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# SOME DECOMPOSITIONS OF THE SPACE OF SYMMETRIC TENSORS ON A RIEMANNIAN MANIFOLD

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## 1. Introduction

In this article we consider a *compact* $C^\infty$ manifold $M$ and endow it with a riemannian structure $g$. For such a riemannian manifold $(M, g)$, the space $A^p$ of exterior differential forms carries an elliptic operator and the de Rham laplacian $\Delta$, and has an orthogonal decomposition

$$A^p = \ker\Delta \oplus dA^{p-1} \oplus \delta A^{p+1}\,, \tag{1.1}$$

orthogonal with respect to the global scalar product on $A^p$. Moreover the dimension of $\ker\Delta$ is equal to the $p$-th Betti number of $M$, thanks to de Rham's theorem.

The simplest space of tensors to consider, besides the $A^p$'s, is the space $S^2$ of symmetric bilinear differential forms on $M$. It is natural to look for a decomposition of $S^2$ like (1, 1). In this article we give all the reasonable decompositions of $S^2$, which we are aware of. Unhappily we have no essential applications of them, because of the lack of some kind of a de Rham theorem, connecting topological invariants of $M$ with the dimension of the kernel of the elliptic operators considered on $S^2$. However we think it is worthwhile to list and prove these decompositions, hoping the reader will find interest in the problems and questions which naturally arise.

After fixing notations in §2 we give in §§3 and 4 the decomposition (3.1), which essentially yields the subspace $\delta^{-1}(0)$ of $S^2$ as the tangent space at $g$ of the space $\mathcal{M}/\mathcal{D}$ of classes of riemannian structures on $M$ under diffeomorphisms. In §5 we give two decompositions for manifolds of constant scalar curvature which are naturally associated to deformations of the scalar curvature. One of these is due to L. Nirenberg. §6 lists four elliptic operators on $S^2$; in §6.c an application is made to minimal surfaces; Corollary 6.2 was suggested to us by J. Simons. §§7 and 8 are concerned with Einstein manifolds. As an application of the decomposition (3.1) of §3, we find that the space of Einstein

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structures is locally finite-dimensional and we also get a result of nondeformability of some Einstein manifolds (Corollary 7.3 and Lemma 7.4).

## 2. Notations

On our fixed compact manifold $M$, we denote by $A^p$ $(p=0, 1, \cdots, \dim M=n)$ the $p$-th exterior power $\overset{p}{\wedge} T^*(M)$ of the cotangent bundle $T^*(M)$ of $M$. Hence $C^\infty(A^p)$, the space of all $C^\infty$-sections of $A^p$, is the space of exterior differential forms of degree $p$ (in particular $A^0$ is the space of $C^\infty$-functions on $M$). *When no confusion is possible*, we write $A^p$ for $C^\infty(A^p)$. Similarly $T^r_s$ represents the space of all tensors on $M$, which are $r$-times covariant and $s$-times contravariant, i.e., $T^r_s = [\overset{r}{\otimes} T(M)] \otimes [\overset{s}{\otimes} T^*(M)]$. *By* $S^2 \subset T^0_2$ *we understand the bundle* (or also $C^\infty(S^2)$) *of symmetric bilinear differential forms.*

We endow $M$ with a fixed riemannian structure $g$, making it a riemannian manifold $(M, g)$. On tensor spaces on $M$ we have the canonical scalar product (*point-wise*) $(\cdot \mid \cdot)$ and on their sections the *global* scalar product $\langle \cdot , \cdot \rangle = \int_M (\cdot \mid \cdot) v_g$, where $v_g$ is the canonical measure of $(M, g)$. If $P$ is a differential operator between some tensor bundles over $M$, its formal adjoint $P^*$ is uniquely defined by $\langle P \cdot , \cdot \rangle = \langle \cdot , P^* \cdot \rangle$.

**Examples.** The covariant derivative $\nabla : T^r_s \to T^r_{s+1}$, whose formal adjoint $\nabla^*$ will be also denoted by $\delta = \nabla^* : T^r_{s+1} \to T^r_s$. In local coordinates:

$$(\delta T)^{j_1 \cdots j_r}_{i_1 \cdots i_s} = - \sum_l \nabla^l T^{j_1 \cdots j_r}_{l i_1 \cdots i_s} \,. \tag{2.1}$$

For the restriction of $\delta$ to $S^2$: $\delta : S^2 \to A^1$, still denoted by $\delta$, the adjoint $\delta^*$ is given by the formula:

$$(\delta^* \xi)(x, y) = \frac{1}{2} [\nabla_x \xi(y) + \nabla_y \xi(x)] = \frac{1}{2} \mathscr{L}_{\xi'} g \,,$$

where $\mathscr{L}$ is the Lie derivative and $\xi'$ the vector field dual (by $g$) to the 1-form $\xi$.

On $S^2$ we have the *trace*: $\operatorname{tr} : S^2 \to A^0$ (with respect to $g$ understood), since $g$ defines canonically a map $S^2 \to T^1_1$ and $T^1_1$ is made up of endomorphisms (which have a cononical trace). We will set

$$TZ = \{h \in S^2 : \operatorname{tr} h = 0\} = \operatorname{tr}^{-1}(0) \subset S^2 \,.$$

**Example.** On $A^0$, we have the double covariant derivative, called the *hessian*: $\operatorname{Hess} = \nabla \circ \nabla : A^0 \to S^2$. Its trace is nothing but $-\Delta$, the usual Laplace-Beltrami operator on $A^0$: $\Delta = - \operatorname{tr} \circ \operatorname{Hess} : A^0 \to A^0$ (in local coordinates $\Delta f = - \sum_l \nabla^l \nabla_l f = - \sum_l \nabla^l d_l f$).

