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SIGN CHANGES, EXTREMA, AND CURVES OF MINIMAL ORDER

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Let $\{p_k(t)\}$ be a set of polynomials with real coefficients indexed by the degree. If the moments of a continuous function $f(t)$ for these polynomials vanish, that is, if

$$\int_0^1 f(t)p_k(t)dt = 0\,, \qquad 0 \leq k \leq n\,,$$

and if $f(t)$ does not vanish identically, then $f(t)$ has at least $n+1$ changes of sign in $(0, 1)$. A similar result holds for trigonometric integrals, namely, if

$$\int_0^1 f(t)\cos 2\pi\, kt\, dt = \int_0^1 f(t)\sin 2\pi\, kt\, dt = 0\,, \qquad 0 \leq k \leq n\,,$$

then $f(t)$ has at least $2n+1$ sign changes in $(0, 1)$ unless it vanishes identically. The theorems and the technique of proof are explained in Pólya-Szegö [1, p. 65, Problems 140 et seq.].

The theorem about the connection between the vanishing of the Fourier coefficients and the number of sign changes of a function has many important applications in geometry. A survey of these applications is given in [2].

In this note we give a general theorem based on an order property of curves in n-space, and during the course of this investigation we are led to an interesting unsolved problem about curves in n-space. We also give, as applications, some new vertex theorems in differential geometry and differential equations, and obtain a much shortened proof and a generalization of a theorem about intermediate values [2, Theorem 3].

1. We say that a continuous curve in Cartesian n-space

$$\boldsymbol{u}\colon [0, 1] \to \boldsymbol{R}^n$$

is of *minimal order* if no $(n-1)$-plane intersects $\{\boldsymbol{u}(t)\}$ in more than n points, or in Haupt's terminology, the set $\{\boldsymbol{u}([0, 1])\}$ is a continuum of POW n for the $(n-1)$-planes in n-space. Hence [3, §5.2.4, 2. Satz], $\boldsymbol{u}$ is a simple arc or a simple closed curve. In fact, the existence of a multiple point can be

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excluded by a simple continuity argument. Take an $(n-1)$-plane through the multiple point and $n-1$ other points. If the other points are kept fixed and the first point is varied, there must exist situations in which more than n points are in one $(n-1)$-plane.

The coordinate functions of $\boldsymbol{u}$ are denoted by $u_i(t)$, $1 \leq i \leq n$.

Theorem 1. *Let $f(t)$ be a continuous, real-valued function not identically zero on* $(0, 1)$, *and $\boldsymbol{u}(t)$ a curve of minimal order in $\boldsymbol{R}^n$. By i_j we denote non-negative integers. If*

$$\int_0^1 f(t)u_1^{i_1}(t)\cdots u_n^{i_n}(t)\,dt = 0\,, \qquad 0 \leq \sum_{j=1}^{n} i_j \leq k\,,$$

then $f(t)$ changes sign at least $kn+1$ times in $(0, 1)$.

We assume first that $f(t)$ has exactly kn sign changes in $(0, 1)$, which occur at t_i, $1 \leq i \leq kn$, and that $t_i < t_{i+1}$. If $f(t)$ should vanish on an interval and changes sign, we arbitrarily assign one of the points of the interval as t_i. Let

$$L_j(\boldsymbol{x}) = 0\,, \qquad 1 \leq j \leq k\,,$$

be the equation of the $(n-1)$-plane through the n distinct points

$$\boldsymbol{u}(t_{j+sk}) \quad 0 \leq s \leq n-1\,.$$

Then no other point $\boldsymbol{u}(t)$ is in the plane. Hence, the sign can be chosen so that $L(\boldsymbol{u}(t_{j+1})) > 0$, and the curve $\boldsymbol{u}(t)$ must cross the plane $L_j(\boldsymbol{x}) = 0$ at all n points of intersection. If $\boldsymbol{u}(t)$ would be in one closed halfplane for t in a neighborhood of t_{j+s_0k}, then for small enough ε the plane through the $\boldsymbol{u}(t_{j+sk})$, $s \neq s_0$, and $\boldsymbol{u}(t_{j+s_0k}+\varepsilon)$ must have $n+1$ points of intersection with $\boldsymbol{u}$. Hence, $L_j(\boldsymbol{u}(t))$ changes sign only at all values t_{j+sk}, $0 \leq s \leq n-1$. The function

$$P(t) = L_1(\boldsymbol{u}(t))\cdots L_k(\boldsymbol{u}(t))$$

changes sign exactly at t_i, $1 \leq i \leq kn$, and $f(t)P(t)$ never changes sign. In particular, $\int_0^1 f(t)P(t)\,dt \neq 0$. Since $P(t)$ is a polynomial of degree k in $u_1(t), \cdots, u_n(t)$, by our hypothesis $\int_0^1 f(t)p(t)\,dt = 0$ for any polynomial $p(t)$ of degree $\leq k$ in the $u_i(t)$. In particular, this holds for $p(t) = P(t)$, and the contradiction shows that $f(t)$ cannot have exactly kn changes of sign.

