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THE NUMBER OF BRANCH POINTS OF SURFACES OF BOUNDED MEAN CURVATURE

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Introduction

Let B be the unit disk $|w|<1$ $(w=u+iv)$, and $\mathfrak{x}=\mathfrak{x}(w)=(x(w), y(w), z(w))$ a vector function of class $C^2(B) \cap C^0(\bar{B})$ satisfying in B the partial differential equations

$$\Delta \mathfrak{x} = 2H(w)(\mathfrak{x}_u \times \mathfrak{x}_v) \;, \tag{0.1}$$

and

$$\mathfrak{x}_u^2 = \mathfrak{x}_v^2 \;, \qquad \mathfrak{x}_u \mathfrak{x}_v = 0 \;, \tag{0.2}$$

where $H = H(w)$ is a bounded function on $\bar{B}$. Furthermore, assume that we have

$$|\mathfrak{x}(w)| \leq 1 \qquad (w \in \bar{B}) \;, \tag{0.3}$$

and

$$A(\mathfrak{x}) = \frac{1}{2}\int\int_B (\mathfrak{x}_u^2 + \mathfrak{x}_v^2) du\, dv < +\infty \;, \tag{0.4}$$

and that $\mathfrak{x}=\mathfrak{x}(w)$ maps the unit circle ∂B topologically onto a closed rectifiable Jordan curve $\Gamma^* \subset R^3$. Geometrically speaking, these conditions express the fact that $\mathfrak{x}=\mathfrak{x}(w)$ represents a surface in R^3 of finite area $A(\mathfrak{x})$, contained in the unit ball $|\mathfrak{x}| \leq 1$, which is bounded by Γ^* and whose mean curvature at each regular point coincides with the function $H(w)$.

The various existence proofs for such surfaces available in the literature ([2], [5], [6], [14], [15], and [16] deal with the case, where $H(w)$ is a constant, while [7] treats the general case) leave the question open, whether, for a given curve Γ^*, there always exists a surface of prescribed mean curvature which is free of branch points. While even for minimal surfaces ($H(w) \equiv 0$) this is an unsolved problem, it is nevertheless possible to estimate the total number of these branch points in terms of geometric quantities associated with Γ^*. The

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principal object of this paper is to obtain such estimates for the case, where Γ^* is a regular curve of class C^3 and $H(w)$ is höldercontinuous on $\bar{B}$ (Theorems 2 and 3). Our method consists in applying the Gauss-Bonnet formula to the singular metric $ds^2 = (d\mathfrak{x})^2$ and corresponds to that used by Sasaki [12] and Nitsche [10] in the case of minimal surfaces. The applicability of the Gauss-Bonnet formula is justified by a certain regularity theorem for surfaces of bounded mean curvature established in [4]. Moreover, in the proofs of Theorems 2 and 3 we require two isoperimetric inequalities. One of them (Lemma 4), which is related to surfaces of constant mean curvature having a certain minimum property, was established in [5]. The other one (Theorem 1′), whose proof we present in § 1, generalizes and partly sharpens an earlier result [3, Theorem 3] proved for the case where $H(w)$ is a constant. In view of its significance for other problems in differential geometry, Theorem 1′ is stated here under hypotheses considerably more general than those made in § 2.

1. An isoperimetric inequality

We first define the function class $\mathfrak{Q}_G(h)$ we are dealing with.

Definition 1. Let G be a simply-connected domain in R^2, which is bounded by a closed Jordan curve $\Gamma = \partial G$, and let h be a nonnegative parameter. Then $\mathfrak{Q}_G(h)$ is the set of vector functions $\mathfrak{x}\colon \bar{G} \to R^3$ with the following properties:

a) $\mathfrak{x} = \mathfrak{x}(w)$ belongs to $C^2(G) \cap C^0(\bar{G})$ and satisfies in G the nonlinear system (0.1)–(0.2), where $H = H(w)$ is a real-valued function in $\bar{B}$ with $h = \sup_{\bar{B}} |H(w)| < +\infty$.

