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NONABELIAN SPENCER COHOMOLOGY AND DEFORMATION THEORY

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In memory of the author's father

Introduction

This work is a continuation of a preceding paper [9] modulo some change of notation. In particular, Chapter I on Lie groupoids is a development of the corresponding chapter of [9]. However, this work has its own interest in introducing the formalism of Nonabelian Spencer Cohomology.

Let V be a compact manifold, and Γ a transitive continuous pseudogroup on V (Definition 1.1, Chap. II). For every large integer k, one defines a fiber bundle of homogenuous spaces, i.e, a fiber bundle whose fiber is a homogenuous space G/H, and denotes this fiber bundle by $C_k(\Gamma)$ (Proposition 2.2, Chap. II). We intend to prove:

(1) There is an involutive differential system S_1 of order 1 in $C_k(\Gamma)$ such that every family of deformations of (V, Γ) (Definition 3.1, Chap. II) induces a family of sections in $C_k(\Gamma)$, which are solutions of the differential system S_1.

(2) In the case where Γ is analytic and elliptic (Definition 1.2, Chap. II), every family of sections in $C_k(\Gamma)$, which are solutions of the differential system S_1, defines inversely a family of deformations of (V, Γ).

The last result is based essentially on the Malgrange-Newlander-Nirenberg theorem (Theorem 2.2, Chap. II), the proof of which not to be given here follows from an argument to be published by B. Malgrange, reproving in particular the well-known theorem of A. Newlander and L. Nirenberg on the "integrability" of almost complex structures.

However, we completely reformulate in our formalism an argument of M. Kuranishi [7] proving the existence of an analytic space K of finite dimension, which is a "locally universal space of deformations" for elliptic pseudogroups (Theorem 4.1, Chap. II). This space was known to M. Kuranishi in the case of complex analytic structures.

This work was developed during a seminar held in 1966–67 under the direction of D. C. Spencer at Stanford University. The central ideas were

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originally expressed in Professor Spencer's fundamental paper on deformation theory [13].

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CHAPTER I

LIE GROUPOID

1. Lie groupoid and associated bundle

Definition 1.1. A set Φ is a groupoid on V, if there are defined a map

$$(a, b): \Phi \to V \times V$$

and an internal law of composition in Φ, which is partial, associative and such that

(1) $z, z' \in \Phi$, $z \cdot z'$ is defined iff $a(z) = b(z')$; and we have

$$a(z \cdot z') = a(z), \qquad b(z \cdot z') = b(z) \ ,$$

(2) $\forall x \in V, \exists \hat{x} \in \Phi, a(\hat{x}) = b(\hat{x}) = x$, and

$$z \cdot \hat{x} = z, \qquad \hat{x} \cdot z' = z' \qquad \text{for all } z, z'$$

such that $a(z) = x$, $b(z') = x$,

(3) $\forall z \in \Phi, \exists z^{-1}, z \cdot z^{-1} = \hat{x}, x = b(z); z^{-1} \cdot z = \hat{y}, y = a(z)$.

The maps a and b are called, respectively, source and target of Φ. For all $x \in V$, the element $\hat{x}$ is clearly unique; $\hat{x}$ is the unity at x of Φ. It may be verified that the set of elements z of Φ, whose source and target are the same element x in V, i.e., $a(z) = b(z) = x$, forms a group G_x, the isotropy-group of Φ at x. If z_0 is an element of Φ whose source is x and target is y, the following map is an isomorphism of groups:

$$z_0 : G_x \to G_y$$
$$z_0 \mapsto z_0 \cdot z \cdot z_0^{-1} \ .$$

The groupoid Φ is *transitive* if the map (a, b) is subjective.

Definition 1.2. Let Φ be a groupoid on V. Φ is a differentiable groupoid if there are on Φ and V structures of (differentiable) manifold[1] such that

(1) the map (a, b) is differentiable,

[1] We consider only paracompact manifolds; differentiable means C^∞-differentiable.

(2) the "inverse" map:

$$\begin{aligned} \Phi &\to \Phi \\ z &\mapsto z^{-1} \end{aligned}$$

is differentiable,

(3) for all couple of differentiable maps f, g from any manifold W to Φ, such that $a \circ f = b \circ g$, the well-defined map

$$\begin{aligned} f \cdot g \colon W &\to \Phi \\ x &\mapsto f(x) \cdot g(x) \end{aligned}$$

is differentiable.

Following Matsushima, we shall define a *Lie groupoid* as a differentiable groupoid whose map (a, b) is a submersion, i.e., a map which is subjective and everywhere is of maximal rank. So a Lie groupoid is transitive.

Proposition 1.1. *If Φ is a Lie groupoid on V, the following statements apply:*

(1) *the isotropy-groups of Φ are isomorphic Lie groups,*

(2) *$\Phi_x = \{z \in \Phi, a(z) = x\}$ is a principal fiber bundle*[2] *on V, fibered by the target map b, whose structural group is the isotropy group G_x.*

Proof.

(i) As the map (a, b) is a submersion by the lemma of Thom, the isotropy groups of Φ are closed submanifolds of Φ, and, by conditions (2) and (3) of the above definition, their algebraic structure is compatible with their manifold structure. Consequently, they are Lie groups which are isomorphic as a Lie groupoid is transitive.

(ii) In the same way, as the source map a is then also a submersion, Φ_x is a submanifold of Φ. Also, it is clearly fibered on V by the submersion b. By condition (3) of the above definition, the isotropy group G_x is a Lie group of transformations on Φ_x, which operates on the right side in a simply transitive way on each fiber. So, by a theorem of Gleason [9], Φ_x is a principal bundle on V whose structural group is the Lie group G_x. q.e.d.

As an immediate consequence, we have

Corollary 1.1. *Every Lie groupoid Φ is locally isomorphic to the trivial Lie groupoid $R^n \times G \times R^n$, where n is the dimension of V, and G is a Lie group which is isomorphic to the isotropy groups of Φ.*

The trivial Lie groupoid $R^n \times G \times R^n$ is the natural Lie groupoid on R^n and has the following law of composition:

$$(z, g', y) \cdot (y, g, x) = (z, g' \cdot g, x) .$$

[2] In the sense of N. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton, 1951.

And precisely, the corollary says that $\forall x \in V$, there are a neighborhood U of x in V and a diffeomorphism

$$\varphi\colon (a, b)^{-1}(U \times U) \to R^n \times G \times R^n$$
$$z \to (y(z), g(z), x(z))$$

such that if $a(z) = b(z')$, we have

$$x(z') = y(z) ,$$
$$\varphi(z \cdot z') = \varphi(z) \cdot \varphi(z') .$$

Definition 1.3. Let Φ be a differentiable groupoid, and E a fibered manifold on V, i.e., admitting a submersion p from E onto V. E is a fiber manifold associated to Φ if

(1) $\forall z \in \Phi$, with $a(z) = x$ and $b(z) = y$, z defines a diffeomorphism $\tilde{z}$ from the fiber $E_x = p^{-1}(x)$ to the fiber E_y:

$$\tilde{z}\colon E_x \to E_y$$
$$e \to \tilde{z}(e) = z \cdot e ,$$

and we have $\widetilde{z \cdot z'} = \tilde{z} \circ \tilde{z}'$.

(2) for every couple of differentiable maps f, g from any manifold W, respectively, into Φ and E such that $a \circ f = p \circ g$ the well-defined map

$$f \cdot g\colon W \to E$$
$$x \to f(x) \cdot g(x)$$

is differentiable.

Condition (1) means that Φ is a groupoid of operators on the fibered E in the sense of C. Ehresmann [4].

Proposition 1.2. *Let E be a fibered manifold associated to a Lie groupoid Φ. E is a (locally trivial) fiber bundle, whose structural group G is isomorphic to the isotropy-group of Φ.*

Indeed, let $F = E_x$. It is immediately seen that E, as fibered manifold, is isomorphic to the fiber bundle, with fiber F, modeled on the principal bundle Φ_x, whose structural group G_x is, by the above definition, a Lie transformation group of F.

We remark that if E is a fiber bundle modeled on the principal bundle Φ_x of a Lie groupoid Φ, E is a fibered manifold associated to the Lie groupoid Φ. Moreover, if each fiber of E, a fibered manifold associated to a Lie groupoid Φ, has an algebraic structure (group, vector space, algebra, etc.) compatible with its structure of manifold such that for every z of Φ, $\tilde{z}$ is also an algebraic isomorphism, then E is a fiber bundle with an algebraic structure (group bundle, vector bundle, algebra bundle, etc.).

Examples.

(1) Let E be a vector bundle (differentiable and locally trivial) over a manifold V, and denote by $\Pi(E)$ the set of all linear isomorphisms from a fiber to another fiber of E. Then $\Pi(E)$ is naturally a Lie groupoid on V, and E is a fiber bundle associated to $\Pi(E)$.

(2) Let V be a manifold, and $\Pi^k(V)$ be the set of the k-jets of all local diffeomorphisms of V. Then $\Pi^k(V)$ is a Lie groupoid on V. Also, let $J_{k-1}(T)$ be the set of the $(k-1)$-jets of all differentiable sections of the tangent bundle of V. Then $J_{k-1}(T)$ is a vector bundle over V, and is associated to the Lie groupoid $\Pi^k(V)$ [9].

