_{kji}{}^h + \phi_j \delta_i{}^h + \phi_i \delta_j{}^h - \phi^h g_{ji} , \tag{1.3}$$

$$(n-2)\nabla_j \phi_i = u^k \nabla_k L_{ji} + L_{ki}\nabla_j u^k + L_{jk}\nabla_i u^k , \tag{1.4}$$

where we put $\phi_j = \nabla_j \phi$, $\phi^h = g^{hi}\phi_i$ and

$$L_{ji} = -K_{ji} + \frac{1}{2(n-1)} K g_{ji} . \tag{1.5}$$

Thus, when $\dim M = n > 2$, u exists if and only if the following system of partial differential equations for $(n+1)^2$ unknowns ϕ, ϕ_j, u^h, $u_i{}^h$ admits a solution $(\phi, \phi_i, u^h, u_i{}^h)$:

$$\begin{aligned} &\nabla_i u^h = u_i{}^h , \\ &u_{ji} + u_{ji} = 2\phi g_{ji} , \\ &\nabla_j \phi = \phi_j , \\ &\nabla_j u_i{}^h = -u^k K_{kji}{}^h + \phi_j \delta_i^h + \phi_i \delta_j^h - \phi^h g_{ji} , \\ &\nabla_j \phi_i = \frac{1}{n-2}(u^k \nabla_k L_{ji} + L_{ki} u_j{}^k + L_{jk} u_i{}^k) . \end{aligned} \tag{1.6}$$

Hence by virtue of the theorem of the unique existence in the theory of differential equations, *a conformal vector field u is uniquely determined by the values of ϕ, $\nabla_i\phi$, u^h, $\nabla_i u^h$ at a point of M.*

On M, the Weyl's conformal curvature tensor field $W(=W_{kji}{}^h)$ is defined by

$$(1.7)\quad \begin{aligned} W_{kji}{}^h = K_{kji}{}^h &+ \frac{1}{n-2}(K_{ki}\delta_j^h - K_{ji}\delta_k^h + g_{ki}K_j^h - g_{ji}K_k^h) \\ &- \frac{K}{(n-1)(n-2)}(g_{ki}\delta_j^h - g_{ji}\delta_k^h) , \end{aligned}$$

and the tensor field $C(=C_{kji})$ by

$$(1.8)\quad \begin{aligned} C_{kji} = &\frac{1}{n-2}(\nabla_k K_{ji} - \nabla_j K_{ki}) \\ &- \frac{1}{2(n-1)(n-2)}(\nabla_k K g_{ji} - \nabla_j K g_{ki}) . \end{aligned}$$

It is well-known that M is conformally flat, i.e., locally conformorphic to a Euclidean space, if and only if $W \equiv 0$ for $n > 3$, and $C \equiv 0$ for $n = 3$. It is also known that

$$(1.9)\quad \mathscr{L}_u W_{kji}{}^h = 0 , \qquad \mathscr{L}_u C_{kji} = W_{kji}{}^h \phi_h .$$

In case $W \equiv 0$, which is automatically true for $n = 3$, we have

$$(1.10)\quad \mathscr{L}_u C_{kji} = 0 .$$

Let T be a tensor field of type (r, s) on M, and $|T|$ denote the length of T so that

$$(1.11)\quad |T|^2 = g_{i_1a_1} \cdots g_{i_ra_r} g^{j_1b_1} \cdots g^{j_sb_s} T^{i_1\cdots i_r}_{j_1\cdots j_s} T^{a_1\cdots a_r}_{b_1\cdots b_r} .$$

Since

$$(1.12)\quad \mathscr{L}_u g^{ji} = -2\phi g^{ji} ,$$

we then have

$$(1.13)\quad \mathscr{L}_u |T|^2 = 2(r-s)\phi|T|^2 + 2\langle \mathscr{L}_u T, T\rangle ,$$

where $\langle , \rangle$ denotes the inner product defined by the Riemannian metric g. If, in particular, T is invariant by u, i.e., $\mathscr{L}_u T = 0$, then we have

$$(1.14)\quad \mathscr{L}_u |T| = (r-s)\phi|T| .$$

If T is further of type (r, r), then $|T|$ is invariant by u, i.e.,

$$(1.15)\quad \mathscr{L}_u |T| = 0 .$$

Now on M let u be a complete vector field, i.e., a vector field generating a global 1-parameter group f_t, $-\infty < t < \infty$, of diffeomorphisms of M onto itself. Assume that u has a singular point p_0, i.e., $u_{p_0} = 0$. At a point p at which $u_p \neq 0$ there exists a unique integral curve $f_t(p)$, $-\infty < t < \infty$, through p such that $f_0(p) = p$; such a curve is called the orbit of u though p. The set

$$W_0 = \{p \in M \mid \lim_{t \to -\infty} f_t(p) = p_0\}$$

is called the *unstable manifold* of u at p_0, and the set

$$W_0^* = \{p \in M \mid \lim_{t \to +\infty} f_t(p) = p_0\}$$

the *stable manifold* of u at p_0. Obviously the stable (unstable) manifold of u is the unstable (stable) manifold of $-u$.

Lemma 1.1 [1], [2], [10]. *On a Riemannian manifold M let u be a complete conformal vector field with a singular point p_0, T be a tensor field, invariant by u, of type (r, s), and W denote the stable (or unstable) manifold of u at p_0. Then*

(i) *the length $|T|$ of T is constant on the closure $\overline{W}$ of W if $r = s$ and*

(ii) *T vanishes identically on $\overline{W}$ if $r < s$.*

Proof. (i) If $r = s$, then from (1.15) we have $\mathscr{L}_u|T| = 0$. It follows that along each orbit $f_t(p)$, $p \in M$, we have $\frac{d|T|}{dt} = 0$, which implies that $|T|$ is constant along each orbit. Since $\lim_{t \to \infty} f_t(p) = p_0$ for $p \in W$ and $|T|$ is constant along $f_t(p)$, we have

$$|T_p| = |T_{p_0}| \qquad \text{for } p \in W .$$

By cotinuity $|T|$ is constant on $\overline{W}$.