Finally, for our riemannian manifold $(M, g)$, we denote by $R$ the curvature tensor, $\rho$ the Ricci curvature ($\rho \in S^2$) and $\tau$ the scalar curvature ($\tau \in A^0$). The signs we are choosing are such that, for the standard sphere $(S^n, g_0)$:

$$R(x, y; x, y) = +1 \quad \text{for any orthonormal } x, y,$$

$$\rho(x, x) = n - 1 \quad \text{for any unit vector},$$

$$\tau = n(n-1)\,.$$

## 3. The first splitting of $S^2$

As is well known, the set $\mathscr{M}$ of riemannian metrics on $M$ is an open (in $C^\infty$ topology) convex positive cone in the linear space $C^\infty(S^2)$. One approach to a study of riemannian geometry is to consider certain distinguished subsets of $\mathscr{M}$; e.g., those metrics which make $M$ a space form, Einstein manifold, homogeneous space, etc.

The diffeomorphism group $\mathscr{D}$ of $M$ acts on $\mathscr{M}$ in a natural way by pull back. (The usual action on the sections of any tensor bundle.) We denote a metric $g$ acted on by $\eta$ as $\eta^*(g)$, and it is easy to see that if $(M, g)$ is a space form, Einstein manifold or homogeneous space, then so is $(M, \eta^*(g))$. This means that each of our distinguished subsets of $\mathscr{M}$ is a union of orbits of $\mathscr{D}$ in $\mathscr{M}$. Hence each one can also be looked at as a subset of the space of orbits $\mathscr{M}/\mathscr{D}$.

In [4], Ebin has analyzed the action of $\mathscr{D}$ on $\mathscr{M}$ and proved the following: Fix $g \in \mathscr{M}$ and let $0_g$ be the orbit of $\mathscr{D}$ containing $g$. Then there exist a neighborhood $U$ of $g$ in $0_g$ and a map $\chi\colon U \to \mathscr{D}$ such that if $\eta^*(g) \in U$, $(\chi(\eta^*(g)))^*(g) = \eta^*(g)$; i.e., $\chi$ is a local section of the map of $\mathscr{D}$ onto its orbit. Also there is a submanifold $S$ of $\mathscr{M}$ containing $g$ such that the map $F\colon U \times S \to \mathscr{M}$ defined by $F(u, s) = (\chi(u))^*(s)$ is a diffeomorphism of $U \times S$ onto a neighborhood of $g$ in $\mathscr{M}$.[1] Furthermore the tangent space of $S$ at $g$ is the kernel of the operator $\delta\colon S^2 \to T^*$.

The existence of $S$ and $F$ is helpful to the study of the local nature of the various distinguished subsets of $\mathscr{M}$. Let $\mathscr{E}$ be such a subset, and $g(t)$ a curve in $\mathscr{M}$, which is contained in $\mathscr{E}$ and such that $g(0) = g$. If $\pi_2\colon U \times S \to S$ is the projection on the second factor, then

$$\pi_2 \circ F^{-1}(g(t)) \quad \text{is a curve in } \mathscr{M}$$

(defined for all $t$ such that $g(t) \in F(U \times S)$). But $h(t) = \pi_2 \circ F^{-1}(g(t))$ and $g(t)$ are in the same orbit of $\mathscr{D}$ because if $F^{-1}(g(t)) = (u, s)$, then $h(t) = s$ and $g(t) = (\chi(u))^*(s)$. Hence $h(t)$ is also a curve in $\mathscr{E}$, so $h(t) \subseteq S \cap \mathscr{E}$. Thus any

[1] $\mathscr{M}$, $\mathscr{D}$, and $0_g$ are all locally Frechét spaces. Therefore we must understand the words "submanifold" and "diffeomorphism" in the sense of ILH-submanifold and diffeomorphism (see [7]).

deformation $g(t)$ of $g$ in the set $\mathscr{E}$ gives rise to a deformation $h(t)$ in $\mathscr{E} \cap S$, such that if $\pi: \mathscr{M} \to \mathscr{M}/\mathscr{D}$ is the natural projection, $\pi g(t) = \pi h(t)$. Since the tangent space of $S$ at $g$ is $\delta^{-1}(0)$, $\left.\frac{d}{dt}h(t)\right|_0 \in \delta^{-1}(0)$. Hence to study deformations in $\mathscr{M}/\mathscr{D}$ we need only study curves in $\mathscr{M}$ whose tangent at $g$ is in $\delta^{-1}(0)$. Here we shall prove not the existence of $S$ and $F$ but an infinitesimal version of their construction, that is, we shall show that

$$C^\infty(S^2) = \delta^{-1}(0) \oplus \delta^*(C^\infty(A^1)) , \tag{3.1}$$

and $\delta^*(C^\infty(A^1))$ is the tangent space of $0_g$ (or $U$) at $g$.

First we give a plausibility argument that $\delta^*(C^\infty(A^1))$ is the tangent space of $0_g$ at $g$ (for details see [4]). We let $\phi_g: \mathscr{D} \to \mathscr{M}$ by $\eta \to \eta^*(g)$, and shall show that at the identity of $\mathscr{D}$, $2\delta^*: C^\infty(A^1) \to C^\infty(S^2)$ is the tangent map of $\phi_g$.