(3) Let Φ and Ψ be two groupoids on the same set V. A *representation* of Φ into Ψ is a map f from Φ into Ψ such that

(a) $a \circ f = a$, $b \circ f = b$;

(b) $\forall z, z' \in \Phi$ such that $z' \cdot z^{-1}$ is defined, we have

$$f(z' \cdot z^{-1}) = f(z') \cdot f(z)^{-1} .$$

In the case where Φ and Ψ are differentiable groupoid, we suppose that f is also differentiable. If E is a fibered manifold associated to a differentiable groupoid Ψ, and another differentiable groupoid Φ admits a representation f into Ψ, then E is also canonically a fibered manifold associated to the groupoid Φ.

(4) *The isotropy group bundle of a Lie groupoid.* Let Φ be a Lie groupoid on V, and denote the set $(a, b)^{-1}(\Delta)$ by $G(\Phi)$, where Δ is the diagonal submanifold of $V \times V$. Then by the transversality theorem, as (a,b) is a submersion, $G(\Phi)$ is a closed submanifold of Φ. Also, $G(\Phi)$ is naturally a fibered manifold on V; each fiber of $G(\Phi)$ is a Lie group, the isotropy group of Φ. We can easily see that $G(\Phi)$ is hence a group bundle associated to the Lie groupoid Φ.

2. Sheaf of Lie algebra and a theorem of representation

Let Φ be a Lie groupoid on V. We have seen that for every $x \in V$, Φ_x is a principal bundle on V. Now let us consider the corresponding exact Atiyah-sequence of vector bundles over V

$$0 \longrightarrow I(\Phi) \longrightarrow A(\Phi_x) \xrightarrow{b} T \longrightarrow 0 ,$$

where $I(\Phi)$ is the isotropic algebra bundle of Φ, i.e., the Lie algebra bundle corresponding to the canonical group bundle $G(\Phi)$ [Example 1.4], T is the tangent bundle of V, and $A(\Phi_x)$ is a vector bundle over V. The sheaf[3] of

[3] Systematically, for every (differentiable) fibre bundle E on V, $\underline{E}$ will denote the corresponding sheaf of (differentiable) sections of E.

sections, $\underline{A(\Phi_x)}$ on V, is the sheaf defined over V of vector fields on Φ_x, which are invariant under the right action of the structural group G_x. It is easy to check that the Lie bracket of vector fields on Φ_x determines canonically on $\underline{A(\Phi_x)}$ a structure of R-Lie algebra sheaf on V, such that the corresponding Atiyah-sequence of sheaves

$$0 \to \underline{I(\Phi)} \to \underline{A(\Phi_x)} \to \underline{T} \to 0$$

is an exact sequence of R-Lie algebra sheaves, where the Lie algebra structure of $\underline{T}$ is also defined by the Lie bracket of vector fields on V. As the vector bundle $A(\Phi_x)$ is determined independently of the choice of x (i.e., for every other point y of V there is an canonical isomorphism of vector bundles between $A(\Phi_x)$ and $A(\Phi_y)$, which is also an isomorphism of R-Lie algebra sheaves), we shall simply denote the vector bundle $A(\Phi_x)$ by $A(\Phi)$ and refer to $\underline{A(\Phi)}$ as the corresponding R-Lie algebra sheaf of the Lie groupoid Φ.

Examples.

(1) Let Φ be the trivial Lie groupoid on V, i.e., the product manifold $V \times G \times V$, where G is a Lie group, admitting the partial law of composition:

$$(z, g, y)\cdot(y, g', x) = (z, g\cdot g', x) .$$

The corresponding R-Lie algebra sheaf $\underline{A(\Phi)}$ is then a sheaf of $\mathcal{O}$-modules ($\mathcal{O}$ being the structural sheaf of differentiable functions on V) whose elements are couples (g, X) with X, a local vector field on V, and g, a local function on V with values in the Lie algebra of G. Its structure of R-Lie algebra is defined by the following bracket

$$[(g, X), (g', X')] = ([g, g'] + X\cdot g' - X'\cdot g, [X, X']) ,$$

where the different notations are classical, e.g., $X\cdot g'$ is the Lie derivative of the function g' by the vector field X, and $[g, g']$ is the new function on V with values in the Lie algebra of G, canonically defined by the bracket of this algebra.

(2) [9] Denote by Π^k the Lie groupoid of all k-jets of local diffeomorphisms of a manifold V, and let T be the tangent bundle of V. Then the vector bundle $A(\Pi^k)$ is the vector bundle $J_k(T)$, the bundle of the k-jets of sections of T, and the Lie algebra structure of $\underline{A(\Pi^k)}$ is the Lie algebra of $\underline{J_k(T)}$ defined by the following bracket:

$$[fj^kX, j^kY] = fj^k[X, Y] - (Y\cdot f)j^kX ,$$

where $Y\cdot f$ is the Lie derivative of the function f by the vector field Y, and $[X, Y]$ is the Lie bracket of the two vector fields X and Y.

(3) Let E be a vector bundle over V, and denote by Π the Lie groupoid

of all linear isomorphisms from a fiber of E to another fiber of E. The corresponding sheaf $\underline{A(\Pi)}$ is then the $\mathcal{O}$-modules sheaf over V of all differential operators δ of order 1 from E to E, such that the symbol of δ, as a section of $E \otimes E^* \otimes T$, is of the form $Id \otimes X$, where Id is the "identity" section of $E \otimes E^*$, and X is the vector field $b(\delta)$ in the exact Atiyah-sequence of sheaves, or more precisely, such that

$$\delta\colon \underline{E} \to \underline{E}$$
$$\delta(fs) = (X \cdot f)s + f\delta(s)$$

for every section s of E and every differentiable function f on V. Also, the structure of R-Lie algebra of $\underline{A(\Pi)}$ is defined by the commutator of differential operators

$$[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta .$$

Let Φ and Ψ be two Lie groupoids on the same manifold V. Then we have the notion of a representation of Φ into Ψ [Example 1.3], but, as in the case of Lie groups, introduce the following definition.

Definition 2.1. A local representation of Φ into Ψ is a differentiable map R defined on an open neighborhood U of the set of unities in Φ with values in the Lie groupoid Ψ such that the following apply:

(1) If a and b are, respectively, the source map and the target map of Φ as of Ψ, we have

$$a \circ R = a , \qquad b \circ R = b .$$

(2) If z and z' are two elements in U such that $z' \cdot z^{-1}$ is defined and also an element in U, we have

$$R(z' \cdot z^{-1}) = R(z') \cdot R(z)^{-1} .$$

In the following, we shall identify two local representations, defined on the neighborhood U and U', respectively, of the set of unities in Φ, if these two representations induce a same representation defined on the open set $U \cap U'$ of Φ.

Definition 2.2. An infinitesimal representation $\mathscr{R}$ from Φ to Ψ is a morphism of $\mathcal{O}$-modules sheaves over V from $\underline{A(\Phi)}$ to $\underline{A(\Psi)}$ such that:

(1) If the same letter b denotes the canonical morphism of $\underline{A(\Phi)}$ and of $\underline{A(\Psi)}$ on $\underline{T}$ in the Atiyah-sequence, we have

$$b \circ \mathscr{R} = b .$$

(2) $\mathscr{R}$ is a morphism of R-Lie algebra sheaves, i.e., if δ, δ' are two sections

of $A(\Phi)$, we have

$$\mathscr{R}([\delta, \delta']) = [\mathscr{R}(\delta), \mathscr{R}(\delta')] \ .$$

Theorem 2.1. *Every local representation of Φ into Ψ induces an infinitesimal representation. Conversely, every infinitesimal representation is induced by one and only one local representation.*

Proof. The first statement is obvious. To prove the second, we have only to make a local study. It is sufficient, indeed, by the assertion of uniqueness. So let U be a simply connected open set of V such that [Corollary 1.1] the set of all elements in Φ (respectively Ψ), whose source and target are in U, makes a trivial Lie groupoid $U \times G \times U$ (respectively $U \times G' \times U$). Over U, the sheaf $\underline{A(\Phi)}$ (respectively $\underline{A(\Psi)}$) is then the $\mathcal{O}$-modules sheaf of couples (g, X), where g is a function on U with values in the Lie algebra $\mathfrak{g}$ of G, and X a vector field on U (respectively (g', X) and $\mathfrak{g}'$ the Lie algebra of G'). Also, if $\mathscr{R}$ is an infinitesimal representation of Φ into Ψ, $\mathscr{R}$ is then a morphism of sheaves of $\mathcal{O}$-modules on U:

$$\mathscr{R}(g, X) = (r(g) + \omega(X), X) \ ,$$

such that [Example 1.1]

(1) $\omega([X, Y]) = [\omega(X), \omega(Y)] + X \cdot \omega(Y) - Y \cdot \omega(X)$,

(2) $r([g_1, g_2]) = [r(g_1), r(g_2)]$,

(3) $r(X \cdot g) = X \cdot r(g) + [\omega(X), r(g)]$,

where the different notations have a clear meaning.