Next, we assume that $f(t)$ has $k_n - l$, $1 \leq l \leq n-1$, sign changes at t_i, $1 \leq i \leq kn - l$. The same contradiction as before is obtained if we put $t_{kn-l+1} = t_{kn-l+2} = \cdots = t_{kn} = 1$. For $l = n$, the first proof applies with k replaced by $k-1$. Hence, it is impossible for $f(t)$ to have less than $kn+1$ changes of sign.

Remark 1.1. If $f(t)$ is periodic of period 1, the number of sign changes in $[0, 1)$ is even. For $kn \equiv 0 (\mathrm{mod}\ 2)$, we obtain an additional sign change.

Remark 1.2. If $\omega(t) > 0$ is a continuous function on $(0, 1)$, the function $f(t)\omega(t)$ can replace $f(t)$ in the hypothesis of Theorem 1.

2. Theorem 2. *Let $u(t)$ be a continuously differentiable curve of minimal order, and $f(t)$ a continuous function of bounded variation, periodic of period 1. If*

$$\sum_{j=1}^{n} i_j \int_0^1 f(t) u_1^{i_1}(t) \cdots u_{j-1}^{i_{j-1}}(t) u_j^{i_j - 1}(t) u_{j+1}^{i_{j+1}}(t) \cdots u_n^{i_n}(t) u_j'(t)\, dt = 0 ,$$

$$1 \leq \sum_j i_j \leq k ,$$

and either $\boldsymbol{u}$ is closed ($\boldsymbol{u}(0) = \boldsymbol{u}(1)$) or $f(0) = f(1) = 0$, then $f(t)$ has at least $kn + 1$ relative extrema in $(0, 1)$.

By Remark 1.1, in the second case the number of relative extrema in $[0, 1)$ is $\geq 2\left[\frac{kn}{2}\right] + 2$.

The theorem is non-trivial only if we assume that $f(t)$ has only a finite number of maxima and minima in one period. This means that $df(t)$ changes sign only a finite number of times. Let the sign changes be fixed at t_i. Then the hypotheses of the theorem are equivalent to

$$\int_0^1 u_1^{i_1}(t) \cdots u_n^{i_n}(t)\, df(t) = 0 , \qquad 0 \leq \sum_j i_j \leq k ,$$

Hence, also $\int_0^1 P(t) df(t) = 0$ for every polynomial of degree $\leq k$ in the $u_i(t)$. If we assume that $df(t)$ has $\leq kn$ sign changes and construct $P(t)$ as in the proof of Theorem 1, it follows that $\int_0^1 P(t) df(t) \neq 0$. The contradiction proves the theorem.

Remark 2.1. A closed curve in R^n meets every $(n - 1)$-plane in an even number of points, and the first alternative in Theorem 2 is possible only for an even n.

3. Many important applications of Theorem 2 refer to the case $k = 1$. If we put $\boldsymbol{u}'(t) = \boldsymbol{v}(t)$, the integral conditions reduce to

$$\int_0^1 f(t) \boldsymbol{v}(t) dt = 0 ,$$

and the curve $\boldsymbol{u}$ does not appear explicitly in the formulation of the theorem. Hence, there is a certain interest in the characterization of the curves $\boldsymbol{v}(t)$ whose integral curves

$$u(t) = u_0 + \int_0^t v(\tau)d\tau$$

are of minimal order. In the much deeper context of order geometry, the problem appears to characterize the cotingent of continua of low orders. In this context, necessary conditions for v have been enunciated by Hjelmslev and proved by Derry and Künneth. They proved [3, §5.3.9] that a continuous curve u of minimal order has one-sided tangents at all points (and two-sided tangents at almost all points), and the intersection of the tangent half-cone (the parallels to the forward halftangents starting from a fixed point) with any $(n - 1)$-plane is a curve piecewise of minimal order in R^{n-1}.

We solve the converse problem for continuous $v(t)$ and $n = 2, 3$. For higher dimensions, we have only a conjecture to offer.