If $\eta(t)$ is a curve in $\mathscr{D}$, with $\eta(0) = Id$, then for each $p \in M$, $\left.\frac{d}{dt}\eta(t)(p)\right|_0$ is an element of $T_p$. Hence the tangent space of $\mathscr{D}$ at $Id$ is the set of functions $\{V\}$ from $M$ to $T$ such that $V(p) \in T_p$. This set is just $C^\infty(T)$. For any element $V$ of $C^\infty(T)$ there is a unique one parameter group of diffeomorphisms $\eta(t)$ such that $\left.\frac{d}{dt}(\eta(t))\right|_0 = V$. Hence to evaluate the tangent map of $\phi_g$ at $V$, it is enough to know $\frac{d}{dt}((\eta(t))^*(g))$. But $\frac{d}{dt}(\eta^*(t)(g))$ is by definition $\mathscr{L}_V(g)$, the Lie derivative of $V$ with respect to $g$. Now using $g$ to identify $A^1$ and $T$ we find by (2.1) that this map is $2\delta^*$.

To prove $C^\infty(S^2) = \delta^{-1}(0) \oplus \delta^*(C^\infty(A^1))$, we must first derive some results on linear partial differential operators.

## 4. Differential operators with injective symbol

Let $E$ be a vector bundle over $M$ with a riemannian structure, i.e., with an inner product on each of the fibres of $E$, and fix a volume element $\mu$ on $M$. Then $C^\infty(E)$ gets an inner product and we call its completion $H^0(E)$. If $J^s(E)$ is the bundle[2] of $s$-jets of $E$, there is a natural map $j_s: C^\infty(E) \to C^\infty(J^s(E))$. Giving $J^s(E)$ a riemannian structure we get an inner product on $C^\infty(J^s(E))$. We define $\langle\ \ \rangle_s$ to be the inner product on $C^\infty(E)$ induced by $j_s$ and the inner product on $C^\infty(J^s(E))$. $H^s(E)$ is defined to be the completion of $C^\infty(E)$ in $\langle\ ,\ \rangle_s$.

Assume $F$ is another such vector bundle, and let $D: C^\infty(E) \to C^\infty(F)$ be a differential operator of order $k$. Then $D$ extends to a continuous linear map $D_s: H^s(E) \to H^{s-k}(F)$. Also $D^*: C^\infty(F) \to C^\infty(E)$ has order $k$, so it can be extended to a map $D^*_{s-k}: H^{s-k}(F) \to H^{s-2k}(E)$. We recall that for any $p \in M$ and

[2] For a definition of $J^s(E)$, $H^s(E)$, and details of this section, see [8].

any cotangent vector $t \in T^*_p$, there is a linear map $\sigma_t(D): E_p \to F_p$ called the symbol of $D$. Also $\sigma_t(D^*): F_p \to E_p$ is the adjoint of $\sigma_t(D)$ with respect to the inner products on $E_p$ and $F_p$. We say that the symbol of $D$ is injective, if $\sigma_t(D)$ is injective for all non-zero $t$.

The goal of this section is to prove:

**Theorem 4.1.**[3] *If $D$ has injective symbol, then*

$$H^{s-k}(F) = D_s(H^s(E)) \oplus (D^*_{s-k})^{-1}(0) ,$$

*and the summands are orthogonal with respect to the inner product $\langle\ ,\ \rangle$ of $H^0(F)$.*

**Corollary 4.2.** *$C^\infty(F) = D(C^\infty(E)) \oplus (D^*)^{-1}(0)$, the summands being orthogonal as above.*

To prove the above theorem we will need the notion of an elliptic operator: we say an operator $\alpha: C^\infty(E) \to C^\infty(F)$ of order $k$ is elliptic if $\sigma_t(\alpha): E_p \to F_p$ is an isomorphism for all non-zero $t$. A fundamental property of elliptic operators is:

**Regularity Theorem 4.3.** *If $\alpha$ is elliptic of order $k$, then*

$$H^{s-k}(F) = \alpha_s(H^s(E)) \oplus \ker \alpha^*_{s-k} ,$$

*and $C^\infty(F) = \alpha(C^\infty(E)) \oplus \ker \alpha^*$. Also if for $l > s - k$, $\alpha_s(x) \in H^l(F)$, then $x \in H^{l+k}(E)$,* and *if $\alpha_s(x) \in C^\infty(F)$, then $x \in C^\infty(E)$.*

*Proof.* [8, Chapter XI].

Also we need two lemmas.

**Lemma 4.4.** *If $D$ is an operator of order $k$ with injective symbol, then $D^*D$ is an elliptic operator of order $2k$.*

*Proof.* $\sigma_t(D^*D) = \sigma_t(D^*)\sigma_t(D) = (\sigma_t(D))^*\sigma_t(D)$, where the "$*$" on the right hand side means the adjoint of the operator $\sigma_t(D): E_p \to F_p$. $\sigma_t(D)$ injective implies $(\sigma_t(D))^*\sigma_t(D)$ is an isomorphism.

**Lemma 4.5.** *Let $X$ and $Y$ be Banach spaces, and $T: X \to Y$ a bounded linear map. Assume $C$, a closed subspace of $Y$, is an algebraic linear complement to $T(X)$. Then $T(X)$ is closed in $Y$, and $Y = T(X) \oplus C$ topologically.*

*Proof.* See [8, Proof of Theorem 1, p. 119].

Now we are ready to prove our theorem.

We first show that $(D_{s-k}D^*_s)^{-1}(0) = (D^*_s)^{-1}(0)$ and $(D^*D)^{-1}(0) = D^{-1}(0)$. Clearly $(D_{s-k}D^*_s)^{-1}(0) \supseteq (D^*_t)^{-1}(0)$. If $D_{s-k}D^*_s(x) = 0$, then $\langle x, D_{s-k}D^*_s x\rangle = \langle D^*_s x, D^*_s x\rangle = 0$ so $D^*_s(x) = 0$. Similarly $(D^*D)^{-1}(0) \supseteq D^{-1}(0)$, and $D^*Dx = 0$ implies $\langle x, D^*Dx\rangle = \langle Dx, Dx\rangle = 0$ so $Dx = 0$.