A. The relation (1) is in fact the classical equation of Maurer-Cartan

$$d\omega + [\omega, \omega] = 0 \ ,$$

where $d\omega$ is the exterior differential of the 1-form ω on U with value in the Lie algebra $\mathfrak{g}'$. By the Frobenius theorem, as U is simply connected, for every x chosen in U there is one and only one differentiable map f:

$$\begin{aligned} f: U &\to G' \\ y &\to f(y, x) \\ x &\to f(x, x) = e', \text{ the unity of } G' \end{aligned}$$

such that with classical notation $\omega = f^{-1} \cdot df$. It is immediate to see that if z is a chosen point of U, the map

$$\begin{aligned} f': U &\to G' \\ y &\to f(y, z) \cdot f(x, z)^{-1} \end{aligned}$$

has the propriety of the map f corresponding to the chosen point x. Hence, by uniqueness, we have

(a) $f(y, x) = f(y, z) \cdot f(x, z)^{-1}$.

B. The given map r can be considered as a function on U with values in the vector space $L(\mathfrak{g}, \mathfrak{g}')$, the space of linear applications from $\mathfrak{g}$ to $\mathfrak{g}'$, such that for every x in U the value $r(x)$ is a representation of Lie algebra; indeed, this is the meaning of condition (2). Let x be a fixed element in U. Then the representation $r(x)$ of Lie algebra determines one and only one local representation of Lie groups ρ_x of G into G', and we denote by W the open neighborhood of the unity in G, where this local representation is defined. For every y of U, let ρ_y be the local representation of Lie groups:

(b) $\rho_y \colon W \to G'$
$u \to f(y, x) \cdot \rho_x(u) \cdot f(y, x)^{-1}$.

Let us consider now the differentiable map

$$R \colon U \times W \times U \to U \times G' \times U$$
$$(y, u, z) \to (y, f(y, z) \cdot \rho_z(u), z) \; .$$

It is easy to see that by (a) and (b) the map R is a local representation of Lie groupoids over U. To show that the local representation R is uniquely defined and induces the given infinitesimal representation, denote by r' the function on U with values in $L(\mathfrak{g}, \mathfrak{g}')$ such that for every y in U, $r'(y)$ is the representation of Lie algebras induced by the local representation of Lie groups ρ_y. This function verifies, as the given function r, the same linear differential equation of order 1, which is the relation (3). Then, by the uniqueness on solution of this differential equation with initial value, the two functions r and r', which have the same value at x, are identical.

3. The exponential mapping

To simplify, we consider now a Lie groupoid Φ on V, which admits a faithful linear representation, i.e., an injective representation into the Lie groupoid $\Pi(E)$ of all linear isomorphisms from a fiber to another fiber of a vector bundle E on V. In other words, Φ is a Lie subgroupoid of $\Pi(E)$ [9]. That is the case for all Lie groupoids which will occur in this work. But the reader can see that, as in the case of Lie groups (there is always a local linear representation which is faithful for Lie groups), this assumption is not restrictive for the following purpose:

Denote by $\Gamma(\Phi)$ the sheaf on V, defined by the germs of differentiable maps σ of some open set U of V into Φ such that $a \circ \sigma =$ Identity, $b \circ \sigma = \varphi$, a diffeomorphism of U into some open set U' of V.

It can be seen that $\Gamma(\Phi)$ is a groupoid on V. Indeed, let σ be a section of $\Gamma(\phi)$ defined on U such that $b \circ \sigma(U) = \varphi(U) = U'$, and σ' another section of $\Gamma(\Phi)$ defined on U'. We have a new section of $\Gamma(\Phi)$ defined on U,

$$\begin{aligned} \sigma' \cdot \sigma \colon U &\to \Phi \\ x &\mapsto [\sigma' \circ \varphi(x)] \cdot \sigma(x) \ . \end{aligned}$$

It is a groupoid of operators on the sheaf $\underline{E}$ and operates on the right in the following way: let σ be a section $\Gamma(\Phi)$ as previously considered, and s a section of E defined on the open set U'. We have a new section of E defined on U,

$$\begin{aligned} \tilde{\sigma}(s) \colon U &\to E \\ x &\mapsto \sigma(x)^{-1} \cdot [s \circ \varphi(x)] \ , \end{aligned}$$

and clearly we have

$$\widetilde{\sigma' \cdot \sigma} = \tilde{\sigma} \circ \tilde{\sigma}' \ .$$

As Φ is a Lie subgroupoid of $\Pi(E)$, $\underline{A(\Phi)}$ is a sheaf of differential operators of E into E [Example 2.3]. If δ is a section of $\underline{A(\Phi)}$, then

$$\begin{aligned} \delta \colon \underline{E} &\to \underline{E} \\ s &\to \delta(s) \ . \end{aligned}$$

Having recalled this fact and given the corresponding notation, we are planning to define a map from the set of global sections of $A(\Phi)$ with compact support into the set of global sections of $\Gamma(\Phi)$. This map will be called the exponential map in order to generalize the following example.

Example. Let $\Phi = V \times \{e\} \times V$, the trivial Lie groupoid with $\{e\}$ as a Lie group with one element (Φ is commonly considered as the Lie subgroupoid of $\Pi(E)$, with $E = V \times R$, the trivial vector bundle). Then we have $\underline{A(\Phi)} = \underline{T}$, the sheaf of vector fields on V, and $\Gamma(\Phi)$, the sheaf defined by the germs of all local diffeomorphisms of V, which we shall denote by Λ. For every vector field X with compact support, it is defined canonically a group of transformation Exp. tX with one parameter on the manifold V. In other words, the classical exponential of vector fields defines a map

$$\begin{aligned} \text{Exp.} \colon H_c^0(V, \underline{T}) &\to H^0(V, \Lambda) \\ X &\to \text{Exp.}\, X \end{aligned}$$

from the set of global sections of the sheaf $\underline{T}$ with compact support into the set of global sections of the sheaf Λ.

Having given this example, we have precisely the following theorem for the more generally considered Lie groupoid Φ.

Theorem 3.1. *There is one and only one exponential map*

$$\mathrm{Exp.}: H^0_c(V, \underline{A(\Phi)}) \to H^0(V, \Gamma(\Phi))$$
$$\delta \to \mathrm{Exp.}\,\delta$$

such that

(1) $b \circ \mathrm{Exp.}\,\delta = \mathrm{Exp.}\,b(\delta)$, *where the second member is the classical exponential of the vector field with compact support* $b(\delta)$,

(2) *if* 0 *is the* 0*-section of* $A(\Phi)$, *then* Exp. 0 *is the unity-section, i.e.,* $\forall x \in V$, $\mathrm{Exp.}\,0(x) = \hat{x}$, *the unity at* x *of* Φ,

(3) *for every section* s *of the vector bundle* E, *then*

$$\lim_{t' \to t} \frac{(\widetilde{\mathrm{Exp.}\,t'\delta})(s) - (\widetilde{\mathrm{Exp.}\,t\delta})(s)}{t' - t} = (\widetilde{\mathrm{Exp.}\,t\delta})[\delta(s)] ,$$

where t *is the real parameter.*

Proof. We shall suppose to simplify that the Lie groupoid Φ is the trivial Lie groupoid $V \times G \times V$, where G is a Lie subgroup of the general linear group $GL(n, R)$. In the general case, the proof is a little more complicated, but it follows from the same argument and presents no essential difficulty.

Under this assumption, the vector bundle E is the trivial vector bundle $V \times R^n$. Moreover, if δ is an element of $H^0_c(V, \underline{A(\Phi)})$ [Example 2.1], then $\delta = (u, X)$, where u is a function on V with values in the Lie algebra of G, and X is a vector field on V and u, X have compact support. If the section Exp. $t\delta$ of $\Gamma(\Phi)$ exists, it is of the form

$$\mathrm{Exp.}\,t\delta = (g(t), \mathrm{Exp.}\,tX) ,$$

where for every t, $g(t)$ is a differentiable function of V with values in the Lie group G.

The third condition of the theorem state that for every function s on V with values in R^n, we must have

$$\lim_{t' \to t} \frac{g(t')^{-1}(s \circ \mathrm{Exp.}\,t'X) - g(t)^{-1}(s \circ \mathrm{Exp.}\,tX)}{t' - t}$$
$$= g(t)^{-1}[(u \cdot s + X \cdot s) \circ \mathrm{Exp.}\,tX] ,$$

where the different notations have a clear meaning.

By an immediate computation the reader can see that this relation is equivalent to saying that for every x fixed in V, the value $g(t, x)$ of the function $g(t)$ must be a differentiable function of the parameter t and verify the following differential equation

$$g(t, x)^{-1}\frac{dg(t, x)}{dt} = -g(t, x)^{-1}[(u \circ \mathrm{Exp.}\,tX)(x)]g(t, x) .$$

Or, this differential equation admits one and only one solution $g(t, x)$ such that $g(0, x) = e$, the neutral element of G. It is well known that this solution depends differentiably on the parameter x. Hence, let $g(t, x)$ be such a solution of the differential equation, and let

$$\text{Exp.}\,\delta = (g(1,), \text{Exp.}\,X) .$$

Clearly, we define in this way the unique exponential mapping of $H^0_c(V, \underline{A(\Phi)})$ into $H^0(V, \Gamma(\Phi))$, which verifies the three conditions of the theorem.

Remark.