For $n = 2$, the Hjelmslev condition says that no point $\neq 0$ is covered more than once by the rays, which start from 0 and carry the vectors $v(t)$. This means that $v(t)$ is a star-shaped curve in R^2. The integral curve of a star-shaped curve is locally convex. It is a *closed* convex curve if

$$(*) \qquad \int_0^1 v(t)\, dt = 0 ,$$

which is the condition on v replacing the condition on u in the first case of Theorem 2.

We denote the determinant of the vectors a and b by $[a, b]$, and also let

$$\varepsilon = \begin{cases} +\,1, & \text{if } \arctan v_2/v_1 \text{ is an increasing function of } t, \\ -\,1, & \text{if } \arctan v_2/v_1 \text{ is a decreasing function of } t. \end{cases}$$

The curve u is convex iff $v(0)$ and $v(1)$ point to different halfplanes of the line which joins $u(0)$ and $u(1)$, i.e., iff

$$(**) \qquad \varepsilon\left[v(0), \int_0^1 v(t)\, dt\right] \geq 0 \geq \varepsilon\left[v(1), \int_0^1 v(t)\, dt\right] .$$

Proposition 3.1. *A plane curve $v(t)$ is the derived curve of a convex curve iff it is star-shaped for the origin and $(**)$ holds.*

For $n = 3$, the Hjelmslev condition says that the intersection of the tangent halfcone and any plane not through 0 is a piecewise strictly convex curve. If u is everywhere differentiable, the intersection curves are strictly convex.

Proposition 3.2. *If the rays carrying the vectors $v(t)$ form a strictly convex cone in R^3, then the integral curve u is of minimal order.*

Let us assume that a given plane intersects u at $u(t_i)$, $0 \leq t_1 < t_2 < t_3 \leq 1$. Without loss of generality we may assume that an eventual fourth point of intersection belongs to a value $T > t_3$ of the parameter. The four points $\boldsymbol{u}(t_i)$, $i = 1, 2, 3$, and $\boldsymbol{u}(T)$ are coplanar iff

$$0 = \begin{vmatrix} 1 & u_1(0) + \int_0^{t_1} v_1 dt & u_2(0) + \int_0^{t_1} v_2 dt & u_3(0) + \int_0^{t_1} v_3 dt \\ 1 & u_1(0) + \int_0^{t_2} v_1 dt & u_2(0) + \int_0^{t_2} v_2 dt & u_3(0) + \int_0^{t_2} v_3 dt \\ 1 & u_1(0) + \int_0^{t_3} v_1 dt & u_2(0) + \int_0^{t_3} v_2 dt & u_3(0) + \int_0^{t_3} v_3 dt \\ 1 & u_1(0) + \int_0^{T} v_1 dt & u_2(0) + \int_0^{T} v_2 dt & u_3(0) + \int_0^{T} v_3 dt \end{vmatrix}$$

$$= \left[\int_{t_1}^{t_2} \boldsymbol{v}(t)dt\,,\ \int_{t_2}^{t_3} \boldsymbol{v}(t)dt\,,\ \int_{t_3}^{T} \boldsymbol{v}(t)dt\right],$$

where the bracket indicates the determinant of the three vectors.

By the convexity condition, the first vector of the last determinant is in the halfspace spanned by $0, \boldsymbol{v}(t_1)$ and $\boldsymbol{v}(t_2)$, and this halfspace does not contain any vector $\boldsymbol{v}(t), t > t_2$. A similar remark holds for the second vector and the plane spanned by $0, \boldsymbol{v}(t_2)$ and $\boldsymbol{v}(t_3)$ so that its halfspace does not contain any vector $\boldsymbol{v}(t), t > t_3$. Hence, the vectors $\boldsymbol{v}(t)$, $t_3 < t < T$, are all in one halfspace of the plane spanned by 0, $\int_{t_1}^{t_2} \boldsymbol{v}dt$, $\int_{t_2}^{t_3} \boldsymbol{v}dt$, whose determinant cannot vanish so that $\boldsymbol{u}$ is of minimal order.

For higher dimensions, a satisfactory result is still missing. Since closed curves of minimal order are possible for $n = 2k$, some relation generalizing (**) is needed. An interesting corollary of Proposition 3.2 is

Remark 3.3. If $\int_{t_0}^{t} \boldsymbol{v}(\tau)d\tau$ is of minimal order in R^3, then $\int_{t_0}^{t} g(\tau)\boldsymbol{v}(\tau)d\tau$ is of minimal order for all continuous $g(t) > 0$. The author conjectures that this property remains true for all odd dimensions.