(ii) If $r < s$, then we consider the tensor field T^* of type (s, s) defined by

$$T^* = T \otimes \underbrace{u \otimes \cdots \otimes u}_{s - r} .$$

Since T and u itself are invariant by f_t, so is T^*. It thus follows from (i) that $|T^*|$ is constant on $\overline{W}$, which contains p_0. Since $u_{p_0} = 0$, we must have $|T^*| \equiv 0$ on $\overline{W}$. Since $u_p \neq 0$ for $p \in W - \{p_0\}$, we have $T_p = 0$ for $p \in W - \{p_0\}$. By continuity of T, $T \equiv 0$ on $\overline{W}$.

Lemma 1.2. *Let u be a complete conformal vector field with a singular point p_0 on a Riemannian n-manifold M, $n \geq 3$. Then the Weyl's conformal curvature tensor field W and the tensor field C vanish identically on the closures of the stable and unstable manifolds W_0^* and W_0 of u at p_0.*

Proof. Since W is of type (1.3) and is invariant by any conformorphism of M, by Lemma 1.1 it vanishes identically on $\overline{W}_0$ and $\overline{W}_0^*$. Then by (1.10) C is invariant by the vector field u, and therefore by Lemma 1.1 again C, being of type (0, 3), vanishes identically on $\overline{W}_0$ and $\overline{W}_0^*$.

2. Main theorems

Lemma 2.1 [1]. *Let u be a complete conformal vector field on a Riemannian n-manifold M, $n \geq 2$. If u has a singular point p_0 at which its divergence ϕ is positive (negative), then*

(i) *p_0 is an isolated singular point, and*

(ii) *the unstable (stable) manifold $W_0(W_0^*)$ of u at p_0 is an open set diffeomorphic to a Euclidean n-space E^n.*

Proof. We need only to consider the case $\phi\,(p_0) > 0$, because the other case $\phi(p_0) < 0$ can be reduced to this one by considering $-u$.

From (1.1), the matrix of the first partial derivatives of u at p_0 in local coordinates has an eigenvalue $\phi(p_0) > 0$ with multiplicity n. Therefore (i) of this lemma is almost obvious. However, in order to see a geometrical meaning of the lemma for our special case, we shall give a proof as follows.

From the assumption $\phi(p_0) > 0$, there is an open geodesic ball B of radius r such that $\phi(p) \geq a > 0$ for all $p \in \bar{B}$ for some positive constant a, and each point of B is covered by one and only one geodesic issuing from p_0 in B, denoted by $\gamma(s)$, where s is the arc length from p_0, and $\gamma(0) = p_0$. Let $\nu(s)$ be the unit tangent vector of $\gamma(s)$. Then $\nu(s) = d\gamma/ds$, and along $\gamma(s)$ we bave

$$\frac{d}{ds}\langle u, \nu\rangle = \left\langle \frac{du}{ds}, \nu\right\rangle = (\nabla_j u_i)\nu^j\nu^i = \phi \geq a$$

in $\bar{B}$, because of (1.1) together with

$$du/ds = \nu^j \nabla_j u^i\ , \qquad d\nu/ds = 0\ .$$

Since $u_{p_0} = 0$, we have $\langle u_{p_0}, \nu(0)\rangle = 0$. Hence along $\gamma(s)$ in $\bar{B}$, we have

$$\langle u, \nu\rangle \geq as, \tag{2.1}$$

which implies at once that $u_p \neq 0$ at $p \in \bar{B} - \{p_0\}$. Thus p_0 is an isolated singular point, and (i) of our Lemma 2.1 is proved.

Let ∂B denote the boundary of B, which is the geodesic sphere of radius r centered at p_0. Then the vector $\nu(r)$ is the unit outer normal to ∂B, and for the unit outer normal ν_p of ∂B at $p \in \partial B$ we have, by (2.1),

$$\langle u_p, \nu_p\rangle \geq ar > 0 \qquad \text{for } p \in \partial B\ . \tag{2.2}$$

Denoting by θ_p the angle between u_p and ν_p, there exists a positive constant θ_0 such that

$$0 \leq \theta_p \leq \theta_0 < \pi/2, \qquad \cos\theta_p \geq \cos\theta_0 > 0 \qquad \text{for } p \in \partial B\ , \tag{2.3}$$

since ∂B is compact and u never vanishes on ∂B.

By the action of the 1-parameter group f_t, $-\infty < t < \infty$, generated by u, we put $B_t = f_t(B)$, $B_0 = B$. Then obviously we have $\partial B_t = f_t(\partial B)$. It follows from (2.2) that

$$\bar{B}_t \subset B_{t'} \qquad \text{for } t < t' . \tag{2.4}$$

It is remarked that B_t is an open ball containing p_0 but not necessarily a geodesic ball for $t \neq 0$.

Take any point p in ∂B and consider the orbit $f_t(p)$ through p. Then, by (2.3), $\{f(p), t \leq 0\}$ is contained in $\bar{B}$, and from (1.14) we have, along $f_t(p)$, $t < 0$,

$$\frac{d}{dt}|u| = \phi|u| \geq a|u| .$$

Therefore we have

$$|u_t| \leq |u_p|e^{at} \qquad \text{for } t < 0 ,$$

where we have put $u_t = u_{f_t(p)}$, and it follows that

$$\lim_{t \to -\infty} |u_t| = 0 .$$

Since p_0 is the only singular point in B we also have

$$\lim_{t \to -\infty} f_t(p) = p_0 .$$

Thus ∂B and therefore $\bar{B}$ are contained in the unstable manifold W_0 of u at p_0. Hence we have

$$W_0 = \bigcup_{-\infty < t < \infty} B_t . \tag{2.5}$$

From (2.4), W_0 is an open set diffeomorphic to a Euclidean n-space E^n.