Secondly we show $D_s(H^s(E)) = D_sD^*_{s+k}(H^{s-k}(F))$. Clearly $D_s(H^s(E)) \supseteq D_sD^*_{s+k}(H^{s+k}(F))$. If $x = D_s y$, $y \in H^s(E)$, then $y = D^*_{s+k}D_s(a) + b$ by the

[3] This theorem is a special case of the statement involving equation (13) on page 447 of [13]. Since the proof of much easier in the special case, we include it.

regularity theorem, where $a \in H^{s+2k}(E)$ and $b \in (D^*D)^{-1}(0)$. But then by the above $b \in D^{-1}(0)$ so

$$x = D_s(y) = D_s D^*_{s+k} D_{s+2k}(a) .$$

By Lemma 4.5, in order to prove our theorem, we need only show $H^s(F) = D_s(H^s(E)) \oplus (D^*_{s-k})^{-1}(0)$ algebraically, since $(D^*_{s-k})^{-1}(0)$ is closed. $(D^*_{s-k})^{-1}(0) \cap D_s(H^s(E)) = \{0\}$ because if $D^*_{s-k}D_s x = 0$ then $\langle x, D^*_{s-k}D_s x\rangle = \langle D_s x, D_s x\rangle = 0$, so $D_s(x) = 0$.

Also

$$\begin{aligned} H^{s-k}(F) &= (D_{s-2k}D^*_{s-k})^{-1}(D_{s-2k}D^*_{s-k})(H^{s-k}(F)) \\ &= (D_{s-2k}D^*_{s-k})^{-1}(D_{s-2k}(H^{s-2k}(E))) \\ &= D_{s-k}(H^{s-k}(E)) \cap H^{s-k}(F) + (D_{s-2k}D^*_{s-k})^{-1}(0) . \end{aligned}$$

But $D_{s-2k}(x) \in H^{s-k}(F)$ implies $D^*_{s-3k}D_{s-2k}(x) \in H^{s-2k}(E)$, which by regularity implies that $x \in H^s(E)$. Also $(D_{s-2k}D^*_{s-k})^{-1}(0) = (D^*_{s-k})^{-1}(0)$ as we have shown above, so $H^{s-k}(F) = D_s(H^s(E)) \oplus (D^*_{s-k})^{-1}(0)$. The summands are orthogonal by the fact that $\langle D_s x, y\rangle = \langle y, D^*_s x\rangle$.

*Proof of Corollary.* First we show that $C^\infty(F) = D(C^\infty(E)) \oplus D^{-1}(0)$ algebraically. From the theorem we know that the two summands have zero intersection, so we need only show that they span. Given $f \in C^\infty(F)$ we know $f = D_s(e) + h$, $e \in H^s(E)$, $h \in (D^*_{s-k})^{-1}(0)$. Therefore $D^*f = D^*_{s-k}D_s(e)$. But $D^*f \in C^\infty(E)$, so by regularity $e \in C^\infty(E)$ and hence $h = f - D(e) \in C^\infty(F)$.

Now we show the sum is topological. In the proof of the theorem we showed that

$$D(H^s(E)) \cap H^t(F) = D(H^{t+k}(E))$$

for any $t \geq s - k$. But $C^\infty(F) = \bigcap_{t \geq s} H^t(F)$, so $D(C^\infty(E)) = C^\infty(F) \cap D_s(H^s(E))$, and $D(C^\infty(E))$ is closed in $C^\infty(F)$. The other summand is also closed, so the sum must be topological.

Now to prove $C^\infty(S^2) = \delta^*(C^\infty(A')) \oplus \delta^{-1}(0)$, we need only look at the symbol of $\delta^*$. A direct calculation shows $\sigma_t(\delta^*)(\xi) = \dfrac{1}{2}(\xi \oplus t + t \oplus \xi)$, which is clearly injective for non-zero $t$.

## 5. Deformations of scalar curvature

Another splitting of $C^\infty(S^2)$ was kindly communicated to us by L. Nirenberg and is based on a conversation between him and L. Fadeev. To define it we need the following two operators: $\alpha\colon A^1 \to S^2$, defined by $\alpha(\xi) = \delta^*\xi + (\delta\xi)g$, and $\beta\colon A^0 \to S^2$, defined by $\beta(f) = \mathrm{Hess}(f) - f\rho + \left(\dfrac{1}{n-1}\right)f\tau g$.

$\sigma_t(\delta)(\xi) = (t, \xi)$, so $\sigma_t(\alpha)(\xi) = \frac{1}{2}(t \otimes \xi + \xi \otimes t) + (t, \xi)g$. It is easy to check that this is injective. $\beta$ is a second order operator, and $\sigma_t(\beta)(f) = \sigma_t(\mathrm{Hess})(f) = f(t \otimes t)$, which is also injective. Hence $C^\infty(S^2) = \alpha(C^\infty(A^1)) \oplus \alpha^{-1}(0)$ and $C^\infty(S^2) = \beta(C^\infty(A^1)) \oplus \beta^{-1}(0)$.

**Lemma 5.1.** *If $(M, g)$ has constant scalar curvature $\tau$, then $\alpha(A^1)$ and $\beta(A^0)$ are orthogonal with respect to the standard inner product $\langle\ ,\ \rangle$ on $C^\infty(S^2T^*)$.*

**Theorem 5.2.** *If $(M, g)$ has constant scallar curvature, then*

$$C^\infty(S^2) = \alpha(C^\infty(A^1)) \oplus \beta(C^\infty(A^0)) \oplus \alpha^{*-1}(0) \cap \beta^{*-1}(0) ,$$

*where the summands are orthogonl.*

*Proof.* The theorem follows immediately from the lemma and the two decompositions of $C^\infty(S^2)$ above.