4. A number of additional theorems can be proved for $k = 1$. Some are restricted for the time being to $n = 2$ or 3, pending a solution of the converse problem of § 3 for higher dimensions.

Proposition 4.1. *Let $f(t)$, $g(t)$ be continuous, of bounded variation and periodic of period* 1, *$g(t) > 0$, and $n = 2$. If $\boldsymbol{v}(t)$ is continuous and star-shaped from the origin, and*

$$\int_0^1 f(t)\boldsymbol{v}(t)\,dt = \int_0^1 g(t)\boldsymbol{v}(t)dt = 0\ ,$$

then $f(t)/g(t)$ has at least four relative extrema in $[0, 1)$.

Here we put $u(t) = u_0 + \int_0^t g(\tau)v(\tau)d\tau$. Then $u(0) = u(1)$ and the hypotheses of Theorem 2 are satisfied for $k = 1$ and the function f/g.

For $n = 3$, the condition $\int_0^1 g(t)v(t)dt = 0$ is impossible for v on a strictly convex cone and $g(t) > 0$. Therefore, we must use the second alternative in Theorem 2. We see that

$$\int_0^1 f(t)v(t)dt = 0 \,, \qquad f(0) = f(1) = 0$$

implies that $f(t)/g(t)$ has at least $3 + 1 = 4$ relative extrema for arbitrary positive continuous $g(t)$. Hence, the extrema must have different signs:

Proposition 4.2. *If* $n = 3$, $f(0) = f(1) = 0$, $\int_0^1 f(t)v(t)dt = 0$, $f(t)$ *is continuous and of bounded variation, and* $v(t)$ *is the derived curve of a smooth curve of minimal order, then* $f(t)$ *changes sign at least four times in* $[0, 1)$.

The last result can also be obtained as a special case of

Proposition 4.3. *If* $v(t)$ *is the derived curve of a differentiable curve of minimal order,* $f(t)$ *is a continuous function of bounded variation, and periodic of period* 1, *and*

$$\int_0^1 f(t)v(t)dt = f(1)\int_0^1 v(t)dt \,,$$

then $f(t)$ *has at least* $n + 1$ *extrema in* $(0, 1)$.

In fact, the hypothesis implies $\int_0^1 u(t)df(t) = 0$, and he proof then proceeds as for Theorem 2.

5. Theorem 5. *If*

$$\int_0^1 f(t)u(t)dt = \bar{f}\int_0^1 u(t)dt \,, \qquad \bar{f} = \int_0^1 f(t)dt \,,$$

then $f(t)$ *is equal to its mean value* $\bar{f}$ *at least* $n + 1$ *times in* $(0, 1)$.

Here again the result can be improved by one for even n and periodic $f(t)$. The function $F(t) = \int_0^t f(\tau)d\tau - \bar{f}t$ is periodic, $F(0) = F(1) = 0$ and $\int_0^1 F(t)u'(t)dt = 0$ by integration by parts. By Theorem 2, $F'(t)$ changes sign at least $n + 1$ times in $(0, 1)$.

Remark 5.1. If $u(t)$ is of minimal order, so is every translate of u. The condition of Theorem 5 can be reworded to say that the conclusion holds if $\int_0^1 f(t)\hat{u}(t)dt = 0$ for that particular translate $\hat{u}$ of u for which $\int_0^1 \hat{u}(t)dt = 0$. In this form, special cases of Theorem 5 appear in the literature [2].

6. If $n = 2m$, a curve of minimal order can be closed. We are interested in curves which are point-symmetric with respect to the origin. Since an $(n-1)$-plane through the origin and $n - 1$ generic points on the curve contains other $n - 1$ points by symmetry, point-symmetric curves of minimal order exist only for $2(n - 1) \leq n$ or $n \leq 2$. A point-symmetric plane convex curve shall always be referred to a parameter σ, $0 \leq \sigma \leq 1$, which expresses the symmetry by

$$u\left(\sigma + \frac{1}{2}\right) = -u(\sigma) .$$

Proposition 6. *If*

$$\int_0^1 f(\sigma)u_1^{j_1}(\sigma)u_1^{j_2}(\sigma)d\sigma = 0 , \qquad 1 \leq j_1 + j_2 \leq k ,$$

for a periodic, continuous $f(\sigma)$ of bounded variation and a point-symmetric plane convex $u(\sigma)$, then the equation

$$f(\sigma) = f\left(\sigma + \frac{1}{2}\right)$$

is satisfied at least $k + 1$ times in $\left[0, \frac{1}{2}\right)$ for an even k and $k + 2$ times for an odd k.