Lemma 2.2. *If a Riemannian n-manifold M, $n \geq 2$, admits a complete conformal vector field u with singular points at each of which its divergence does not vanish, then M is homeomorphic to either a Euclidean n-sphere S^n or a Euclidean n-space E^n (or a punctured Euclidean n-sphere $S^n - \{p_\infty\}$).*

Proof. Let p_0 be a singular point of u. We may assume that the divergence ϕ is positive at p_0. Then by Lemma 2.1 the unstable manifold W_0 of u at p_0 is diffeomorphic to E^n.

Case I. $\partial W_0 \neq \emptyset$. We first show that ∂W_0 consists only of singular points of u. We assume the contrary. Let q_0 be a point in ∂W_0 at which u does not vanish so that $u_{q_0} \neq 0$, and choose a sequence $\{q_n\}$ in W_0 converging to q_0 so that $\lim_{n \to \infty} q_n = q_0$. Then by (2.4) and (2.5) each q_n lies on one and only one ∂B_{t_n}. Taking the outer unit normal ν_n to ∂B_{t_n} at q_n, and choosing a subsequence if necessary, we may assume that $\lim_{n\to\infty} \nu_n$ exists, and denote it by ν_0 so that $\lim_{n\to\infty} \nu_n = \nu_0$. ν_0 is normal to the tangent vectors of ∂W_0 if there exist any, and by continuity we obtain, since $\lim_{n\to\infty} u_{q_n} = u_{q_0}$,

$$\lim_{n\to\infty} \langle u_{q_n}, \nu_n \rangle = \langle u_{q_0}, \nu_0 \rangle . \tag{2.6}$$

On the other hand, we have

$$\langle u_{q_n}, \nu_n \rangle = \cos\theta_n |u_{q_n}| , \tag{2.7}$$

where θ_n denotes the angle between u_{q_n} and ν_n. Since f_t preserves the angle, θ_n is the same as the angle between u_{p_n} and the unit outer normal ν_{p_n} to ∂B at p_n, where $p_n = f_{-t_n}(q_n) \in \partial B$. Thus we have, by (2.3),

$$\cos\theta_n \geq \cos\theta_0 > 0 , \tag{2.8}$$

and from (2.6), (2.7) and (2.8) it follows that

$$\langle u_{q_0}, \nu_0 \rangle = \lim_{n\to\infty} \cos\theta_n |u_{q_n}| \geq \cos\theta_0 |u_{q_0}| > 0 . \tag{2.9}$$

However, from (2.5) we must have

$$f_t(\partial W_0) \subset \partial W_0 , \qquad \langle u_{q_0}, \nu_0 \rangle = 0 ,$$

contradicting (2.9). Hence q_0 is a singular point, and ∂W_0 consists only of singular points which are denoted by p_λ, $\lambda \in \Lambda$. From (2.4), we have $\lim_{t\to\infty} u_{f_t(p)} = 0$ for $p \in W_0 - \{p_0\}$. It follows that $\lim_{t\to\infty} f_t(p)$ exists in ∂W_0 and is one of p_λ. Then by our assumption $\phi(p_\lambda) \neq 0$ and the stable manifold W_λ^* at p_λ is non-empty since $p \in W_\lambda$. Therefore $\phi(p_\lambda) < 0$ and W_λ^* is diffeomorphic to E^n by Lemma 2.1, Since $\lim_{t\to\infty} f_t(p)$ exists in ∂W_0 for any $p \in W_0 - \{p_0\}$, we have

$$W_0 - \{p_0\} = \bigcup_\lambda (W_0 \cap W_\lambda^*) ,$$

or $W_0 - \{p_0\}$ is a union of open sets. Furthermore, if $\lambda \neq \lambda'$, then the intersection

$$(W_0 \cap W_\lambda^*) \cap (W_0 \cap W_{\lambda'}^*) = \emptyset .$$

In fact, if there is a point p in the intersection, we must have $\lim_{t\to\infty} f_t(p) = p_\lambda$ and $\lim_{t\to\infty} f_t(p) = p_{\lambda'}$, contradicting each other. Therefore $W_0 - \{p_0\}$ is a disjoint union of open sets. Since for at least one λ, say $\lambda = 1$, $W_0 \cap W_1^* \neq \emptyset$ and

$W_0 - \{p_0\}$ is connected for $n > 2$, we have

$$W_0 - \{p_0, p_1\} = W_0 \cap W_1^* , \qquad W_0 \cap W_\lambda^* = \emptyset \qquad \text{for } \lambda \neq 1 .$$

Thus the set $M' = \{p_0\} \cup (W_0 \cap W_1^*) \cup \{p_1\}$ is homeomorphic to an n-sphere S^n. Since M' can be written as

$$M' = \overline{W}_0 = W_0 \cup W_1^* ,$$

it is open and closed in M, which is connected. Therefore we have $M = M'$, and M is homeomorphic to S^n.

Case II. $\partial W_0 = \emptyset$. In this case, W_0 is open and closed in the connected manifold M, and hence $M = W_0$, which is diffeomorphic to E^n. q.e.d.

It is noted that Lemmas 2.1 and 2.2 are true even for $n = 2$, and therefore are applicable to a holomorphic vector field on a Kähler manifold of complex dimension 1.

Theorem 2.3. *If a Riemannian n-manifold M, $n > 2$, admits a complete conformal vector field u with singular points at each of which its divergence does not vanish, then M is conformorphic to either a Euclidean n-sphere S^n or a Euclidean n-space E^n (or a punctured n-sphere $S^n - \{p_\infty\}$).*

Proof. Since by Lemma 1.2 the Weyl's conformal curvature tensor field W vanishes on the closure $\overline{W}_0$ of the unstable manifold W_0 of the vector field u at a singular point, W vanishes identically on $M = \overline{W}_0$ by Lemma 2.2, and therefore also the tensor field C. By Lemma 2.2, being simply connected and conformally flat, M is conformorphic to an open set of a Euclidean sphere S^n, [6]. If M is compact, it is conformorphic onto S^n. If M is not compact it is conformorphic to a region of S^n, which is diffeomorphic to E^n and therefore conformorphic to $S^n - \{p_\infty\}$, where p_∞ is a point on S^n.