To prove Lemma 5.1 we must introduce a second order operator $\gamma : S^2 \to A^0$. $\gamma$ is defined as follows: given $h \in S^2$, let $g(t)$ be a curve in $\mathscr{M}$ such that $g(0) = g$ and $\left.\frac{d}{dt}g(t)\right|_0 = h$. Let $\tau(t)$ be the scalar curvature of the metric $g(t)$ and define $\gamma(h)$ to be $\left.\frac{d}{dt}\tau(t)\right|_0$. From [2, (3.4)] we find

$$\gamma(h) = \Delta\,\mathrm{tr}\,(h) + \delta\delta(h) - (h, \rho) . \tag{5.3}$$

Since $\delta^*(\xi) = \frac{1}{2}\mathscr{L}_{\xi'}(g)$ for $\xi \in A^1$, it is clear that $\gamma \circ \delta(\xi) = \frac{1}{2}\mathscr{L}_{\xi'}(\tau)$, so in particular if $\tau$ is constant $\gamma \circ \delta^* = 0$.

It is easy to check that the symbol of $\gamma$ is surjective, so $\sigma_t(\gamma^*)$ must be injective. Hence we get an orthogonal splitting $C^\infty(S^2) = \gamma^*(A^0) \oplus \gamma^{-1}(0)$. Since $\gamma \circ \delta^* = 0$, $\gamma^*(A^0)$ and $\delta^*(A^1)$ orthogonal, so that we can get a finer splitting:

$$S^2 = \gamma^*(A^0) \oplus \delta^*(A') \oplus \delta^{-1}(0) \cap \gamma^{-1}(0) . \tag{5.4}$$

This is very similar to the splitting of Theorem 5.2, to which we now return. We prove Lemma 5.1.

We need only show $\beta(A^0) \subseteq \alpha^{*-1}(0)$, for $\alpha^{*-1}(0)$ is orthogonal to $\alpha(A^1)$. First we remark that from (5.3) we easily get

$$\gamma^*(f) = (\Delta f)g - f\rho + \delta^* df . \tag{5.5}$$

Also since $\gamma \circ \delta^* = 0$, $\delta \circ \gamma^* = 0$ so that

$$\delta((\Delta f)g) - \delta(f\rho) + \delta\delta^* df = 0. \tag{5.6}$$

To prove the lemma we must show $\alpha^* \circ \beta = 0$, and from the formulas for $\alpha$ and $\beta$,

$$\alpha^* \circ \beta(f) = (\delta + d \circ tr)\left(\mathrm{Hess}(f) - f\rho + \frac{1}{n-1} f\tau g\right)$$

(5.7)
$$= \delta\delta^* d(f) - \delta(\rho f) + \left(\frac{1}{n-1}\right)\tau\delta(fg) - d\Delta f$$
$$- \tau df + \left(\frac{1}{n-1}\right)\tau df\,.$$

But,

(5.8)
$$\delta(fg) = -\,df\,,$$

so

(5.9)
$$\alpha^* \circ \beta(f) = \delta\delta^* d(f) - \delta(\rho f) - d\Delta f + \left(\frac{-1}{n-1} - 1 + \frac{n}{n-1}\right)\tau df\,.$$

Using (5.6) and (5.8), we find that $\delta\delta^* d(f) - \delta(\rho f) - d\Delta f = 0$, and the last term on the right hand side is also zero, so the lemma is proved.

**Remark 5.10.** S. Deser discusses decompositions like (3.1), (5.2) and (5.4) in some articles on general relativity for the purpose of finding a canonical form for the gravitational field. See [11, pp. 158-162] and [12].

**Remark 5.11.** C. Barbance [1] proved our original splitting $S^2 = \delta^{-1}(0) + \delta^*(A^1)$ in the case where $M$ is an Einstein manifold. She also gave a more refined splitting for this case.

**Remark 5.12.** E. Calabi [3] has investigated the operator $\delta^*: A^1 \longrightarrow S^2$ in the case where $M$ has constant curvature. He formed a sequence of differential operators

$$(\delta^*)^{-1}(0) \longrightarrow A^1 \xrightarrow{\delta^*} S^2 \xrightarrow{D_1} S^2(A^2) \xrightarrow{D_2} B_3 \xrightarrow{D_3} \cdots \xrightarrow{D_{n-1}} B_n \longrightarrow 0\,,$$

which is locally exact and resolves the sheaf defined by $(\delta^*)^{-1}(0)$. $D_1$ is defined just as $\gamma$ is, except that the full curvature tensor replaces the scalar curvature. $S^2(A^2)$ means the symmetric tensor of $A^2$ with itself, and the $\{B_i\}$ are certain subbundles of $A^2 \otimes A^i$.

## 6. Four elliptic operators on $S^2$

**a. The rough laplacian $\bar{\Delta}$.** For any $r, s$ it is easy to check the covariant derivative $\nabla: T^r_s \to T^r_{s+1}$ has injective symbol. Hence by Lemma 4.4 the operator $\bar{\Delta} = \nabla^*\nabla = \delta\nabla: T^r_s \to T^r_s$ is an elliptic operator on $T^r_s$, called the *rough laplacian*. In local coordinates:

$$(\bar{\Delta} t)^{j_1 \cdots j_r}_{i_1 \cdots i_s} = -\sum_l \nabla^l \nabla_l t^{j_1 \cdots j_r}_{i_1 \cdots i_s}\,.$$

Moreover $\bar{\Delta}$ is non-negative: $\langle \bar{\Delta} t, t\rangle = \| \Delta t \|^2 \geq 0$, and so $\ker \bar{\Delta} = \ker \Delta$. For $T_0^0 = A^0$ the rough laplacian $\bar{\Delta}$ is nothing but $\Delta$, the usual Laplace-Beltrami operator for functions on $M$.