(1) The exponential map Exp. determines clearly a morphism of sheaves on V:

$$\text{Exp.}: \underline{I(\Phi)} \to \underline{G(\Phi)} ,$$

where $I(\Phi)$, $G(\Phi)$ are, respectively, the isotropy Lie-algebra bundle and the isotropy Lie group bundle of the Lie groupoid Φ. This morphism of sheaves is the one defined by the classical exponential morphism of bundles from the Lie algebra bundle to the Lie group bundle [9].

(2) As in the case of the exponential map for vector fields, we have

$$\text{Exp.}\,(t + t')\delta = (\text{Exp.}\,t\delta)\cdot(\text{Exp.}\,t'\delta) ,$$

where the law of composition in the second member is naturally the law of composition in $\Gamma(\Phi)$, with its structure of groupoid on V as we have defined in the beginning of this section.

In particular, we have also the *Campbell-Hausdorff formula*. Specifically, let s be a section of the vector bundle E; then the section

$$s(t) = (\widetilde{\text{Exp.}\,t\delta}) \circ (\widetilde{\text{Exp.}\,t\delta'})(s)$$

depends differentiably on the real parameter t, and has the Taylorian expansion relative to the parameter t of the form, which we give only up to the second order,

$$s(t) = s + t(\delta + \delta')(s) + \frac{1}{2}t^2([\delta, \delta'] + (\delta + \delta')^2)(s) + 0(t^3) ,$$

or

$$s(t) = \widetilde{\text{Exp}}.\left(t(\delta + \delta') + \frac{1}{2}t^2[\delta, \delta']\right)(s) + 0(t^3) .$$

4. Prolongation of a Lie groupoid

If Φ is a Lie groupoid on V, then Φ is a fiber bundle on V by the source map a. As there is some risk of confusion, we shall denote this bundle by

(Φ, a, V). Hence, the sheaf $\Gamma(\Phi)$ is a subsheaf of the sheaf $\underline{(\Phi, a, V)}$ of sections of this bundle. Further, we shall denote[4] by Φ_k the subset of $\overline{J_k(\Phi, a, V)}$ whose elements are the jet of sections of $\Gamma(\Phi)$. Let us recall [9] that the structure of groupoid on $\Gamma(\Phi)$ naturally induces on Φ_k a structure of Lie groupoid on V such that (Φ_k, a, V) is a differentiable sub-bundle of $J_k(\Phi, a, V)$. And, if Φ is supposed, as in the preceding section, to be a Lie subgroupoid of the groupoid $\Pi(E)$, then Φ_k is a Lie subgroupoid of the Lie groupoid $\Pi(J_k(E))$ of all linear isomorphisms from a fiber to another fiber of the vector bundle $J_k(E)$ on V. As a Lie groupoid, Φ_k has the corresponding sheaf of Lie algebra $\underline{A(\Phi_k)}$. The purpose of this section is to relate this sheaf with the sheaf $\underline{J_k[A(\Phi)]}$.

For every vector bundle E on V, we shall denote to simplify $\overline{A(E)}$ the sheaf of Lie algebra of the Lie groupoid $\Pi(E)$. If Φ is a Lie subgroupoid of $\Pi(E)$, then $\underline{A(\Phi)}$ is a Lie algebra subsheaf of $\underline{A(E)}$, whose sections are differential operators on E. More generally, we have the following lemma where we denote by $E \otimes \Lambda^p T^*$ the Whitney tensor product of E with the vector bundle of p-forms on the manifold V:

Lemma 4.1. *Every section δ of $A(\Phi)$ canonically defines a differential operator of order 1 from $E \otimes \Lambda^p T^*$ into itself:*

$$\delta \colon \underline{E \otimes \Lambda^p T^*} \to \underline{E \otimes \Lambda^p T^*}$$
$$s \otimes \omega \mapsto \delta(s) \otimes \omega + s \otimes [b(\delta) \cdot \omega] \ ,$$

where $b(\delta) \cdot \omega$ is the Lie derivative of the p-form ω by the vector field $b(\delta)$.

Proposition 4.1. *The sheaf of $\mathcal{O}$-modules $\underline{J_k[A(\Phi)]}$ is a $\mathcal{O}$-modules subsheaf of the sheaf of all differential operators of order* 1 *from $J_k(E)$ into itself, especially if δ is a section of $\underline{A(\Phi)}$, then*

$$j^k\delta \colon \underline{J_k(E)} \to \underline{J_k(E)}$$
$$fj^k s \to fj^k[\delta(s)] + (b(\delta) \cdot f) j^k s \ ,$$

where $b(\delta) \cdot f$ is the Lie derivative of the function f on V by the vector field $b(\delta)$.

Proof. As the sheaf $\underline{J_k[A(\Phi)]}$ is a sheaf of $\mathcal{O}$-modules generated by the integrable sections $j^k\delta$, it is sufficient to verify that every integrable section $j_k\delta$ of $J_k[A(\Phi)]$ defines a differential operator on $J_k(E)$ in the indicated way. This also means that $j^k\delta$ is a section of $\underline{A[J_k(E)]}$. Or it is true for $k = 0$, $j^0\delta = \delta$. We shall prove the assertion by recurrence on the integer k. Hence, let σ be a section of $J_k(E)$:

$$\sigma = f^i j^k s_i \qquad \text{(finite summation)} \ ,$$

[4] Systematically, for every fibered manifold E on V, $J_k(E)$ denotes the set of k-jets of sections of E, which is naturally also a fibered manifold on V by the source mapping [9].

and define $(j^k\delta)(\sigma)$ by the equation

$$(j^k\delta)(\sigma) = f^i j^k[\delta(s_i)] + [b(\delta)\cdot f^i]j^k s_i \ .$$

If π is the canonical morphism from $J_k(E)$ into $J_{k-1}(E)$, which associates to every k-jet its jet of inferior order, and D is the Spencer operator on jet-bundle [9],

$$D\colon \underline{J_k(E)} \to \underline{J_{k-1}(E) \otimes T^*} \ ,$$

we have

$$\pi[(j^k\delta)(\sigma)] = f^i j^{k-1}[\delta(s_i)] + [b(\delta)\cdot f^i]j^{k-1}s_i \ ,$$
$$D[(j^k\delta)(\sigma)] = j^{k-1}[\delta(s_i)] \otimes df^i + j^{k-1}s_i \otimes [b(\delta)\cdot df^i] \ .$$

Then, by the hypothesis of recurrence,

$$\pi[(j^k\delta)(\sigma)] = (j^{k-1}\delta)[\pi(\sigma)] \ ,$$
$$D[(j^k\delta)(\sigma)] = (j^{k-1}\delta)[D(\sigma)] \ ,$$

where the second members are, respectively, the transformation of the sections $\pi(\sigma)$ and $D(\sigma)$ [Lemma 4.1] under the differential operator $j^{k-1}\delta$, considered as a section of $\underline{A[J_{k-1}(E)]}$. These relations mean that $\pi[(j^k\delta)(\sigma)]$ and $D[(j^k\delta)(\sigma)]$ are defined independently of the choice of the sections s_i to express σ. Hence, the same is true of $(j^k\delta)(\sigma)$. Consequently, we have realized $j^k\delta$ as a differential operator on $J_k(E)$. q.e.d.

If we consider the commutator of differential operators, we have the following corollary.

Corollary 4.1. *The sheaf* $\underline{J_k[A(\Phi)]}$ *is a sheaf of Lie algebra such that*

$$[j^k\delta, fj^k\delta'] = fj^k[\delta, \delta'] + (b(\delta)\cdot f)j^k\delta' \ .$$

In other words, with this structure of Lie algebra $\underline{J_k[A(\Phi)]}$ is a Lie algebra subsheaf of the sheaf $\underline{A[J_k(E)]}$. Let δ be a section with compact support of $A(\Phi)$, we have immediately by the uniqueness assertion of Theorem 3.1 the following important lemma:

Lemma 4.2.

$$\text{Exp.}\,(j^k\delta) = j^k(\text{Exp.}\,\delta) \ .$$

This lemma states that as $j^k\delta$ is a section with compact support of $\underline{A[J_k(E)]}$, we can apply the exponential mapping. In this way, we attain a section of $\Gamma(\Phi_k)$. Hence, $j^k\delta$ is a section of $\underline{A(\Phi_k)}$. As $\underline{A(\Phi_k)}$ and $\underline{J_k[A(\Phi)]}$ are two subsheafs of $\mathcal{O}$-modules of $\underline{A[J_k(E)]}$, and these integrable sections $j^k\delta$ generate the sheaf of $\mathcal{O}$-modules $\underline{J_k[A(\Phi)]}$, we must have the inclusion

$$\underline{J_k[A(\Phi)]} \subset \underline{A(\Phi_k)} \ .$$

An immediate computation on the dimension of these two vector bundles shows that we have in fact the equality. In other words, we have proved

Theorem 4.2.

$$J_k[A(\Phi)] = A(\Phi_k) \ .$$

Remark. (1) In a general way, we shall denote by π the canonical application which associates to every k-jet of a differentiable mapping its jet of inferior order. Then, if Φ_1 is the first-order prolongation groupoid of Φ, the the application π defines a subjective representation of groupoid

$$\Phi_1 \xrightarrow{\pi} \Phi \longrightarrow 1 \ .$$

The corresponding infinitesimal representation $\mathscr{R}(\pi)$ is the canonical morphism π of vector bundles, if we identify by the last theorem $A(\Phi_1)$ with $J_1[A(\Phi)]$

$$\mathscr{R}(\pi) = \pi : J_1[A(\Phi)] \to A(\Phi) \ .$$

(2) The representation of groupoids π induces a subjective morphism of group bundles on V

$$G(\Phi_1) \xrightarrow{\pi} G(\Phi) \longrightarrow 1 \ ,$$

where $G(\Phi_1)$ and $G(\Phi)$ are, respectively, the isotropy group bundle of Φ_1 and Φ. $N(\Phi)$ will denote the group bundle which is the kernel of this subjective morphism. Then the exact sequence of jet bundle

$$0 \longrightarrow A(\Phi) \otimes T^* \longrightarrow J_1[A(\Phi)] \xrightarrow{\pi} A(\Phi) \longrightarrow 0$$

immediately shows that the corresponding Lie-algebra bundle of the Lie-group bundle $N(\Phi)$ is isomorphic to the vector bundle $A(\Phi) \otimes T^*$, the Whitney tensor product of $A(\Phi)$ with the cotangent vector bundle T^* of the manifold V.