Remark. On S^n a conformal transformation, which leaves a pair of antipodal points $\{p_0, p_\infty\}$ invariant, can be expressed as a homothetic transformation on $S^n - \{p_\infty\}$ with Euclidean metric; this will be seen in § 3. Therefore, if M is not compact, then the conformal vector field in Theorem 2.3 is homothetic with respect to a Euclidean metric which is conformal to the original Riemannian metric.

The assumption of the existence of a singular point is not a restriction in the following sence:

Theorem 2.4 [1]. *On a Riemannian manifold every essential conformal vector field has a singular point.*

Proof. Let u be a conformal vector field with no singular point on a Riemannian manifold (M, g). Then the length $|u|$ of u is positive everywhere on M and satisfies $\mathscr{L}_u |u| = \phi |u|$, where $\mathscr{L}_u g = 2\phi g$. Thus

$$\mathscr{L}_u(|u|^{-2} g) = 0 ,$$

and u is an infinitesimal isometry with respect to the Riemannian metric $|u|^{-2}g$ conformally related to g. Hence u is not essential. q.e.d.

The values of the divergence ϕ of a conformal vector field u may vanish at its singular points, and an inessential conformal vector field may have a singular point. However, the divergence of an inessential conformal vector field vanishes at each of its singular points. In fact, in general let g and $g^* = e^{2\rho}g$ be conformally related, and u a conformal vector field, and let ϕ and ϕ^* denote the divergences of u with respect to g and g^* respectively. On account of the fact that

$$\nabla_j^* u^h = \nabla_j u^h + (\rho_j \delta_i^h + \rho_i \delta_j^h - g^{ha}\rho_a g_{ji})u^i ,$$

where ∇^* denotes the covariant differentiation with respect to g^* and $\rho_j = \nabla_j \rho = \nabla_j^* \rho$, we then obtain

$$\phi^* = \nabla_a^* u^a / n = \phi + \rho_a u^a = \phi + \mathscr{L}_u \rho , \tag{2.10}$$

from which it follows that at a singular point p_0 of u we have

$$\phi(p_0) = \phi^*(p_0) ,$$

which means that the values of the divergence at singular points remain the same under any conformal change of metric. If u is inessential, then u is isometric and hence has vanishing divergence with respect to some conformally related Riemannian metric. Hence we have

Theorem 2.5. *On a Riemannian manifold the values of the divergence of a conformal vector field at its singular points are unchanged by any conformal change of metric. If a conformal vector field has non-vanishing divergence at one of its singular points, then it is essential. An inessential conformal vector field has vanishing divergence at each of its singular points.*

Remark. There is, however, an essential conformal vector field with vanishing divergence at each of its singular points; this will be seen in § 3.

3. Conformal vector fields on a sphere

In this section we shall consider a Euclidean sphere S^n of radius 1 as a model of Riemannian manifolds admitting the essential conformal transformation group (this seems to be the only known example of such compact Riemannian manifolds up to conformal changes of metric), and shall study particularly the behaviors of the singular points of conformal vector fields and their divergences at the singular points.

It is convenient to imbed S^n into the real projective $(n+1)$-space P^{n+1} in the following standard manner. Let E^{n+1} be a Euclidean $(n+1)$-space with a coordinate system $(y^0, y^1, \cdots, y^n)$, and S^n be given by the equation

$$(y^0)^2 + (y^1)^2 + \cdots + (y^n)^2 = 1 . \tag{3.1}$$

Let E^{n+2} be a Euclidean $(n+2)$-space with a coordinate system $(X^0, X^1, \cdots, X^n, X^\infty)$, and P^{n+1} the projective $(n+1)$-space with a homogeneous coordinate system $(X^0, X^1, \cdots, X^n, X^\infty)$. We give P^{n+1} the Riemannian metric with sectional curvature $1/2$; this means that the natural projection π of $E^{n+2} - \{0\}$ onto P^{n+1} gives a local isometry of the sphere S^{n+1} of radius $\sqrt{2}$ centered at 0 in E^{n+2} onto P^{n+1}. The isometric embedding of S^n into P^{n+1} is given by the equations

$$(3.2) \qquad X^0 = (1 + y^0)/\sqrt{2}\,, X^i = y^i\,(1 \leq i \leq n), X^\infty = (1 - y^0)/\sqrt{2}\,,$$

and the image of S^n in P^{n+1} is the quadric Q called the *Möbius space* and defined by

$$(3.3) \qquad (X^1)^2 + \cdots + (X^n)^2 - 2X^0X^\infty = 0\ .$$

In E^{n+2}, Q is nothing but the intersection of the cone defined by (3.3) with the sphere S^{n+1}

$$(3.4) \qquad (X^0)^2 + (X^1)^2 + \cdots + (X^n)^2 + (X^\infty)^2 = 2\ ,$$

or the hyperplane defined by

$$(3.5) \qquad X^0 + X^\infty = \sqrt{2}\ .$$

Then the group $\tilde{O}(n+2)$ of the linear transformations of E^{n+2} leaving the quadratic form

$$(3.6) \qquad (X^1)^2 + \cdots + (X^n)^2 - 2X^0X^\infty$$

invariant is a transformation group acting on Q with kernel $\{e, -e\}$, e being the identity, the effective group $\tilde{O}(n+2)/\{e, -e\} = CO(n)$ is called the *Möbius group or the conformal transformation group* of Q, and the Lie algebra CO^L of $CO(n)$, and hence of $\tilde{O}(n+2)$, consists of $(n+2) \times (n+2)$ matrices of the form:

$$(3.7) \qquad \begin{pmatrix} \alpha & {}^t a & 0 \\ b & A & a \\ 0 & {}^t b & -\alpha \end{pmatrix}, \qquad A + {}^t A = 0\ ,$$

where A is a skew symmetric $n \times n$ matrix, a and b are column n-vectors, and α is a real number.