Note also that $\bar{\Delta} S^2 \subset S^2$, so we can consider $\bar{\Delta}: S^2 \to S^2$ as an operator on $S^2$ (using the same notation $\bar{\Delta}$). From $\nabla g = 0$:

$$\bar{\Delta} g = 0 . \tag{6.1}$$

It is also elementary that:

$$\ker \nabla = \boldsymbol{R}\cdot g \text{ iff } (M, g) \text{ locally irreducible} .$$

Using the trace $tr: S^2 \to A^0$, from (6.1) one deduces:

$$tr\cdot \bar{\Delta} = \Delta \circ tr . \tag{6.2}$$

As a consequence of (6.2), we can consider $\bar{\Delta}$ as an operator $\bar{\Delta}: TZ \to TZ$.

**b. The operator $\Theta$.** Define $\square: S^2 \to S^2 \otimes A^1$ by

$$(\square h)(x, y; z) = \nabla_z h(x, y) - \nabla_x h(z, y) - \nabla_y h(z, x) , \tag{6.3}$$

and consider $(\square, \sqrt{2}\,\delta): S^2 \to (S^2 \otimes A^1) \times A^1$. It is straightforward to check this operator has injective symbol. Hence if we set $\Theta = (\square, \sqrt{2}\,\delta)^*(\square, \sqrt{2}\,\delta) = \square^*\square + 2\delta^*\delta$, the so-gotten operator $\Theta$ is elliptic non-negative (Lemma 4.4): $\langle \Theta h, h)\rangle = \| \square h \|^2 + 2 \| \delta h \|^2 \geq 0$ and $\ker \Theta = \ker \square \cap \ker \delta$. In fact: $\ker \square = \ker \nabla = \ker \bar{\Delta}$; for by (6.3), $\square h = 0$ implies that $\nabla h$ is antisymmetric in the two last entries, and being symmetric in the two first entries it has to be zero.

To get an explicit formula for $\Theta$, we perform explicit computations (straightforward ones with the Ricci commutation formulas) in local coordinates:

$$(\square^* k)_{ab} = \sum_l (\nabla^l k_{lab} + \nabla^l k_{lba} - \nabla^l k_{abl}) .$$

Therefore

$$\Theta = 3\bar{\Delta} + K , \tag{6.4}$$

where $K: S^2 \to S^2$ is the zero-order differential operator defined explicitly by

$$(Kh)_{ab} = \sum_l (\rho_{al} h^l{}_b + \rho_{bl} h^l{}_a) - 2 \sum_{l,m} R_{albm} h^{lm} . \tag{6.5}$$

Direct verification yields

$$K\cdot g = 0 , \quad \text{so} \quad \Theta \cdot g = 0 ,$$

$$\operatorname{tr} \circ K = 0 . \tag{6.6}$$

Hence by (6.2),

(6.7) $$\operatorname{tr} \circ \Theta = \varDelta \circ \operatorname{tr} ,$$

so that $\Theta$ also preserves $TZ$ and can be considered as an operator $\theta\colon TZ \to TZ$. On $TZ$ the operator $K$ is intimately related to the sectional curvature of $(M, g)$ as follows:

**Proposition 6.1.** *The operator K is positive definite on TZ if $(M, g)$ is of strictly positive sectional curvature.*

*Proof.* At any point $m \in M$, diagonalize $h$ with respect to $g$, using an orthonormal basis $\{e_a\}$. Letting the sectional curvature of the plane of $T_m(M)$ generated by $e_a$ and $e_b$ be $\sigma(e_a, e_b)$ and expressing $\sigma(e_a, e_b)$ in terms of $R$, by a direct computation from (6.5) we get

$$(Kh \mid h) = \sum_{a \neq b} \sigma(e_a, e_b)(h_{aa} - h_{bb})^2 .$$

Hence $(Kh \mid h) \geq 0$, and $(Kh \mid h) = 0$ implies that all the $h_{aa}$'s are equal; but $\operatorname{tr} h = \sum_a h_{aa}$, so all the $h_{aa}$'s have to be zero.

**c. The operator $\varPsi$.** This operator is the one associated to a differential system introduced by J. Simons in [9, pp. 96–97]. Define $\sigma\colon S^2 \to A^2 \otimes A^1$ by $(\sigma h)(x, y; z) = \nabla_x h(y, z) - \nabla_y h(x, z)$ and $(\sigma, \sqrt{2}\,\delta)\colon S^2 \to (A^2 \otimes A^1) \times A^1$. One can check it has injective symbol, so $\varPsi = (\sigma, \sqrt{2}\,\delta)^*(\sigma, \sqrt{2}\,\delta) = \sigma^*\sigma + 2\delta^*\delta$ is elliptic, non-negative: $\langle \varPsi h, h \rangle = \|\sigma h\|^2 + 2\|\delta h\|^2 \geq 0$ and $\ker \varPsi = \ker \sigma \cap \ker \delta$. Direct computations yield

$$(\sigma^* k)_{ab} = -\sum_l (\nabla^l k_{lab} + \nabla^l k_{lba}) ,$$

$$\varPsi = 2\bar{\varDelta} + K .$$

By (6.7) we still have $\operatorname{tr} \circ \varPsi = \varDelta \circ \operatorname{tr}$ and a restricted operator $\varPsi\colon TZ \to TZ$.