b) We have the inequalities

$$|\mathfrak{x}(w)| \leq 1 \qquad (w \in \bar{G}) , \tag{1.1}$$

$$A_G(\mathfrak{x}) = \frac{1}{2}\int\int_G (\mathfrak{x}_u^2 + \mathfrak{x}_v^2)du\, dv < +\infty , \tag{1.2}$$

$$L_\Gamma(\mathfrak{x}) = \int_\Gamma |d\mathfrak{x}| < +\infty . \tag{1.3}$$

For $\mathfrak{x} \in \mathfrak{Q}_G(h)$ we introduce the quantity

$$R_\Gamma(\mathfrak{x}) = \inf_{\mathfrak{a} \in R^3} [\sup_{w \in \Gamma} |\mathfrak{x}(w) - \mathfrak{a}|] . \tag{1.4}$$

Because of (1.1) we have

$$R_\Gamma(\mathfrak{x}) \leq 1 . \tag{1.5}$$

Evidently $R_\Gamma(\mathfrak{x})$ can be considered as the radius of the smallest ball in R^3

containing the space curve $\mathfrak{x}(\Gamma)$. The main result of this section is

Theorem 1. *Let $\mathfrak{x} \in \mathfrak{Q}_G(h)$, where $h < 1$. Then we have the inequality*

$$A_G(\mathfrak{x}) \leq \frac{R_\Gamma(\mathfrak{x})L_\Gamma(\mathfrak{x})}{2(1 - hR_\Gamma(\mathfrak{x}))} . \tag{1.6}$$

Now an elementary consideration shows that the estimate

$$R_\Gamma(\mathfrak{x}) \leq \frac{1}{4}L_\Gamma(\mathfrak{x}) \tag{1.7}$$

holds. Thus Theorem 1 implies

Theorem 1′. *Under the hypotheses of Theorem 1 we have*

$$A_G(\mathfrak{x}) \leq \frac{L_\Gamma(\mathfrak{x})^2}{8(1 - h)} . \tag{1.8}$$

In proving Theorem 1 we follow the method used by Courant [1, pp. 129–131] for the case of minimal surfaces. In order to carry out this procedure, we require the following lemma which is related to the maximum principle for subharmonic functions and which seems to be of independent interest. In addition, use is made of a general convergence theorem for the total variations of vector functions $\mathfrak{x}: \bar{G} \to R^3$ satisfying the differential inequality

$$|\Delta\mathfrak{x}| \leq c|\mathfrak{x}_u \times \mathfrak{x}_v| , \tag{1.9}$$

c being a fixed positive constant (see [4, Theorem 2]). First we have

Lemma 1. *Let G be a bounded domain in R^2, and $\mathfrak{x}: \bar{G} \to R^3$ a vector function of class $C^2(G) \cap C^0(\bar{G})$ satisfying in G the differential inequality*

$$|\Delta\mathfrak{x}| \leq h(\mathfrak{x}_u^2 + \mathfrak{x}_v^2) , \tag{1.10}$$

where $0 < h < 1$. Furthermore assume that we have

$$|\mathfrak{x}(w)| \leq 1 \qquad (w \in \bar{G}) , \tag{1.11}$$

$$|\mathfrak{x}(w) - \mathfrak{a}| \leq R \leq 1 \qquad (w \in \partial G) , \tag{1.12}$$

where $\mathfrak{a}$ is a fixed vector in R^3. Then the inequality

$$|\mathfrak{x}(w) - \mathfrak{a}| \leq R \tag{1.13}$$

holds in $\bar{G}$.

Proof. We apply a continuity method. Consider the family of vector functions

$$\mathfrak{x}_\lambda(w) = \mathfrak{x}(w) - \lambda\mathfrak{a} \qquad (0 \leq \lambda \leq 1) . \tag{1.14}$$

Then in virtue of (1.11) and (1.12) we have, for $w \in \partial G$ and $\lambda \in [0, 1]$, the estimate

$$\begin{aligned} |\mathfrak{x}_\lambda(w)| &= |\lambda(\mathfrak{x}(w) - \mathfrak{a}) + (1 - \lambda)\mathfrak{x}(w)| \\ &\leq \lambda|\mathfrak{x}(w) - \mathfrak{a}| + (1 - \lambda)|\mathfrak{x}(w)| \leq 1 . \end{aligned} \tag{1.15}$$