In particular, the Lie algebra G^L of the isotropy group G at the point $P_0 = (1, 0, \cdots, 0)$ consists of the matrices of the form

$$(3.8) \qquad \begin{pmatrix} \alpha & {}^t a & 0 \\ 0 & A & a \\ 0 & 0 & -\alpha \end{pmatrix}, \qquad A + {}^t A = 0\ ,$$

and the isotropy group G itself as a subgroup of $\tilde{O}(n+2)$ consists of the matrices of the form:

$$(3.9)\qquad \begin{pmatrix} \lambda & \lambda\, {}^t a & \lambda |a|^2/2 \\ 0 & T & Ta \\ 0 & 0 & 1/\lambda \end{pmatrix},$$

where $T \in O(n)$, $|a|$ is the length of a, and λ is a nonzero real number.

On account of (3.7), a conformal vector field U at each point $P = (X^0, X^1, \cdots, X^n, X^\infty)$ on Q has the following form:

$$(3.10)\qquad U = \begin{pmatrix} U^0 \\ U^* \\ U^\infty \end{pmatrix} = \tilde{U}P = \begin{pmatrix} \alpha & {}^t a & 0 \\ b & A & a \\ 0 & {}^t b & -\alpha \end{pmatrix} \begin{pmatrix} X^0 \\ X \\ X^\infty \end{pmatrix},$$

where we have put

$$X = \begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix}, \qquad P = \begin{pmatrix} X^0 \\ X \\ X^\infty \end{pmatrix}.$$

Although U is defined in P^{n+1}, it is tangent to Q at P because of

$$(3.11)\qquad {}^t U^* X - U^0 X^\infty - U^\infty X^0 = 0 .$$

U vanishes at P if and only if, for some real number μ,

$$(3.12)\qquad U = \mu P .$$

Thus we have

Lemma 3.1. *P is a singular point of U if and only if P, considered as a vector in E^{n+2}, is a real eigenvector of the matrix $\tilde{U}$ corresponding to U.*

Now assume that U has a singular point, which may be assumed, without loss of generality, to be the point $P_0 = (1, 0, \cdots, 0)$.

Then by (3.8), U takes the form

$$(3.13)\qquad U = \tilde{U}P = \begin{pmatrix} \alpha & {}^t a & 0 \\ 0 & A & a \\ 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} X^0 \\ X \\ X^\infty \end{pmatrix} = \begin{pmatrix} aX^0 + {}^t aX \\ AX + aX^\infty \\ -\alpha X^\infty \end{pmatrix}.$$

Let F_t, $-\infty < t < \infty$, be the 1-parameter group generated by U. Then we have

Lemma 2.1. *If a vector field U in* (3.13) *has a singular point $P^* \neq P_0$, then there is an element $\sigma \in G$ such that $\sigma^{-1}F_t\sigma$ leaves P_0 and $P_\infty =$* (0, 0, $\cdots$, 0, 1) *invariant.*

Proof. Put $P^* = (b^0, b^1, \cdots, b^n, 1)$, $b^0 = \frac{1}{2}\sum_{i=1}^{n}(b^i)^2$. Then the element σ given by

$$\sigma = \begin{pmatrix} 1 & {}^t b & \frac{1}{2}|b|^2 \\ 0 & I & b \\ 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} b^1 \\ \vdots \\ b^n \end{pmatrix},$$

I being the identity matrix, has the property that $\sigma(P_0) = P_0$, $\sigma(P_\infty) = P^*$. Therefore $\sigma \in G$ and $\sigma^{-1}F_t\sigma$ leaves both P_0 and P_∞ invariant. q.e.d.

In this case $\sigma^{-1}F_t\sigma$ gives the vector field of the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -\alpha \end{pmatrix}\begin{pmatrix} X^0 \\ X \\ X^\infty \end{pmatrix} = \begin{pmatrix} \alpha X^0 \\ AX \\ -\alpha X^\infty \end{pmatrix}.$$

To study the singular points of U in (3.13), we consider the following two cases.

Case I: $\alpha \neq 0$. By Lemma 3.1, the singular points of U correspond to the real eigenvalues of $\tilde{U}$, which are α and $-\alpha$, both of multiplicity 1. Therefore the vector field U has exactly two singular points, namely, P_0 and some other point P^*. By Lemma 3.2, we may assume, by transforming F_t by an element of G, that F_t leaves P_0 and P_∞ invariant and U takes the form

$$U = \tilde{U}P = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -\alpha \end{pmatrix}\begin{pmatrix} X^0 \\ X \\ X^\infty \end{pmatrix}. \tag{3.14}$$

Thus the orbit $F_t(P)$ through a point P is expressed by

$$F_t(P) = \begin{pmatrix} e^{t\alpha} & 0 & 0 \\ 0 & \exp tA & 0 \\ 0 & 0 & e^{-t\alpha} \end{pmatrix}\begin{pmatrix} X^0 \\ X \\ X^\infty \end{pmatrix}. \tag{3.15}$$

If $P \in Q - \{P_0, P_\infty\}$, then we have

$$\lim_{t\to-\infty} F_t(P) = P_0\,, \qquad \lim_{t\to+\infty} F_t(P) = P_\infty\,, \tag{3.16}$$

provided that $\alpha > 0$.

If U is restricted to the open subset $Q - \{P_\infty\}$, which is conformorphic to E^n, then by taking the coordinate system $(x^1, \cdots, x^n)$ with $x^i = X^i/X^0$, U is expressed as a vector field with coordinates

$$u = (A - \alpha I)\tilde{X}\,, \quad \text{where} \quad \tilde{X} = X/X^0\,, \tag{3.17}$$

and the orbit $f_t(p)$ has the coordinates:

(3.18) $$f_t(p) = \exp t(A - \alpha I)\tilde{X} ,$$

which shows that f_t *is a* 1*-parameter group of homothetic transformations on* E^n.