(3) Let R be a representation of Lie groupoids on V from Φ to Ψ, and denote by $\mathscr{R}$ its infinitesimal representation

$$\mathscr{R} : A(\Phi) \to A(\Psi) \ .$$

The representation R naturally induces a representation of prolongation

$$j^k R : \Phi_k \to \Psi_k \ .$$

The infinitesimal representation of $j^k R$ is the canonical prolongation $j^k \mathscr{R}$, when we identify $A(\Phi_k)$ and $A(\Psi_k)$ with $J_k[A(\Phi)]$ and $J_k[A(\Psi)]$:

$$j^k\mathscr{R}\colon J_k[A(\Phi)] \to J_k[A(\Psi)] \ .$$

(4) In finishing, we give the following remark: $\underline{J_1[A(\Phi)]}$ is a sheaf of Lie algebra [Corollary 4.1], isomorphic to $\underline{A(\Phi_1)}$. The $\mathcal{O}$-modules subsheaf $\underline{A(\Phi)\otimes T^*}$ is then a subsheaf of ideals. In fact, let ω be a section of $A(\Phi)\otimes T^*$; then the $ad\omega$ is a $\mathcal{O}$-linear morphism of sheaves of $\mathcal{O}$-modules:

$$\begin{aligned} ad\omega\colon \underline{J_1[A(\Phi)]} &\to \underline{A(\Phi)\otimes T^*} \\ \delta &\to [\omega, \delta] \ . \end{aligned}$$

5. The nonabelian Spencer cohomology

If Φ_1 is the first prolongation groupoid of the Lie groupoid Φ, we shall denote by $\mathscr{D}$ the following morphism of sheaves on V

$$\begin{aligned} \mathscr{D}\colon \Gamma(\Phi_1) &\to \Gamma(\Phi_1) \\ \sigma &\to \sigma^{-1}\cdot j^1[\pi(\sigma)] \ , \end{aligned}$$

where the composition $\sigma^{-1}\cdot j^1[\pi(\sigma)]$ is naturally the composition in $\Gamma(\Phi_1)$ with its structure of groupoid as we have defined in the beginning of § 3. This morphism is the so-called non-linear Spencer operator for Lie groupoid, and, in fact, has values in the sheaf $\underline{N(\Phi)}$, subsheaf of $\Gamma(\Phi_1)$ [Remark 4.2]. The following proposition characterizes this operator.

Proposition 5.1. *There is one and only one morphism of sheaves*

$$\begin{aligned} \mathscr{D}\colon \Gamma(\Phi_1) &\to \underline{N(\Phi)} \\ \sigma &\to \mathscr{D}(\sigma) \end{aligned}$$

such that the following are true:

(1) $\mathscr{D}(\sigma) = 1$, *the neutral section of* $N(\Phi)$, *iff* σ *is an integrable section, i.e., there is a local section* s *of* $\Gamma(\Phi)$ *such that*

$$\sigma = j^1 s \ .$$

(2) *We have*

$$\mathscr{D}(\sigma'\cdot\sigma) = [ad(\sigma^{-1})\mathscr{D}(\sigma')]\cdot\mathscr{D}(\sigma) \ .$$

(3) *If* σ *is a section of the subsheaf* $N(\Phi)$, *then*

$$\mathscr{D}(\sigma) = \sigma^{-1} \ .$$

(We have denoted by $ad(\quad)$ the natural operation of the groupoid $\Gamma(\Phi_1)$ on

$\underline{N(\Phi)}$:

$$ad(\sigma^{-1})\mathscr{D}(\sigma') = \sigma^{-1}\cdot\mathscr{D}(\sigma')\cdot\sigma \ .$$

Proof. Since the non-linear Spencer operator evidently verifies these three conditions, we have only to prove the uniqueness assertion. So let $\mathscr{D}$ be the Spencer operator

$$\sigma\cdot\mathscr{D}(\sigma) = j^1[\pi(\sigma)], \text{ for every section } \sigma \text{ of } \Gamma(\Phi_1) \ .$$

If $\mathscr{D}'$ is another morphism which verifies the conditions of the proposition, by the first condition we have

$$\mathscr{D}'[\sigma\cdot\mathscr{D}(\sigma)] = \mathscr{D}'(j^1[\pi(\sigma)]) = 1 \ .$$

Then the second and third conditions assert

$$\mathscr{D}'[\sigma\cdot\mathscr{D}(\sigma)] = \mathscr{D}(\sigma)^{-1}\cdot\mathscr{D}'(\sigma) = 1 \ .$$

Hence $\mathscr{D}'(\sigma) = \mathscr{D}(\sigma)$. q.e.d.

For every integer k, Φ_k is the groupoid of prolongation of order k of the Lie groupoid Φ, and we have a natural injection

$$\Phi_{k+1} \hookrightarrow [\Phi_k]_1 \ .$$

so that Φ_{k+1} is a Lie subgroupoid of the Lie groupoid $[\Phi_k]_1$. We shall denote by the same letter $\mathscr{D}$ the restriction to $\Gamma(\Phi_{k+1})$ of the Spencer operator in $\Gamma([\Phi_k]_1)$

$$\mathscr{D}\colon \Gamma(\Phi_{k+1}) \to \underline{N(\Phi_k)} \ .$$

If a section σ of $\Gamma(\Phi_{k+1})$ verifies

$$\mathscr{D}(\sigma) = 1, \text{ the neutral section of } N(\Phi_k) \ ,$$

we have

$$\sigma = j^1[\pi(\sigma)] \ .$$

It is well known that the last relation implies σ to be an integrable section, i.e., there is a section s of $\Gamma(\Phi)$ such that

$$\sigma = j^{k+1}s \ .$$

In other words, the operator $\mathscr{D}$ of Spencer defines a non-linear exact sequence of sheaves

$$1 \longrightarrow \Gamma(\Phi) \xrightarrow{j^{k+1}} \Gamma(\Phi_{k+1}) \xrightarrow{\mathscr{D}} \underline{N(\Phi_k)} \ ,$$

where by exactness we mean that the morphism j^{k+1} is injective, and if σ and σ' ane two sections in $\Gamma(\Phi_{k+1})$ with

$$\mathscr{D}(\sigma) = \mathscr{D}(\sigma') ,$$

there is a section s in $\Gamma(\Phi)$ such that

$$\sigma = j^{k+1}s \cdot \sigma' .$$

For every integer l, the Spencer operator naturally defines a morphism $h(\mathscr{D})$ of fiber bundles on V

$$h(\mathscr{D}) \colon ([\Phi_{k+1}]_{l+1}, a, V) \to J_l[N(\Phi_k)] ,$$

where the first term is the Lie groupoid $[\Phi_{k+1}]_{l+1}$ considered as fiber bundle on V by its source mapping a, that is, the Spencer operator is a non-linear differential operator of order 1, whose tangential linear differential operator along a local section σ of $\Gamma(\Phi_{k+1})$ will be denoted by D_σ. If φ is the local diffeomorphism $b \circ \sigma$, then D_σ is a differential operator of order 1 defined on the open set U on which the section σ is given:

$$D_\sigma \colon \underline{\varphi^*[A(\Phi_{k+1})]} \to \underline{A(\Phi_k) \otimes T^*} ,$$

where $\varphi^*[A(\Phi_{k+1})]$ is the pull-back vector bundle of $A(\Phi_{k+1})$ by the local diffeomorphism φ on U, and $A(\Phi_k) \otimes T^*$ is the Lie-algebra bundle of the Lie-group bundle $N(\Phi_k)$. We shall identify $A(\Phi_{k+1})$ with $J_{k+1}[A(\Phi)]$ for every integer k.

Lemma 5.1. *Let δ be a section of $J_{k+1}[A(\Phi)]$. Then*

$$D_\sigma[\varphi^*(\delta)] = (Adj^1[\pi(\sigma)])^{-1}D(\delta) ,$$

where D is the linear Spencer operator on jet bundles

$$D \colon \underline{J_{k+1}[A(\Phi)]} \to \underline{J_k[A(\Phi)] \otimes T^*} ,$$

and Adu, for every section u of $\Gamma([\Phi_k]_1)$, denotes the natural operation of the groupoid $\Gamma([\Phi_k]_1)$ on $\underline{J_k[A(\Phi)] \otimes T^*}$, sheaf of sections of the Lie-algebra bundle of the group bundle $N(\Phi_k)$.