In [9] J. Simons introduced the system $\delta h = \sigma h = 0$, because if $(M, g) \subset (S^{n+1}, g_0)$ is a minimal hypersurface of the standard sphere $(S^{n+1}, g_0)$, then its second fundamental form $h$ satisfies $\delta h = \sigma h = 0$ (in fact $\sigma h = 0$ and $\operatorname{tr} h = 0$, which implies immediately $\sigma h = 0$). Combining this with Proportion 6.1, we get:

**Corollary 6.2.** *If $(M, g) \subset (S^{n+1}, g_0)$ is a compact minimal hypersurface, and $(M, g)$ is of strictly positive sectional curvature, then $(M, g)$ has to be an equator of $(S^{n+1}, g_0)$.*

Note this is a best possible result, since the flat standard square two-dimensional torus has a minimal imbedding in $(S^3, g_0)$.

**d. The operator $\varDelta$.** This operator was introduced by Lichnerowicz in [6, p. 27]. It is defined in an explicit way as $\varDelta = \bar{\varDelta} + K$. If we write out $\varDelta$ in local coordinates, the formula has the property that, when one replaces $h \in S^2$ by $\alpha \in A^2$ then $\varDelta h$ becomes $\varDelta \alpha$, the usual laplacian of de Rham on exterior forms. We do not known if $\varDelta$, which is of course elliptic, is still non-negative. We again have $\operatorname{tr} \circ \varDelta = \varDelta \circ \operatorname{tr}$ (see [6, last half of p. 27]).

**e. Orthogonal decompositions.** By the general theorem on elliptic operators (4.3), we have the orthogonal decompositions: $S^2 = \ker \nabla \oplus \bar{\Delta}(S^2)$, $S^2 = \ker \nabla \oplus \Theta(S^2)$, $S^2 = \ker \Psi \oplus \Psi(S^2)$, $S^2 = \ker \Delta \oplus \Delta(S^2)$. But, contrary to the case of the laplacian on exterior forms, the sums $\Theta(S^2) = \delta^* A^1 + \square^*(S^2 \otimes A^1)$, $\Psi(S^2) = \delta^* A^1 + \sigma^*(A^2 \otimes A^1)$ are neither direct nor orthogonal, for $\delta \circ \sigma^* \neq 0$ and $\delta \circ \square^* \neq 0$.

## 7. Deformations of Einstein manifolds

An *Einstein manifold* is a riemannian manifold $(M, g)$ for which the Ricci curvature satisfies $\rho = k \cdot g$ (for some real number $k$).

By a *deformation of Einstein structures through g* we mean a smooth curve $g(t)$ ($t$ running through some open interval containing 0) in $\mathcal{M}$ with $g(0) = g$ and such that, for all $t$, there exists $k(t)$, a real number, with $\rho_{g(t)} = k(t) \cdot g(t)$. If $k(0) \neq 0$, by normalization we can replace our deformation by another one such that $\rho_{g(t)} = \varepsilon \cdot g(t), \varepsilon = \pm 1$. In the following we assume this is done. In the case $k(0) = 0$, we do not know if such a deformation exists with $k(t)$ not identically 0; it is an interesting problem. We now compute consequences of the equation $\rho_{g(t)} = k(t) \cdot g(t)$. The formula giving $d\rho_g(t)/dt$ is classical; we write it down as in [2, Formula (3.3) for $t = 0$], all invariants (like $\delta, \delta_*, \Delta$, Hess) being understood to be with respect to $g = g(0)$:

$$\frac{d\rho_{g(t)}}{dt}(0) = \frac{1}{2}(\Delta h + 2\delta^* \delta h - \mathrm{Hess}\,(\mathrm{tr}\, h)),\ h = \frac{dg(t)}{dt}(0)\,.$$

From § 3, we know we can assume $\delta h = 0$, so

$$\frac{d\rho_{g(t)}}{dt}(0) = \frac{1}{2}(\Delta h - \mathrm{Hess}\,(\mathrm{tr}\, h))\,.$$

**Lemma 7.1.** *The function* $\mathrm{tr}\, h$ *is necessarily constant.*

The proof depends on whether $k(0)$ is $1, -1$, or 0.

a. $k(0) = 0$. By hypothesis we have $\Delta h - \mathrm{Hess}\,(\mathrm{tr}\, h) = 2k'(0) \cdot g + 2k(0)h = 2k'(0) \cdot g$. Taking the trace of both sides and using 6$d$ and $\Delta = -\mathrm{tr} \circ \mathrm{Hess}$ we get $\Delta(\mathrm{tr}\, h) = nk'(0)$ which implies $k'(0) = 0$ and $\mathrm{tr}\, h$ is constant.

b. $k(0) = -1$. In this case, $\rho_{g(t)} = -g(t)$ for all $t$ so that taking traces again: $\Delta(\mathrm{tr}\, h) = -\mathrm{tr}\, h$, which implies $\mathrm{tr}\, h = 0$.

c. $k(0) = 1$. Now we have $\Delta(\mathrm{tr}\, h) = \mathrm{tr}\, h$. We apply [5, p. 135] to the effect that $\lambda_1$, the first eigenvalue of $\Delta$ on $A^0$, is $\geq \dfrac{n}{n-1} \cdot \alpha$, where $\alpha$ is a lower bound for $\rho$, i.e., $\rho \geq \alpha \cdot g$. But here one can take $\alpha = 1$, so $\lambda_1 \geq \dfrac{n}{n-1} > 1$, and hence $\mathrm{tr}\, h = 0$.