Case II: $\alpha = 0$. In this case, if there is a singular point P^* different from P_0, then we may assume that U takes the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X^0 \\ X \\ X^\infty \end{pmatrix},$$

and F_t is a 1-parameter group of isometries. Therefore an essential conformal vector field U with $\alpha = 0$ cannot have a singular point other than P_0, and at P_0 it takes the form:

(3.19) $$U = \begin{pmatrix} 0 & {}^t a & 0 \\ 0 & A & a \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X^0 \\ X \\ X^\infty \end{pmatrix}, \quad a \neq 0 .$$

By Lemma 3.1, *U of the form* (3.19) *is essential if and only if the linear equation*

(3.20) $$AX + aX^\infty = 0$$

has no solution for $X^\infty \neq 0$. Therefore A is a singular matrix, and we may assume, without loss of generality, that A has the form

(3.21) $$A = \begin{pmatrix} 0 & 0 \\ 0 & A'' \end{pmatrix}, \quad A'' + {}^t A'' = 0 .$$

Corresponding to this decomposition of A, we write a and X in the form:

(3.22)

$$a = a' + a'' = \begin{pmatrix} a^1 \\ \vdots \\ a^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a^{k+1} \\ \vdots \\ a^n \end{pmatrix}, \quad X = X' + X'' = \begin{pmatrix} X^1 \\ \vdots \\ X^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ X^{k+1} \\ \vdots \\ X^n \end{pmatrix}.$$

Since (3.20) has no solution, we have

(3.23) $$a' \neq 0 .$$

Now let $F_t(P)$ be the orbit through $P(\neq P_0)$, and $(Y^0(t), Y'(t), Y''(t), Y^\infty(t))$ represent the coordinates of $F_t(P)$ corresponding to (3.21) and (3.22). Then from (3.19) we obtain

$$\frac{dY^0}{dt} = {}^t a' X' + {}^t a'' X'' ,$$
$$\frac{dY'}{dt} = a' X^\infty ,$$
$$\frac{dY''}{dt} = A'' X'' + a'' X^\infty ,$$
$$\frac{dY^\infty}{dt} = 0 ,$$

with the initial condition $(Y^0(0), Y'(0), Y''(0), Y^\infty(0)) = (X^0, X', X'', X^\infty)$. Since $P \neq P_0$, we may assume $Y^\infty(t) = 1 = X^\infty$. Then we have

$$Y'(t) = a't + X' ,$$
$$Y^0(t) = \tfrac{1}{2}|Y'(t) + Y''(t)|^2 \geq \tfrac{1}{2}|a't + X'|^2 ,$$

and therefore

$$\lim_{t\to\pm\infty} \frac{Y^\infty(t)}{Y^0(t)} = 0 , \qquad \lim_{t\to\pm\infty} \frac{Y(t)}{Y^0(t)} = 0 .$$

Hence we conclude that

$$\lim_{t\to\pm\infty} F_t(P) = P_0 ,$$

which shows that the stable and unstable manifolds are both Q itself.

Summarizing the above results we arrive at

Proposition 3.3. *Let U be an essential conformal vector field on Q.*

(i) *Then U has either exactly two distinct singular points or exactly one singular point.*

(ii) *If U has two singular points, then they may be assumed to be P_0 and P_∞ by a suitable conformal transformation of Q so that α is nonzero and the orbit $F_t(P)$ through a point $P \in Q - \{P_0, P_\infty\}$ has the property*

$$\lim_{t\to-\infty} F_t(P) = P_0 , \qquad \lim_{t\to+\infty} F_t(P) = P_\infty ,$$

or

$$\lim_{t\to-\infty} F_t(P) = P_\infty , \qquad \lim_{t\to+\infty} F_t(P) = P_0 .$$

(iii) *If U has just one singular point P_0, then $\alpha = 0$, and the orbit $F_t(P)$ through $P \in Q$ has the property*

$$\lim_{t \to \pm\infty} F_t(P) = P_0 .$$

Next we consider S^n as a Riemannian manifold of sectional curvature 1. It is well-known [16] that on S^n, $n > 2$, a conformal vector field u is uniquely decomposed into the sum of an infinitesimal isometry v and a gradient vector field, namely,

$$u = v - \operatorname{grad} \phi , \tag{3.24}$$

provided $n > 2$. The divergence ϕ satisfies

$$\nabla_j \nabla_i \phi + \phi g_{ji} = 0 . \tag{3.25}$$

As was remarked in § 1, a conformal vector field u with $\nabla_j u_i + \nabla_i u_j = 2\phi g_{ji}$ is uniquely determined by the values of ϕ, $\nabla_i \phi$, u^h and $\nabla_i u^h$ at a point, and at a point p we make u correspond to an element of the Lie algebra CO^L of $\tilde{O}(n+2)$ in the following manner:

$$u \leftrightarrow \begin{pmatrix} \phi & \phi_i & 0 \\ -u^h & \phi\delta_i^h - \nabla_i u^h & \phi^i \\ 0 & -u_i & -\phi \end{pmatrix} \text{ evaluated at } p . \tag{3.26}$$

It is not difficult to verify [7] that this correspondence is an isomorphism between the Lie algebra L of the conformal vector fields on S^n and CO^L.

If u is an essential conformal vector field on S^n, then it has a singular point p_0 by Theorem 2.4, and corresponds, by the correspondence (3.26), to the element of G^L of the form

$$u \leftrightarrow \begin{pmatrix} \phi(p_0) & \phi_i(p_0) & 0 \\ 0 & -\nabla_i v^h(p_0) & \phi^i(p_0) \\ 0 & 0 & -\phi(p_0) \end{pmatrix} , \tag{3.27}$$

where $v = u + \operatorname{grad} \phi$, and we assume $g_{ij} = \delta_{ij}$ at p_0. Since $\phi(p_0)$ is the divergence of u at p_0, the number α appearing in the matrix expression of a conformal vector field U on Q is considered as the value of the divergence of U at p_0.