Proof. We can suppose the section δ with compact support. Then, by the definition of the linear tangential operator D_σ, we have

$$\begin{aligned} D_\sigma[\varphi^*(\delta)] &= \mathscr{D}(\sigma)^{-1} \cdot \left. \frac{d\mathscr{D}[(\mathrm{Exp.}\ t\delta)\cdot\sigma]}{dt} \right|_{t=0} \\ &= (Adj^1[\pi(\sigma)])^{-1} \left. \frac{d\mathscr{D}(\mathrm{Exp.}\ t\delta)}{dt} \right|_{t=0} . \end{aligned}$$

By Lemma 4.2 we have

$$\mathscr{D}(\text{Exp. } t\delta) = (\text{Exp. } t\delta)^{-1}\cdot(\text{Exp. } tj^1\pi(\delta)) ,$$

and the Campbell-Hausdorff formula (Remark 3.2) yields immediately

$$\left.\frac{d\mathscr{D}(\text{Exp. } t\delta)}{dt}\right|_{t=0} = j^1\pi(\delta) - \delta = D(\delta) . \qquad \text{q.e.d.}$$

Let σ and σ' be two sections of $\Gamma(\Phi_{k+1})$ such that

$$\mathscr{D}(\sigma) = \mathscr{D}(\sigma') .$$

By the exactness of the non-linear Spencer sequence, we have

$$\sigma = j^{k+1}s\cdot\sigma'$$

with s to be a section of $\Gamma(\Phi)$.

Lemma 5.2. *For every section δ of $J_{k+1}[A(\Phi)]$, we have*

$$D_\sigma[\varphi^*(\delta)] = (Adj^1[\pi(\sigma')])^{-1}D[(Adj^{k+1}s)^{-1}(\delta)] ,$$

where $Adj^{k+1}s$ denotes the natural operation of $\Gamma(\Phi_{k+1})$ on the sheaf $\underline{A(\Phi_{k+1})}$.

Proof. By Lemma 5.1, we have

$$D_\sigma[\varphi^*(\delta)] = (Adj^1[\pi(\sigma')])^{-1}(Adj^{k+1}s)^{-1}D(\delta) .$$

As $j^{k+1}s$ is an integrable section, we thus evidently have the commutation

$$(Adj^{k+1}s)^{-1}D = D(Adj^{k+1}s)^{-1} . \qquad \text{q.e.d.}$$

Having proved these two lemmas, we are going now to define, as in the case of linear Spencer cohomology, the "*nonnaive*" *operator of Spencer* for Lie groupoid. So let us recall that the canonical mapping π defines a surjective representation of Lie groupoids

$$\Phi_{k+1} \xrightarrow{\pi} \Phi_k \longrightarrow 1 .$$

It induces a surjective morphism of group bundles

$$G(\Phi_{k+1}) \xrightarrow{\pi} G(\Phi_k) \longrightarrow 1,$$

and we shall denote by N_{k+1} the group bundle which is the kernel of this surjective morphism:

$$1 \longrightarrow N_{k+1} \longrightarrow G(\Phi_{k+1}) \xrightarrow{\pi} G(\Phi_k) \longrightarrow 1 .$$

The representation π also induces a subjective morphism of sheaves

$$\Gamma(\Phi_{k+1}) \to \Gamma(\Phi_k) \to 1$$

in the sense that for every *germ* of section σ of $(\Gamma(\Phi_k)$ there is a *germ* of section ζ of $\Gamma(\Phi_{k+1})$ such that

$$\pi(\zeta) = \sigma .$$

Also, if ζ' is another germ of section of $\Gamma(\Phi_{k+1})$ such that

$$\pi(\zeta') = \pi(\zeta) = \sigma ,$$

we have

$$\zeta' = \eta \cdot \zeta$$

with η a germ of section of $\underline{N_{k+1}}$.

Clearly, N_{k+1} is a Lie subgroup bundle of the Lie group bundle $N(\Phi_k)$. We shall denote by $C(\Phi_k)$ the associated bundle of homogenuous space, the fiber bundle whose fiber is the homogenuous space defined by the group fiber of $N(\Phi_k)$ modulo *on the left* the subgroup which is the corresponding fiber of N_{k+1}, and also by the symbol 1 the class-section of the neutral section of $N(\Phi_k)$.

Proposition 5.2. *There is a canonical morphism of sheaves*

$$\underline{\mathscr{D}}\colon \Gamma(\Phi_k) \to \underline{C(\Phi_k)}$$

such that the following sequence of sheaves

$$1 \longrightarrow \Gamma(\Phi) \xrightarrow{j^k} \Gamma(\Phi_k) \xrightarrow{\underline{\mathscr{D}}} \underline{C(\Phi_k)}$$

is exact in the sense that the morphism j^k is injective, and

(1) *σ being a section of $\Gamma(\Phi_k)$, $\underline{\mathscr{D}}(\sigma) = 1$ iff σ is integrable*, i.e., $\sigma = j^k s$,

(2) *if σ and σ' are two sections of $\Gamma(\Phi_k)$, we have $\underline{\mathscr{D}}(\sigma) = \underline{\mathscr{D}}(\sigma')$, iff $\sigma = j^k s \cdot \sigma'$.*

Proof. Let σ be a *germ* of $\Gamma(\Phi_k)$. Denote by $\underline{\mathscr{D}}(\sigma)$ the class in $\underline{C(\Phi_k)}$ of the germ of section $\mathscr{D}(\zeta)$, with ζ any germ of section of $\Gamma(\Phi_{k+1})$ such that $\pi(\zeta) = \sigma$. The section $\underline{\mathscr{D}}(\sigma)$ is independent of the choice of the germ of section ζ; indeed, let ζ' be another germ of section of $\Gamma(\Phi_{k+1})$ such that

$$\pi(\zeta) = \pi(\zeta') = \sigma .$$

We have $\zeta' = \eta \cdot \zeta$, with η being a germ of section of N_{k+1}. Hence,

$$\mathscr{D}(\zeta') = [ad(\zeta^{-1})\eta^{-1}] \cdot (\zeta) \qquad \text{[Proposition 5.1]} .$$

It is immediate to see that $ad(\zeta^{-1})\eta^{-1}$ is again a section of $\underline{N_{k+1}}$ proving the assertion. Because of this, one sees easily that we have defined in this way a morphism of sheaves

$$\underline{\mathscr{D}}: \Gamma(\Phi_k) \to \underline{C(\Phi_k)} .$$

To prove this canonical morphism $\underline{\mathscr{D}}$ verifies the property of the proposition, by the exactness of the sequence defined by the Spencer operator $\mathscr{D}$ itself, we have only to prove that

$$\underline{\mathscr{D}}(\sigma) = 1 \text{ iff } \mathscr{D}(\sigma) = 1 \text{, the neutral section of } \underline{N(\Phi_{k-1})} .$$

Indeed, the representation of Lie groupoids

$$\pi: \Phi_k \to \Phi_{k-1}$$

induces, by prolongation, a representation of Lie groupoids

$$j^1\pi: [\Phi_k]_1 \to [\Phi_{k-1}]_1 ,$$

which defines a morphism of group bundles

$$j^1\pi: G([\Phi_k]_1) \to G([\Phi_{k-1}]_1)$$

such that we have the commutative diagram of group bundles

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N(\Phi_k) & \longrightarrow & G([\Phi_k]_1) & \xrightarrow{\pi} & G(\Phi_k) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle j^1\pi} & & \downarrow{\scriptstyle j^1\pi} & & \downarrow{\scriptstyle \pi} & & \\
1 & \longrightarrow & N(\Phi_{k-1}) & \longrightarrow & G([\Phi_{k-1}]_1) & \xrightarrow{\pi} & G(\Phi_{k-1}) & \longrightarrow & 1 .
\end{array}$$

Furthermore, the restriction of $j^1\pi$ to the subgroup bundle $G(\Phi_{k+1})$ of $G([\Phi_k]_1)$ is exactly the morphism π:

$$j^1\pi = \pi: G(\Phi_{k+1}) \to G(\Phi_k) .$$

In particular, the group bundle N_{k+1} is contained in the kernel of the morphism $j^1\pi$, and if ζ is a section of $\Gamma(\Phi_{k+1})$, we have

$$\mathscr{D}[\pi(\zeta)] = j^1\pi[\mathscr{D}(\zeta)] .$$

So let σ be a section of $\Gamma(\Phi_k)$ such that

$$\underline{\mathscr{D}}(\sigma) = 1$$

for every germ of section ζ of $\Gamma(\Phi_{k+1})$ such that $\pi(\zeta) = \sigma$; then we have $\mathscr{D}(\zeta)$ a section of N_{k+1}. Hence,

$$\mathscr{D}(\sigma) = j^1\pi[\mathscr{D}(\zeta)] = 1 .$$

Conversely, if $\mathscr{D}(\sigma) = 1$, by the property of the operator $\mathscr{D}$, we have

$$\sigma = j^k s \ ,$$
$$\underline{\mathscr{D}}(\sigma) = \mathscr{D}(j^{k+1}s) \bmod . N_{k+1} = 1 \ . \qquad \text{q.e.d.}$$

The morphism $\underline{\mathscr{D}}$ is the so-called "non-naive" Spencer operator. It is clearly a differential operator of order 1, and its tangential linear differential operator along a section σ of $\Gamma(\Phi_k)$ will be denoted by $\underline{D}_\sigma$. If φ is the local diffeomorphism $b \circ \sigma$, then $\underline{\mathscr{D}}_\sigma$ is a differential operator of order 1 defined on the open set U on which the section σ is given:

$$\underline{D}_\sigma\colon \underline{\varphi^*[J_k(A(\Phi))]} \to \underline{F}_\sigma \ ,$$

where $\varphi^*[J_kA(\Phi)]$ is the pull-back vector bundle of $J_k(A(\Phi))$ by the diffeomorphism φ on U, and F_σ is vector bundle on U of vertical tangent vectors along the section $\underline{\mathscr{D}}(\sigma)$ of $C(\Phi_k)$.