Define $D(\sigma) = f(\sigma) - f\left(\sigma + \frac{1}{2}\right)$. Then the function $D(\sigma)$ satisfies the conditions of Theorem 1. In fact, if $D(\sigma) = 0$ identically, there is nothing to prove. Otherwise, we may assume $D(0) \neq 0$. Then by Remark 1.1, $D(\sigma)$ changes sign at least $2k + 2$ times in $(0, 1)$. Since the sign changes appear in pairs, there are at least $k + 1$ changes in $\left(0, \frac{1}{2}\right)$. But the number of sign changes must be odd, hence the result.

7. The algebraic curve $u = (t, t^2, \cdots, t^n)$ of order n in n-space is of minimal order. In fact, the determinant whose vanishing implies the linear dependence of $u(t)$ on $u(t_1), \cdots, u(t_n)$ is a van der Monde determinant and vanishes only for $t = t_i$, $1 \leq i \leq n$. Theorem 1 for this curve and $k = 1$ yields the theorem on moments quoted in the introduction.

The circle is a symmetric plane convex curve. The known theorems on

Fourier coefficients result from the fact that the monomials in $\cos t$ and $\sin t$ are linear functions of the trigonometric functions of multiples of the argument. The vanishing of the Fourier coefficients up to order k implies the hypothesis of Theorem 1 for $\boldsymbol{u} = (\cos 2\pi t, \sin 2\pi t)$, and that of orders 1 to k implies the hypothesis of Theorem 2 (see [2] for references and applications). The special case of Theorem 6 for the circle is contained in [4]; for $k = 1$ it is contained in Süss' proof of the assertion of Blaschke that a smooth convex plane curve admits three pairs of points with equal radii of curvature and parallel tangents.

The six vertex theorem of unimodular affine geometry [5] is a special case of Remark 1.1 for $k = 2$ and $n = 2$. Proposition 4.1 for the circle is a basic theorem of relative differential geometry [6], and it is also known for a closed convex curve $\boldsymbol{v}$ [6].

8. Let C be a closed convex curve, which is the union of a straight segment C_0 and a strictly convex and differentiable arc C_1, and let the origin 0 be a point in the interior of C. The couple $(C, 0)$ defines a norm in the plane for which the vectors $\overrightarrow{OP}$, $P \in C$, are the unit vectors. This norm satisfies the positivity condition and the triangle inequality but $\|\alpha \boldsymbol{x}\| = \alpha \|\boldsymbol{x}\|$ is true only for $\alpha \geq 0$.

The polar reciprocal of C for the euclidean unit circle of center 0 is rotated by $-\pi/2$. The resulting curve T is the isoperimetrix of the geometry [7], and is closed convex and differentiable except for one cusp at which the half-tangents make an angle supplementary to the angle of the rays from 0 to the endpoints of C_1. Therefore, T with a suitable parametrization can serve as a curve $\boldsymbol{u}$. We choose the cusp $\boldsymbol{u}_0$ as the point $t = 0$ on T and for the parameter of $\boldsymbol{u} \in T$ twice the area of the sector $\boldsymbol{u}_0 0 \boldsymbol{u}$ of T. It is easily checked [8] that $\boldsymbol{u}'(t) = \boldsymbol{v}(t)$ is a unit vector in the $(C, 0)$ norm, and the curve $\boldsymbol{v}$ is a parametrization of C_1 and satisfies $(*)$.

A differentiable curve $\boldsymbol{x}$ in the plane has a tangent image (under the Gauss map) by its unit tangent vectors. We call a curve *admissible* in this Minkowski geometry if its tangent image is in C_1. If an admissible curve is convex, it can be parametrized by t as parameter of its tangent directions. The Minkowski radius $R(t)$ of curvature of a curve $\boldsymbol{x}(t)$ is defined by

$$\frac{d\boldsymbol{x}}{dt} = R(t)\boldsymbol{v}(t)\ .$$

If the curve is closed, $\int_0^{2 \text{ area } T} R(t)\boldsymbol{v}(t)dt = 0$. Hence

Proposition 8. *In a Minkowski geometry whose isoperimetrix is rough at one point, the radius of curvature of an admissible closed convex curve has at least three extrema.*

9. Next, in euclidean n-space we consider curves referred to the arc length s as parameter, and assume that the curve admits a continuous moving frame