By Proposition 3.1, if $\phi(p_0) \neq 0$, then the antipodal point p_∞ of the singular point p_0 of u may be assumed to be a singular point of u as well. Furthermore, we then have $\nabla_j \phi = 0$ at both p_0 and p_∞. If $\phi(p_0) > 0$, then ϕ takes the maximum and minimum at p_0 and p_∞ with the same absolute value but opposite singns [9], [10].

If $\phi(p_0) = 0$, then p_0 is the only singular point of u, and the gradient of ϕ cannot vanish at p_0.

Theorem 2.3 corresponds to the Case 1 in the above discussion. A result including both Cases I and II will be given in a forthcoming paper.

4. Sufficient conditions

Ishihara and Tashiro [5], [13] proved that if a compact Riemannian n-manifold M, $n \geq 2$, admits a gradient conformal vector field u, then M is conformal to a Euclidean n-sphere. However, since a gradient field may not be a gradient field by a conformal change of metric. The assumption is not of conformal nature and can be replaced by a conformal one. In fact, let ξ be the 1-form corresponding to u by means of the Riemannian metric g, and assume that at each point of M there are a neighborhood V and a 1-form η on V such that

$$d\xi = \eta \wedge \xi \ . \tag{4.1}$$

This assumption is of conformal nature and obviously weaker than the above one. Moreover, if u satisfies (4.1), then it automatically satisfies the assumption of Theorem 2.3.

Proposition 4.1. *Let u be a conformal vector field with singularities on a Riemannian n-manifold M, $n > 2$, and ξ the 1-form corresponding to u by means of the Riemannian metric. If there is a 1-form η in some neighborhood of each singular point of u satisfying* (4.1), *then u has nonvanishing divergence at each singular point.*

Proof. If u has a vanishing divergence at its singular point p_0, then $u = 0$, $\phi = 0$ at p_0. From our assumption, in a neighborhood of p_0 we have

$$\nabla_i u_h = \phi g_{ih} + \eta_i u_h - \eta_h u_i \ , \tag{4.2}$$

from which it follows that

$$\nabla_i u_h = 0 \qquad \text{at } p_0 \ . \tag{4.3}$$

Covariant differentiation of (4.2) gives

$$\nabla_j \nabla_i u_h = \phi_j g_{ih} \qquad \text{at } p_0 \ . \tag{4.4}$$

On the other hand, by (1.3) we have

$$\nabla_j \nabla_i u_h = \phi_j g_{ih} + \phi_i g_{jh} - \phi_h g_{ji} \qquad \text{at } p_0 \ . \tag{4.5}$$

From (4.4) and (4.5) we then obtain, by contravecting with g^{ij},

$$\phi_h = (2 - n)\phi_h \qquad \text{at } p_0 \ ,$$

which implies $\phi_h = 0$ at p_0. Thus ϕ, ϕ_j, u^h, $\nabla_i u^h$ all vanish at the point p_0, and u becomes the zero vector field. Hence ϕ cannot vanish at its singular points. q.e.d.

Since a gradient conformal vector field obviously satisfies the assumption of Proposition 4.1, Theorem 2.3 generalizes the result of Ishihara and Tashiro.

Tanno and Weber [12] considered a conformal vector field u with $d\xi = 0$; such u also satisfies the assumption of Proposition 4.1, and hence a compact manifold admitting such a vector fieled u is conformorphic to a sphere, and u is a gradient vector field.

As a conformal vector field on the Möbius space, a vector field u with $d\xi = 0$ corresponds to a vector field of the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} X^0 \\ X \\ X^\infty \end{pmatrix}$$

without the rotation part, which is nothing but a gradient vector field satisfying (3.25). Furthermore, on E^n a complete conformal vector field u with the property in Proposition 4.1 is automatically a gradient field.

On account of the above facts we know that if u is a complete conformal vector field on a Riemannian manifold, then the property (4.1) is equivalent to the condition

$$d\xi = d\rho \wedge \xi \tag{4.6}$$

for some scalar ρ on the whole M. In fact, if (4.1) is satisfied, then M is conformorphic to a sphere or a punctured sphere, and in either case the corresponding vector field is a gradient vector field. If h is the conformorphism of M onto S^n with $h^*g^* = e^{2\rho}g$, g^* being the standard Riemannian metric on S^n, then the function ρ can be used in (4.6). The verification of this is a routine computation, since the corresponding vector field u^* on S^n may be assumed to be $d\xi^* = 0$, ξ^* being the 1-form corresponding to u^* on S^n.

5. Final miscellany

In this section we shall give several remarks on the two conjectures mentioned in the introduction and some results related to them.

5.1. Conjecture I is an implication of Conjecture II by means of Yamabe's theorem [15]. However, the converse is not known. In fact, under the additional conditions i) M is compact and ii) the scalar curvature of M is 1, the conformorphism established in Theorem 2.3 is not known to be an isometry of M to S^n.

If the conformal vector field under consideration is a gradient vector field, then the above diffeomorphism can be taken as an isometry [5], [13], [18].

Since the existence of a gradient conformal vetor field on such M is a necessary and sufficient condition for M to be isometric to a Euclidean sphere [5], [9], [10], the Conjecture II follows from Conjecture I if and only if a Riemannian manifold M, with constant scalar curvature and conformorphic to S^n, always admits a gradient conformal vector field. Although the existence of such a field is obvious on S^n with the standard Riemannian metric, the above equivalence of Conjectures I and II is not so clear.

5.2. As for Conjecture II, there have been published many papers, most of which, however, have given sufficient conditions for the existence of a gradient conformal vector field under the assumption that M is a compact Riemannian manifold of constant scalar curvature.

In general, if the scalar curvature is a constant k, then for any conformal vector field u with divergence ϕ, we have [16]

$$\Delta\phi = nk\phi , \tag{5.1}$$

where

$$\Delta = -g^{ji}\nabla_j\nabla_i . \tag{5.2}$$

This property seems to have been most used in the papers just mentioned. In the following we shall assume that $k = 1$.