Denote by C_k^1 the quotient bundle of $J_k[A(\Phi)] \otimes T^*$ by the subvector bundle $\partial(A(\Phi) \otimes S^{k+1}(T^*))$. The bundle C_k^1 is isomorphic to the vector bundle of vertical tangent vectors along the section 1 to the bundle $C(\Phi_k)$. Then the bundle F_σ is isomorphic to the bundle $(Ad\, j^1\sigma)^{-1}(C_k^1)$, which is the quotient bundle of $(Ad\, j^1\sigma)^{-1}(J_kA(\Phi) \otimes T^*)$ by its sub-bundle $(Ad\, j^1\sigma)^{-1}\partial[A(\Phi) \otimes S^{k+1}(T^*)]$. Remark that the last two bundles are well defined, as we have precedingly pointed out the operation noted Ad of the Lie groupoid $[\Phi_k]_1$ on the vector bundle $J_k[A(\Phi)] \otimes T^*$ [Lemma 5.1]). So the following proposition has a clear meaning:

Proposition 5.3. *Let δ be a section of $J_k[A(\Phi)]$. Then*

$$\underline{D}_\sigma[\varphi^*(\delta)] = (Ad\, j^1\sigma)^{-1}\underline{D}(\delta) \ ,$$

where D is the "non-naive" linear Spencer operator [12]:

$$\underline{D}\colon \underline{J_k[A(\Phi)]} \to \underline{C_k^1} \ .$$

The proof of this proposition does not present any difficulty, as it is a consequence of Lemma 5.1 and of the preceding remarks.

The differential operator $\underline{\mathscr{D}}$ defines naturally for every integer l a morphism of fiber bundles

$$h(\underline{\mathscr{D}})\colon ([\Phi_k]_{l+1}, a, V) \to J_l[C(\Phi_k)] \ .$$

Proposition 5.4. *We have the following exact sequence of bundles*

$$1 \longrightarrow (\Phi_{k+l+1}, a, V) \xrightarrow{\ i\ } ([\Phi_k]_{l+1}, a, V) \xrightarrow{h(\underline{\mathscr{D}})} J_l[C(\Phi_k)] \ ,$$

where i is the canonical injection and exactness means that:

(1) $h(\underline{\mathscr{D}})(X) = 1$, *the* l-jet *of the section* 1 *of* $C(\Phi_k)$, *iff* $X \in \Phi_{k+l+1}$,

(2) $h(\underline{\mathscr{D}})(X) = h(\underline{\mathscr{D}})(X')$ *iff* $X = Y \cdot X'$ *with* $Y \in \Phi_{k+l+1}$.

Proof. We look first at the case where $l = 0$. Then

$$h(\underline{\mathscr{D}}) \colon ([\Phi_k]_1, a, V) \to C(\Phi_k)$$
$$X \to Y^{-1} \cdot X \bmod N_{k+1}$$

with Y to be any element in Φ_{k+1} such that $\pi(Y) = \pi(X) \in \Phi_k$. So

$$Y^{-1} \cdot X \in N_{k+1}, \text{ iff } X \in \Phi_{k+1} .$$

Let X_1 be an element of $[\Phi_k]_1$ such that

$$h(\underline{\mathscr{D}})(X) = h(\underline{\mathscr{D}})(X_1)$$
$$Y^{-1} \cdot X = Y_1^{-1} \cdot X_1 \bmod. N_{k+1} .$$

Hence, we have

$$X_1 \cdot X^{-1} = Y \cdot Y_1^{-1} \bmod. N_{k+1} \in \Phi_{k+1} ,$$

and have proved the proposition for this case. In the general case, it suffices to remind that the differential system of prolongation of order l of the differential system Φ_{k+1} of order 1 in the bundle (Φ_k, a, V) is exactly the Lie groupoid Φ_{k+l+1}. q.e.d.

For every integer l, denote by S_l the direct image of the bundles morphism

$$h(\underline{\mathscr{D}}) \colon ([\Phi_k]_{l+1}, a, V) \to J_l[C(\Phi_k)] .$$

As an immediate consequence of the last proposition, S_l is a subfiber bundle of $J_l[C(\Phi_k)]$ (in fact, $S_0 = C(\Phi_k)$). S_l is a differential system in the bundle $C(\Phi_k)$. It is immediate to verify that we have the subjective morphism

$$\pi \colon S_{l+1} \to S_l ,$$

and for $l \geq 1$, S_{l+1} is the prolongation differential system of the differential system S_l. Moreover, the differential system S_l is involutive. In other words, we have the following proposition.

Proposition 5.5. *For every section σ of $\Gamma(\Phi_k)$, the section $\underline{\mathscr{D}}(\sigma)$ of $C(\Phi_k)$ is a solution of an involutive differential system S_1 of order* 1; *namely, the sub-bundle of $J_1[C(\Phi_k)]$ defined as the direct image of the bundle morphism*

$$h(\underline{\mathscr{D}}) \colon ([\Phi_k]_2, a, V) \to J_1[C(\Phi_k)] .$$

We remark that, by definition, the differential system S_1 is completely integrable.

Proposition 5.6. *Let ω be a section of $C(\Phi_k)$, which is a solution of the differential system S_1. Then the set of sections σ of $\Gamma(\Phi_k)$ such that $\underline{\mathscr{D}}(\sigma) = \omega$*

is the set of solutions of an involutive differential system $P_1(\omega)$ *of order* 1 *namely, the sub-bundle of* $([\Phi_k]_1 a, V)$ *defined as the inverse image of the section* ω *by the bundle morphism*

$$h(\underline{\mathscr{D}}): ([\Phi_k]_1, a, V) \to C(\Phi_k) .$$

Proof.

(1) For every section s of the bundle S_{l-1}, denote by $P_l(s)$ the inverse image of the section s in $([\Phi_k]_l, a, V)$ by the bundle morphism $h(\underline{\mathscr{D}})$. Proposition 5.4 asserts immediately that $P_l(s)$ is a sub-bundle of the bundle $([\Phi_k]_l, a, V)$, and, evidently, if we denote by $\pi(s)$ the section in S_{l-2}, the image of the section s by the subjective morphism

$$\pi: S_{l-1} \to S_{l-2} ,$$

then we have the surjective morphism of bundles

$$\pi: P_l(s) \to P_{l-1}[\pi(s)] .$$

(2) Hence, $P_1(\omega)$ is a differential system of order 1 in the bundle (Φ_k, a, V). Clearly, $P_2(j^1\omega)$ is the prolongation differential system of $P_1(\omega)$, and, as we have remarked, the morphism π is surjective

$$\pi: P_2(j^1\omega) \to P_1(\omega) .$$

Then the system $P_1(\omega)$ is involutive, if its linear symbol is involutive, or its linear symbol is isomorphic to the linear symbol of the differential system defined by the "non-naive" linear operator $\underline{D}$ of Spencer (Proposition 5.3):

$$\underline{D}: \underline{J_k[A(\Phi)]} \to \underline{C_k^1} ,$$

which is involutive. q.e.d.

The main question is whether for every solution ω of the differential system S_1, the corresponding differential system $P_1(\omega)$ is completely integrable. The answer is positive. That is to say

Theorem 5.1. (*Malgrange-Newlander-Nirenberg*). *Denote by* Σ *the sheaf of solutions of the differential system* S_1 *in* $C(\Phi_k)$; *we have the exact nonlinear sequence of sheaves*

$$1 \longrightarrow \Gamma(\Phi) \xrightarrow{j^k} \Gamma(\Phi_k) \xrightarrow{\underline{\mathscr{D}}} \Sigma \longrightarrow 1 ,$$

where the surjectivity of the morphism $\underline{\mathscr{D}}$ *means that for every germ of section* ω *of* Σ, *there is a germ of section* σ *of* $\Gamma(\Phi_k)$ *such that* $\underline{\mathscr{D}}(\sigma) = \omega$. (*The exactness of the rest of the sequence has the same meaning as stated in Proposition* 5.2.).

If the manifold V is analytic, and the groupoid Φ is analytic, i.e., Φ has an analytic structure such that the mapping (a, b) is analytic, then the exactness of the sequence (this result is known to Spencer)

$$1 \longrightarrow \Gamma_{an}(\Phi) \xrightarrow{j^k} \Gamma_{an}(\Phi_k) \xrightarrow{\mathscr{D}} \Sigma_{an} \longrightarrow 1$$

of sheaves of analytic sections is an immediate consequence of Proposition 5.6 as the differential system $P_1(\omega)$, corresponding to every analytic solution ω of the analytic system S_1, is an analytic involutive differential system, hence completely integrable.