Moreover, on account of (1.10), the differential inequality

$$\begin{aligned} \Delta(\mathfrak{x}_\lambda^2) &= 2(\mathfrak{x}_{\lambda u}^2 + \mathfrak{x}_{\lambda v}^2) + 2(\mathfrak{x}_\lambda, \Delta\mathfrak{x}\lambda) \\ &\geq 2(1 - h\varphi(\lambda))(\mathfrak{x}_{\lambda u}^2 + \mathfrak{x}_{\lambda v}^2) \end{aligned} \tag{1.16}$$

holds, where $\varphi(\lambda) = \max_{w \in \bar{G}} |\mathfrak{x}\lambda(w)|$. This shows that the function $\mathfrak{x}_\lambda^2$ is subharmonic in G if $\varphi(\lambda) \leq 1/h$. In conjunction with (1.15) this implies that we have either $\varphi(\lambda) \leq 1$ or $\varphi(\lambda) > 1/h$. Since $\varphi(0) \leq 1$, and $\varphi(\lambda)$ is continuous in $[0, 1]$, we must have $\varphi(1) \leq 1$. Using now (1.12) and applying again the maximum principle for subharmonic functions we conclude that the inequality (1.13) holds in $\bar{G}$, which proves the lemma.

Proof of Theorem 1. According to Theorem 2 of [4] there exists, for every $\varepsilon > 0$ and every compact subset $C \subset G$, a closed analytic Jordan curve $\gamma \subset G$, whose interior domain D contains C such that

$$\left| \int_\gamma |d\mathfrak{x}(w)| - L_\Gamma(\mathfrak{x}) \right| < \varepsilon . \tag{1.17}$$

Furthermore, in virtue of (1.4) and (1.5), there exists a fixed vector $\mathfrak{a} \in R^3$ such that

$$|\mathfrak{x}(w) - \mathfrak{a}| \leq R_\Gamma(\mathfrak{x}) \leq 1 \tag{1.18}$$

holds on Γ.

Applying Lemma 1 we infer

$$|\mathfrak{x}(w) - \mathfrak{a}| \leq R_\Gamma(\mathfrak{x}) \tag{1.19}$$

for $w \in \bar{G}$. Putting $\mathfrak{y}(w) = \mathfrak{x}(w) - \mathfrak{a}$, by Green's Theorem we then have

$$\begin{aligned} &\iint_D (\mathfrak{y}_u^2 + \mathfrak{y}_v^2 + \mathfrak{y}\Delta\mathfrak{y}) du\, dv \\ &= \int_\gamma \mathfrak{y} \frac{\partial \mathfrak{y}}{\partial \nu} d\sigma \leq R_\Gamma(\mathfrak{x}) \cdot \int_\gamma \left| \frac{\partial \mathfrak{x}}{\partial \nu} \right| d\sigma , \end{aligned} \tag{1.20}$$

where ν is the outer normal on D and $d\sigma$ is the arc element on γ. Now in virtue of (0.2) the equation

$$\left| \frac{\partial \mathfrak{x}}{\partial \nu} \right| = \left| \frac{\partial \mathfrak{x}}{\partial \sigma} \right| \tag{1.21}$$

holds on γ. Together with (1.17), (1.19) and (1.20), this entails

$$(1.22)\qquad \begin{aligned}&(1 - R_\Gamma(\mathfrak{x})h)\iint_D(\mathfrak{x}_u^2 + \mathfrak{x}_v^2)du\,dv\\ &\le \iint_D(\mathfrak{y}_u^2 + \mathfrak{y}_v^2 + \mathfrak{y}\Delta\mathfrak{y})du\,dv \le R_\Gamma(\mathfrak{x})(L_\Gamma(\mathfrak{x}) + \varepsilon)\ ,\end{aligned}$$

and therefore

$$(1.23)\qquad \iint_C(\mathfrak{x}_u^2 + \mathfrak{x}_v^2)du\,dv \le \frac{R_\Gamma(\mathfrak{x})(L_\Gamma(\mathfrak{x}) + \varepsilon)}{1 - R_\Gamma(\mathfrak{x})h}$$

for every $\varepsilon > 0$ and every compact subset $C \subset B$. Letting ε tend to zero, we hence arrive at the conclusion of the theorem.

2. The branch points

We are now prepared to establish the main results of this paper. In order to formulate them, we make use of the following definition.