On the Euclidean sphere S^n of radius 1, the eigenfunction of Δ belonging to the eigenvalue n is the divergence of a conformal vector field, indeed of $-\operatorname{grad}\phi = \operatorname{grad}(-\phi)$. However, this is not true in general. In fact, the following is an example showing the existence of a compact Riemannian manifold with scalar curvature 1, which admits eigenfunctions of Δ with eigenvalue n but no nonisometric conformal vector field:

Example 5.1. Let V and W be compact Riemannian manifolds with scalar curvature 1 of dimension n_1 and n_2 respectively. For example, V and W may be spheres of radius 1 of dimension n_1 and n_2. Let $V(\alpha)$, $0 < \alpha < 1$, denote the Riemannian manifold with the same underlying manifold V and the Riemannian metric g/α, g being the metric of V. Then $V(\alpha)$ has the scalar curvature α. Similarly, we can define the manifold $W(\beta)$, $0 < \beta < 1$. Let $M(\alpha) = V(\alpha) \times W(1-\alpha)$ be the Riemannian product. Then $M(\alpha)$ has the scalar curvature 1.

Now let f be a non-constant function on V with $\Delta_1 f = \lambda f$, where Δ_1 is the Laplacian on V and λ a real number $> n = n_1 + n_2$, and put

$$f^*(x, y) = f(x), \qquad x \in V, y \in W.$$

Then f^* is a function on $M = V \times W$. Let $\Delta(\alpha)$ and $\Delta_1(\alpha)$ denote the Laplacians on $M(\alpha)$ and $V(\alpha)$ respectively. Then we have

$$\Delta(\alpha)f^* = \Delta_1(\alpha)f = \alpha\Delta f = \alpha\lambda f = \alpha\lambda f^* ,$$

and therefore $\Delta(\alpha)f^* = nf^*$ for $\alpha = n/\lambda$. But such a product manifold $M(\alpha)$ cannot admit a non-isometric conformal vector field [11], [14]. This shows that a condition is needed for a function f with $\Delta f = nkf$ to be the gradient of a conformal vector field on a Riemannian manifold with constant scalar curvature k. However, only a sufficient condition is known for compact M in the following theorem [9], [10]:

Let M be a compact Riemannian n-manifold, $n \geq 2$. If on M a non-constant function ϕ satisfies $\Delta\phi = nk\phi$, $k > 0$, then

$$\int_M [K_{ji} - (n-1)kg_{ji}]\phi^j\phi^i dM \leq 0 \ ,$$

where the equality holds if and only if

$$\nabla_j\phi_i + k\phi g_{ji} = 0 \ ,$$

and hence if and only if M is isometric to a sphere of radius $1/\sqrt{k}$.

It should be noted that the scalar curvature of M in the above theorem is not assumed to be constant. Therefore in Example 5.1 we must have

$$\int_{M(\alpha)} [K_{ji} - (n-1)g_{ji}]f^jf^i dM(\alpha) < 0 \ .$$

5.3. In the set of all the compact Einstein n-spaces with scalar curvature 1, the possible minimum eigenvalue of the Laplacian is just n, and n is attained only on S^n [9], [10]. However, in the set of all the compact Riemannian n-manifolds with scalar curvature 1, there does not exist such a minimum eigenvalue.

In fact, in Example 5.1 we consider the family of Riemannian manifolds $M(\alpha)$, $0 < \alpha < 1$, with scalar curvature 1. Then for any eigenvalue λ of Δ_1 on V, there is an eigenvalue $\alpha\lambda$ of $\Delta(\alpha)$ on $M(\alpha)$. As $\alpha \to 0$, $\alpha\lambda$ tends to zero and hence there is no minimum eigenvalue in the set $\{M(\alpha)\}$.

5.4. With regard to Conjecture II, there have been published many results, in some [3], [17], [19] of which the following types of assumptions have been made:

(i) $\mathcal{L}_u F = c = \text{const.}$,
(ii) $\mathcal{L}_u\mathcal{L}_u F \leq 0$ (or ≥ 0),
(iii) $\mathcal{L}_u\mathcal{L}_u F < 0$,

for a conformal vector field u and a certain function F on a compact Riemannian manifold M. However, each of the assumptions (i), (ii) is equivalent to $\mathcal{L}_u F \equiv 0$ for any vector field u on any compact manifold M, and has nothing to do with the conformal vector fields or the Riemannian metric of M.

In fact, for (i) since M is compact, u generates a global 1-parameter group f_t, $-\infty < t < \infty$, of diffeomorphisms of M. Along each orbit $f_t(p)$ through a point p, we have $dF/dt = c$, or $F(t) = ct + c_0$, which gives that $\lim_{t\to\infty} F(t) = \pm\infty$ unless $c = 0$. Hence $\mathscr{L}_u F \equiv 0$. Or we can say in the following way: Since F takes a maximum at a point on M, $\mathscr{L}_u F = 0$ at that point, and therefore $\mathscr{L}_u F \equiv 0$.

For (ii), assume that there is a point p_0 on M, at which $\mathscr{L}_u F \neq 0$. Then p_0 is not a singular point of u, and we can consider the orbit $f_t(p_0)$ of u through p_0. Without loss of generality, we may assume $\mathscr{L}_u F = c < 0$ at p_0, for otherwise we use $-u$ and the assumption $\mathscr{L}_u \mathscr{L}_u F \leq 0$. Then along the orbit $f_t(p_0)$, we have $d^2F/dt^2 \leq 0$, and therefore for $t > 0$, $dF/dt \leq c < 0$ or $F(t) \leq ct + F(p_0)$, which gives $\lim_{t\to\infty} F = -\infty$, a contradiction. Hence $\mathscr{L}_u F \equiv 0$. From (ii) we see immediately that it is impossible to assume (iii).

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