$\{e_i(s), 1 \leq i \leq n$, whose Frenet equations are (see [9])

$$e_i' = -k_{i-1}e_{i-1} + k_i e_{i+1}, \qquad k_0 = k_n = 0.$$

If in addition the curvatures are continuous, the curve is called a Frenet curve. By Theorem 2, we have

Proposition 9. *Given a Frenet arc in n-space with parallel tangents at its endpoints, if the curve $e_2(s)$ is of minimal order, then the first curvature $k_1(s)$ has at least $n+1$ relative extrema. If n is even and the curve is closed, the minimal number of extrema is $n+2$. The same statement holds for $k_{n-1}(s)$ if $e_n(s)$ is a closed curve of minimal order.*

10. Outside of geometry, our theorems have interesting applications to differential equations. For a Sturm-Liouville problem

$$y'' + \lambda p(t)y = 0, \qquad a \leq t \leq b,$$

with *positive* $p(t)$, let $\phi_i(t)$ be the eigenfunctions. Then the eigenfunction expansion of a continuous function $f(t)$ defined on $[a, b]$ is $f \sim \sum c_i\phi_i$. Since $p(t) > 0$, the acceleration of the point

$$u(t) = (\phi_1(t), \phi_2(t))$$

is always directed towards the origin. This means that the arc $u(t)$ is convex in a neighborhood of any of its interior points, and therefore is a simple closed curve completely in one of the closed halfplanes defined by the u_2-axis by the known properties of eigenfunctions [10, p. 395]. Hence $u(t)$ is convex (of minimal order).

Proposition 10.1. *Given a Sturm-Liouville operator on a finite interval with $p(t) > 0$. If the coefficients of indices $0, 1, 2$ of the eigenfunction expansion of a continuous function vanish, then the function changes sign at least three times in the interval of the definition of the operator.*

We leave for the reader the proof of the assertion that

$$f(a) = f(b) = \int_a^b f'(t)\phi_1(t)dt = \int_a^b f'(t)\phi_2(t)dt = 0$$

implies that $f(t)$ has at least four relative extrema in $[a, b]$.

Another type of result can be obtained for a Hill equation

$$y'' + Q(t)y = 0,$$

$$Q(t) \text{ periodic of period } \pi, Q(t) > 0 \text{ and continuous.}$$

We assume that the equation admits two coexisting periodic solutions and, in particular, that these solutions correspond to a collapsed *first* interval of instability [11, Chap. VII]. Then it is known that any solution of the equation satisfies $y(t+\pi) = -y(t)$, and hence the curve $\boldsymbol{u}(t) = (y_1(t), y_2(t))$ constructed on two linearly independent solutions of our equation is closed convex (by the argument leading to Prop. 10.1) and symmetric with respect to the origin so that $\boldsymbol{u}(t+\pi) = -\boldsymbol{u}(t)$. Then it is easily checked that $Q(t)$ and $\boldsymbol{u}(t)$ satisfy the hypotheses of the first case of Theorem 2 for $k = n = 2$, $0 \leq t \leq 2\pi$. Hence $Q(t)$ has at least six extrema in a double period. Since a periodic function must have an even number of extrema in one period, we obtain

Proposition 10.2. *If the first zone of instability of a Hill equation with continuous positive coefficient $Q(t)$ is reduced to a point, then the coefficient has at least four relative extrema in a half-open interval of periodicity.*

The same result could have been obtained from the six-vertex theorem of unimodular centro-affine differential geometry [6], but no similar statements hold for the higher zones of instability. An example is $Q(t) = (1 + a\cos t)^{-3}$, $|a| < 1$, which has only two extrema and for which the second zone of instability collapses.

The same method leads to the more general theorem:

Proposition 10.3. *If the differential equation*

$$x^{(2m)} + \sum_{i=1}^{m-1} c_i x^{(2i)} + \lambda Q(t)x = 0\,,$$

$$Q(t) > 0, Q(t+\pi) = Q(t), Q(t) \text{ continuous}, c_i \text{ constant},$$

admits a $2m$-fold eigenvalue of the Liapounoff boundary value problem

$$x(\pi) = -x(0)\,,$$

and if no hyperplane through the origin in $2m$-space intersects the curve

$$\boldsymbol{x}(t) = (x_1(t), \cdots, x_{2m}(t))$$

constructed on $2m$ linearly independent solutions $x_i(t)$ of that boundary value problem in more than $2(2m-1)$ points, then the function $Q(t)$ has at least $4m$ relative extrema in $[0, \pi)$.

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