Definition 2. Let Γ^* be a closed rectifiable Jordan curve of total length $l(\Gamma^*)$ contained in the unit ball $|\mathfrak{x}| \le 1$ in R^3, and h be a nonnegative parameter. Then $\mathfrak{Q}^*(\Gamma^*, h)$ is the subset of vector functions $\mathfrak{x}\colon \bar{B} \to R^3$ of $\mathfrak{Q}_B(h)$ with the following properties:

a) $\mathfrak{x} = \mathfrak{x}(w)$ maps ∂B topologically onto Γ^*.

b) The curvature function $H(w)$ occurring in (0.1) is höldercontinuous on $\bar{B}$.

Our analysis is based upon the following lemma.

Lemma 2. *Let $\mathfrak{x} \in \mathfrak{Q}^*(\Gamma^*, h)$, where Γ^* is a regular closed Jordan curve of class C^3 Then $\mathfrak{x}$ belongs to $C^2(\bar{B})$, and moreover, in the neighborhood of any branch point $w^* \in \bar{B}$ we have the asymptotic expansion*

$$(2.1)\qquad \mathfrak{x}_u - i\mathfrak{x}_v = \mathfrak{k}(w - w^*)^k + o(|w - w^*|^k) \qquad (w \to w^*, w \in \bar{B})\ ,$$

where $\mathfrak{k}$ is a complex vector $\neq 0$, and k is a positive integer.

Proof. See Theorem 4 of [4].

As an immediate consequence of this lemma we observe that the total number of branch points of the surface $\mathfrak{x}\colon \bar{B} \to R^3$ is finite, each branch point w^* having a finite order $k = k(w^*)$. Moreover, at each point $w_0 \in \bar{B}$ the surface possesses a generalized normal $\mathfrak{N} = \mathfrak{N}(w_0)$ defined by the equation

$$(2.2)\qquad \mathfrak{N} = \lim_{\substack{w\to w_0\\ w\neq w_0}} \frac{\mathfrak{x}_u \times \mathfrak{x}_v}{|\mathfrak{x}_u \times \mathfrak{x}_v|}\ .$$

Evidently $\mathfrak{N} = \mathfrak{N}(w)$ is continuous in $\bar{B}$. To proceed further, we have to make use of some fundamental concepts of differential geometry. Let $\mathfrak{x} = \mathfrak{x}(s)$ $(0 \leq s \leq l = l(\Gamma^*))$ be the representation of the curve Γ^* in terms of its arc length. If we assume Γ^* to be positively oriented with respect to the surface $\mathfrak{x}\colon \bar{B} \to R^3$, then its generalized geodesic curvature is given by the formula

$$\kappa_g = (\mathfrak{N}, \mathfrak{x}', \mathfrak{x}'') \; . \tag{2.3}$$

κ_g coincides with the ordinary geodesic curvature at all points of the curve Γ^*, where $\mathfrak{x}_u \neq 0$. Moreover, for the total curvature of Γ^* we have the equation

$$\kappa(\Gamma^*) = \int_0^l |\mathfrak{x}''(s)| \, ds \; . \tag{2.4}$$

Now let $\{w_\alpha\}$ and $\{\tilde{w}_\beta\}$ be the branch points of the surface lying in B and on ∂B, respectively. Furthermore, let K be the Gauss curvature of the surface $\mathfrak{x} = \mathfrak{x}(w)$ defined in the punctured disk $B' = B - \{w_\alpha\}$, $K^- = \frac{1}{2}(|K| - K)$ be its negative part, and $d\omega = E \, du \, dv$ be the surface element. Then our first main result can be stated as follows.

Theorem 2. *Let* $\mathfrak{x}\colon \bar{B} \to R^3$ *be a surface of class* $\mathfrak{L}^*(\Gamma^*, h)$, *where* Γ^* *is a regular closed Jordan curve of class* C^3 *and* $h < 1$. *Then we have the inequality*

$$\begin{aligned} &1 + \sum_\alpha k(w_\alpha) + \frac{1}{2} \sum_\beta k(\tilde{w}_\beta) \\ &\leq \frac{\kappa(\Gamma^*)}{2\pi} + \frac{h^2}{16\pi(1-h)} l(\Gamma^*)^2 - \frac{1}{2\pi} \iint_{B'} K^- d\omega. \end{aligned} \tag{2.5}$$