From the lemma we nave now the standard equation for $h = \frac{dg(t)}{dt}(0)$ of a deformation of Einstein structures:

$$\Delta h = \varepsilon \cdot h \ , \qquad \varepsilon = 1, -1, 0 \colon \rho = \varepsilon \cdot g \ . \tag{7.1}$$

Since $\Delta$ is elliptic, it has finite dimensional kernel, so from § 3 we get

**Corollary 7.2.** *The subset $\pi(\xi)$ of Einstein structures in $\mathcal{M}/\mathcal{D}$ is finite dimensional.*

**Remark 1.** We can define a solution of (7.1) as an *infinitesimal deformation of Einstein structures.*

**Remark 2.** We proved, precisely, that the tangent direction $h$ has to belong to $\ker(\Delta - \varepsilon)$. The converse might be true: if $h \in \ker(\Delta - \varepsilon)$, then there exists a deformation of Einstein structures with $h$ as tangent direction. This existence theorem extending an infinitesimal deformation into a local one seems to be out of the reach of the present results of non-linear analysis.

Using the definition $\Delta = \bar{\Delta} + K$, the definition of $K$ in (6.5), and the fact that $\rho = \varepsilon \cdot g$ we can write (7.1) more explicitly:

$$\Delta - \varepsilon = \bar{\Delta} + L \ , \tag{7.2}$$

where $L$ is defined explicitly by

$$(Lh)_{ab} = - \sum_{l,m} R_{albm} h^{lm} \ . \tag{7.3}$$

In the case $\varepsilon = -1$ or $\varepsilon = 0$ there exist deformations of Einstein structures, for example the flat riemannian structures on tori, or the families of constant negative curvature on a surface of genus greater than one. In the case $\varepsilon = +1$, we have, by the above:

**Corollary 7.3.** *If L is positive-definite on TZ, then $(M, g)$ is not infinitesimally deformable,* i.e., $\dim_g \mathscr{E} = 0$.

This happens, for example, in the following:

**Lemma 7.4.** *If the sectional curvature of $(M, g)$ ranges in the interval $\left]\frac{p-1}{p}, 1\right]$ for $n = 2p$ or in the interval $\left]\frac{2p^2-1}{2p(p+1)}, 1\right]$ for $n = 2p+1$, then L is positive-definite on TZ.*

*Proof.* This lemma is very elementary. At a given point, we diagonalize $h$ with respect to $g$, using an orthonormal basis $\langle e_a \rangle$; set $h_{aa} = x_a$, $\sigma(e_a, e_b) = \alpha_{ab}$. Then

$$\begin{aligned} \frac{1}{2}(Lh \mid h) &= - \sum_{a \neq b} \alpha_{ab} x_a x_b = \Big( \sum_a x_a \Big)^2 - \sum_{a \neq b} \alpha_{ab} x_a x_b \\ &= \sum_a x_a^2 + \sum_{a \neq b} (1 - \alpha_{ab}) x_a x_b \ . \end{aligned}$$

Separate the $x_a$'s in $\geq 0$ and $\leq 0$: $y_i \geq 0$ $(i = 1, \cdots, p)$, $z_j \leq 0$ $(j = 1, \cdots, q)$ and set $A = \sum_i y_i = -\left(\sum_j z_j\right)$. We have also $1 - \alpha_{ab} \geq \varepsilon \geq 0$, $\varepsilon$ to be found later. But

$$\frac{1}{2}(Lh|h) \geq \sum_i y_i^2 + \sum_j z_j^2 + \sum_{i,j} \varepsilon y_i z_j$$
$$= \sum_i y_i^2 + \sum_j z_j^2 + 2\varepsilon\left(\sum_i y_i\right)\left(\sum_j z_j\right) \geq \frac{A^2}{p} + \frac{A^2}{q} - 2\varepsilon A^2,$$

which is positive if $\varepsilon \leq \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)$. Taking the minimum value for this, i.e., $p = q = \frac{n}{2}$ if $n = 2p$, and $q = p + 1$ if $n = 2p + 1$, the lemma is proved.

**Remark.** In the interval $\left]\frac{n-2}{n-1}, 1\right]$, see [2, Proposition (6.4)].

## 8. Corrections and addition to [2]

We wish to take the opportunity, writing on Einstein manifolds, to correct some mistakes made by Berger in [2].

**a.** In formula (3.2), p. 38, read at the end a "+". In formula (3.3) put a "−" in front of $\mathscr{D}dh$, in (3.4) "+" in front of $\delta\delta h$.

**b.** The number $i(\gamma)$ defined on the last line of p. 39 makes sense only when $n/2$, and so does everything which follows in § 4 of [2].

**c.** The end of p. 40 is incorrect, in the sense one should add everywhere the condition "$\tau$ is nowhere zero". In fact the best way to get Einstein structures as critical is to take $i(\gamma) = \int_M \tau_\gamma \cdot v_\gamma$ under the normalization $\int_M \tau_\gamma = 1$.

**d.** The first remark of p. 54 (and so the Proposition (8.1)) is simply proved by Bochner's theorem [14, Theorem 2.9., p.37]. If $(M, g)$ is flat and compact, then it has [10, Theorem 3.3.1, p. 105] a covering $(\tilde{M}, \tilde{g})$ which is a flat torus. But

**Addition 8.1.** *On a torus, there are no Einstein structures with* $\varepsilon = 1$ (i.e., $\rho = g$), *and any Einstein structure with* $\rho = 0$ *is necessarily flat.*

*Proof.* By the quoted result of Bochner, every harmonic 1-form has zero covariant derivative if $\rho = 0$, and is zero if $\rho > 0$. By the de Rham theorem and the Hodge-de Rham theorem there are $n(= \dim M =$ first Betti number of $M)$ such linearly independent such harmonic 1-forms; hence our manifold carries $n$ linearly independent 1-forms with zero covariant derivative, so it has to be flat.

**Remark.** It would be interesting to decide whether or not a torus can carry an Einstein structure with $\varepsilon = -1$, i.e., $\rho = -g$ (this question is a particular case of the problem which arose at the beginning of § 5).

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