In the case, where $h = 0$ and Γ^* is analytic, this inequality is due to Nitsche [10, pp. 235–236] who improved an earlier result of Sasaki [12] (see also [9] for a corrected version of Sasaki's proof). As an immediate consequence of this theorem we obtain the inequality

$$1 + \sum_\alpha k(w_\alpha) + \frac{1}{2} \sum_\beta k(\tilde{w}_\beta) \leq \frac{\kappa(\Gamma^*)}{2\pi} + \frac{h^2}{16\pi(1-h)} l(\Gamma^*)^2 \tag{2.6}$$

For the case of minimal surfaces and arbitrary Jordan curves the weaker estimate

$$1 + \sum_\alpha k(w_\alpha) \leq \frac{\kappa(\Gamma^*)}{2\pi} \tag{2.7}$$

was established by R. Schneider [13] by using different methods related to earlier results of T. Radó [11]. Here $\kappa(\Gamma^*)$ is to be taken in the sense of Milnor [8]. The methods of Schneider are more elementary than ours since they do not involve an analysis of the boundary behavior of the surfaces in question. On the other hand, they do not seem to be applicable to the case considered here because of the nonlinear structure of the system (0.1).

In proving our theorem we make use of Theorem 1′ and an appropriate generalization of the Gauss-Bonnet formula.

Lemma 3. *Let $\mathfrak{x}$ belong to $\mathfrak{Q}^*(\Gamma^*, h)$, where Γ^* is a regular curve of class C^3. Then the integral $\int\int_{B'} |K| d\omega$ exists and we have the equation*

$$(2.8) \qquad 1 + \sum_\alpha k(w_\alpha) + \frac{1}{2} \sum_\beta k(\tilde{w}_\beta) = \frac{1}{2\pi} \int_{\Gamma^*} \kappa_g ds + \frac{1}{2\pi} \int\int_{B'} K d\omega \ .$$

Proof. Let $\{r_\alpha^{(n)}\}$ and $\{\tilde{r}_\beta^{(n)}\}$ be two sequences of positive numbers tending to zero for $n \to \infty$. Consider the sequence of domains $\{G_n\}$ defined by the inequalities $|w| < 1$, $|w - w_\alpha| > r^{(n)}$, and $|w - w_\beta^{(n)}| > \tilde{r}_\beta^{(n)}$. In virtue of Green's formula and the Theorema egregium we have the equation

$$(2.9) \qquad \int\int_{G_n} K d\omega = -\frac{1}{2} \int_{\partial G_n} \frac{\partial \log E}{\partial \nu} d\sigma \ ,$$

where ν is the outer normal of G_n, and $d\sigma$ is the arc element on ∂G_n. Now for $w \in \partial G_n \cap \partial B$ we have the classical formula

$$(2.10) \qquad -\frac{1}{2} \frac{\partial \log E}{\partial \nu} = 1 - \sqrt{E} \kappa_g \ .$$

Moreover, from Lemma 2 we infer that the sequences $\{r_\alpha^{(n)}\}$ and $\{\tilde{r}_\beta^{(n}\}$ can be chosen such that the relations

$$(2.11) \qquad -\frac{1}{2} \int_{\substack{|w-w_\alpha| = r_\alpha^{(n)} \\ w \in \bar{B}}} \frac{\partial \log E}{\partial \nu} d\sigma \to 2\pi k(w_\alpha) \qquad (n \to \infty) \ ,$$

$$(2.12) \qquad -\frac{1}{2} \int_{\substack{|w-\tilde{w}_\beta| = \tilde{r}_\beta^{(n)} \\ w \in \bar{B}}} \frac{\partial \log E}{\partial \nu} d\sigma \to \pi k(\tilde{w}_\beta) \qquad (n \to \infty)$$

hold. Combining (2.9)–(2.12) we obtain the equation

$$(2.13) \qquad \lim_{n\to\infty} \int\int_{G_n} K d\omega = -\int_{\Gamma^*} \kappa_g ds + 2\pi + 2\pi \sum_\alpha k(w_\alpha) + \pi \sum_\beta k(\tilde{w}_\beta) \ .$$

Together with the inequality

$$(2.14) \qquad K(w) \leq H(w)^2 \leq h^2 \qquad (w \in B') \ ,$$

this implies that $\iint_{B'} |K| d\omega$ is finite and that equation (2.8) holds, which proves the lemma.

In the special case, where the sets $\{w_\alpha\}$ and $\{\tilde{w}_\beta\}$ are empty, equation (2.8) just reduces to the ordinary Gauss-Bonnet formula.

Proof of Theorem 2. For abbreviation, let us put $K^+ = \frac{1}{2}(|K| + K)$. Then on account of Theorem 1′ and (2.14) we have

$$\begin{aligned} \frac{1}{2\pi}\iint_{B'} K d\omega &= \frac{1}{2\pi}\iint_{B'} K^+ d\omega - \frac{1}{2\pi}\iint_{B'} K^- d\omega \\ &\leq \frac{h^2}{2\pi} A(\mathfrak{x}) - \frac{1}{2\pi}\iint_{B'} K^- d\omega \\ &\leq \frac{h^2}{16\pi(1-h)} l(\Gamma^*)^2 - \frac{1}{2\pi}\iint_{B'} K^- d\omega \, . \end{aligned} \tag{2.15}$$

Moreover, from (2.3) we infer $|\kappa_g| \leq |\mathfrak{x}''|$, and hence

$$\frac{1}{2\pi}\int_{\Gamma^*} \kappa_g ds \leq \frac{\kappa(\Gamma^*)}{2\pi} \, . \tag{2.16}$$

Applying now Lemma 3, we obtain the inequality (2.6), which proves the theorem.

To state our second result, we introduce the class $\mathfrak{C}(\Gamma^*)$ of surfaces $\mathfrak{x} \in C^1(B) \cap C^0(\bar{B})$ satisfying (0.3) and (0.4) and mapping ∂B monotonically onto Γ^*. Again Γ^* is a closed rectifiable Jordan curve of length $l(\Gamma^*)$ contained in the unit ball $|\mathfrak{x}| \leq 1$. We have the following

Lemma 4. *Suppose that* $\mathfrak{x} \in \mathfrak{Q}^*(\Gamma^*, h)$ *minimizes the functional*

$$E(\mathfrak{z}) = \iint_B \left[\mathfrak{z}_u^2 + \mathfrak{z}_v^2 + \frac{4H}{3}(\mathfrak{z}, \mathfrak{z}_u, \mathfrak{z}_v)\right] du\, dv \tag{2.17}$$

among all vectors $\mathfrak{z} \in \mathfrak{C}(\Gamma^*)$ *and that H is a constant satisfying* $h = |H| < 3/2$. *Then we have*

$$A(\mathfrak{x}) \leq \frac{1}{4}\,\frac{1 + 2h/3}{1 - 2h/3}\, l(\Gamma^*)^2 \, . \tag{2.18}$$

Proof. See [5, Theorem 5.1].

The same conclusions as in the proof of Theorem 2 lead to the following result:

Theorem 3. *Let* $\mathfrak{x}: \bar{B} \to R^3$ *be a surface of class* $\mathfrak{Q}^*(\Gamma^*, h)$ *minimizing* $E(\mathfrak{z})$ *in the class* $\mathfrak{C}(\Gamma^*)$, *and suppose that H is a constant satisfying* $h = |H| < 3/2$. *Then we have the inequality*

$$(2.19)\qquad \begin{aligned} &1 + \sum_{\alpha} k(w_\alpha) + \frac{1}{2} \sum_{\beta} k(\tilde{w}_\beta) \\ &\qquad \leq \frac{\kappa(\Gamma^*)}{2\pi} + \frac{h^2}{8\pi} \frac{1 + 2h/3}{1 - 2h/3} l(\Gamma^*)^2 - \frac{1}{2\pi} \iint_{B'} K^- d\omega . \end{aligned}$$

Added in proof

1. In his recent note (*The nonexistence of branch points in the classical solution of Plateau's problem*, Bull. Amer. Math. Soc. **75** (1969) 1247–1248) R. Osserman states that the Douglas solution of the Plateau problem for minimal surfaces is free of branch points.

2. The factor 1/4 in the isoperimetric inequality (2.18) can be replaced by $1/(4\pi)$. Correspondingly, one can replace the factor $h^2/(8\pi)$ in (2.19) by $h^2/(8\pi^2)$.

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