The proof of Theorem 5.1 follows from an argument due to B. Malgrange, and will not be given here (see a forthcoming work of Malgrange).

The Malgrange-Newlander-Nirenberg theorem has its main interest in deformation theory (see the next chapter), but the deformation theory of differentiable principal bundles is trivial. We are planning to show in another work how the nonabelian Spencer cohomology can be applied to the Griffiths deformation theory of holomorphic principal bundles.

CHAPTER II

Γ-STRUCTURES AND DEFORMATION

1. Transitive continuous pseudogroups

Given a manifold V, recall that we denote the pseudogroup of local diffeomorphisms of V by Λ, and the Lie groupoid on V of all k-jets of local diffeomorphisms of V by Π_k.

Definition 1.1. A subpseudogroup Γ of the pseudogroup Λ is called a transitive continuous pseudogroup on V, if for every integer $k(\geq 0)$, the set of its k-jets

$$\Psi_k = \{j_x^k f \mid x \in V \quad \text{and} \quad f \in \Gamma\}$$

is a Lie subgroupoid of the Lie groupoid Π_k on V.

Let us consider the trivial fiber bundle $(V \times V, p, V)$, the product manifold $V \times V$ with its first projection p on V. Then (Ψ_k, a, V), i.e., the Lie groupoid Ψ_k considered as bundle on V by its source mapping a, is a subfiber bundle of the bundle $J_k(V \times V, p, V)$. In other words, for $k \geq 1$, (Ψ_k, a, V) is a differential system of order k in the bundle $(V \times V, p, V)$. By definition the differential system (Ψ_k, a, V) corresponding to a transitive continuous pseudogroup on V is completely integrable. Also, for every integer k, the differential system (Ψ_{k+1}, a, V) is contained in the differential system of prolongation of the differential system (Ψ_k, a, V). By a well-known theorem of Cartan and Kuranishi, there is an integer m such that if $k \geq m$, the differential system (Ψ_{k+1}, a, V) is exactly the differential system of prolongation of the differential system (Ψ_k, a, V). The integer m will be referred to as the order of the transitive continuous pseudogroup Γ. The pseudogroup Γ is *complete* if every element of Λ, which is a solution of the differential system (Ψ_m, a, V), is in Γ. In the following, we consider only transitive continuous pseudogroup, which is complete in this sense.

Examples.

(1) Let G be a Lie subgroup of the general linear group $GL(R^n)$. The pseudogroup $\Gamma(G)$ on R^n, composed of local diffeomorphisms f of R^n such that the Jacobian of f, referred to the canonical coordinates of R^n, at every point x in the domain of definition of f is an element of G, is a transitive continuous pseudogroup on the manifold R^n of order 1.

(2) In R^{2n+1}, consider the form of contact

$$\alpha = dx_{2n+1} - \sum_{i=1}^{n} x_i dx_{n+i} ,$$

where $(x_1, \cdots, x_{2n+1})$ is the canonical system of coordinate in R^{2n+1}. The pseudogroup of contact Γ, composed of local diffeomorphisms which leave invariant the form of contact α, is a transitive continuous pseudogroup of order 1.

(3) More generally, let V be an analytic manifold. Consider a Lie subgroupoid Ψ_k of Π_k such that (Ψ_k, a, V) is an analytic involutive differential system in the trivial bundle $(V \times V, p, V)$. Then the set of elements in the pseudogroup Λ, which are local solutions of the differential system (Ψ_k, a, V), is a transitive continuous pseudogroup (Γ, Ψ_k) of order k on V. Such a pseudogroup will be said to be an analytic pseudogroup. We remark that the set (Γ_{an}, Ψ_k), composed of solutions which are analytic transformations, is also a transitive continuous pseudogroup, but in general is not complete $[(\Gamma, \Psi_k) \neq (\Gamma_{an}, \Psi_k)]$.

(4) Let V be a homogenuous space G/H, and Γ_G the pseudogroup obtained by localizing to open sets the group of transformations G on V. Γ_G is a transitive continuous pseudogroup of order 2. It is of finite type, as the canonical mapping

$$\pi : \Psi_2 \to \Psi_1$$

is an isomorphism.

(5) Let Γ be a transitive continuous pseudogroup on a manifold W. Let then a Γ-structure S be given on a manifold V, and denote the pseudogroup of local automorphisms of the structure S by $\Gamma(S)$ (see [4] for the exact definition of a Γ-structure S and the pseudogroup of local automorphisms $\Gamma(S)$). The pseudogroup $\Gamma(S)$ is also a transitive continuous pseudogroup on V, and has the same order as Γ.

In § 2 of the first chapter we indicated, as an example, that the vector bundle $A(\Pi_k)$ is the k-jet-bundle $J_k(T)$ of the tangent bundle T of V (that is indeed Theorem 4.2, Chap. I, where we take for Φ the trivial Lie groupoid $V \times \{e\} \times V$). If Γ is a transitive continuous pseudogroup on V, for every integer $k(\geq 0)$, the corresponding Lie groupoid Ψ_k has its bundle $A(\Psi_k)$ canonically as a subvector bundle of the vector bundle $J_k(T)$. That is to say, the bundle $A(\Psi_k)$ is a linear differential system of order k in the tangent bundle T. So there will be no risk of confusion to denote the bundle $A(\Psi_k)$ by A_k, and to refer to it as the linear differential system of order k associated to the transitive continuous pseudogroup.

It can be shown easily that[5] the differential system A_{k+1} is contained in the prolongation system of A_k, and for $k \geq m$, the order of the transitive continuous pseudogroup Γ, A_{k+1} is exactly the prolongation system of the differential system A_k. By definition, as the canonical mapping π is subjective:

$$A_{k+1} \xrightarrow{\pi} A_k \longrightarrow 0$$

the differential system A_k (for every $k \geq 1$) is formally completely integrable [9], but we do not know whether our Definition 1.1 of a transitive continuous pseudogroup implies that the differential systems A_k are completely integrable. However, that is the case for different known examples (Examples 1, 2, 3, and 4). We remark that in the definition originally given by Cartan and Spencer of a transitive continuous pseudogroup, one supposes that the associated linear differential systems are completely integrable.

So let Γ be a transitive continuous pseudogroup of order m. We denote the sheaf of solutions of the associated linear differential system A_m by Θ. As the sheaf $\underline{A_m}$ is a Lie algebra subsheaf of the sheaf $\underline{J_m(T)}$ [Corollary 4.1, Chap. I], the sheaf Θ is a Lie algebra subsheaf of the sheaf of vector fields $\underline{T}$, and is named the sheaf of Γ-vector fields. As a matter of fact, by Lemma 4.2, Chap. I, every local vector field X of Θ is easily seen to define a local group of transformations with one parameter Exp. tX on V such that for every fixed t, Exp. tX is an element of Γ (see [9], and note that Γ is supposed complete).

Definition 1.2. A transitive continuous pseudogroup Γ of order m is said to be elliptic iff the canonical linear differential operator δ of order m from the tangent bundle T into the quotient bundle $J_m(T)/A_m$ is elliptic[6]:

$$\delta = p \circ j^m : \underline{T} \xrightarrow{j^m} \underline{J_m(T)} \xrightarrow{p} \underline{J_m(T)/A_m},$$

where p is the canonical bundle morphism from $J_m(T)$ to the quotient bundle $J_m(T)/A_m$.

[5] By applying Lemma 4.2 and Theorem 4.2 of Chapter I.

[6] Elliptic in the sense that its symbol associated to every nonnull cotangent vector is injective.

Example 2.

(1) The pseudogroup $\Gamma(G)$ of Example 1.1 is elliptic iff the Lie algebra $\mathfrak{g}$ of the Lie group G has no element of rank one, i.e., no element of the form $a \otimes u$, $a \in R^n$ and $u \in R^{n*}$, the dual space. That is the case, for example, where G is the orthogonal group $0(n)$, or where $n = 2k$, R^n is identified with the complex space C^k and G is a Lie subgroup of the linear complex group $GL(k, C)$.

(2) A transitive continuous pseudogroup Γ of finite type (i.e. if m is its order, the morphism $\pi: \Psi_{m+1} \to \Psi_m$ is an isomorphism) is elliptic. In particular, the pseudogroup Γ_G of Example 1.4 is elliptic.

(3) If Γ is an elliptic transitive continuous pseudogroup on a manifold W, for every Γ-structure S on another manifold V, the transitive continuous pseudogroup of automorphisms $\Gamma(S)$ is also elliptic.

To conclude this section, the following results should be noted:

(a) (L. Nirenberg) If the analytic transitive continuous pseudogroup (Γ, Ψ_k), see Example 1.3, is elliptic, every local diffeomorphism of the analytic manifold V, which is an element of (Γ, Ψ_k), is analytic, i.e., we have

$$(\Gamma_{an}, \Psi_k) = (\Gamma, \Psi_k)$$

(see §§ 15, 16 of [5]). Hence, if a manifold W admits a (Γ, Ψ_k) structure S, the manifold W is analytic and the pseudogroup of automorphisms $\Gamma(S)$ is analytic.

(b) (R. Palais) Let Γ be an elliptic transitive continuous pseudogroup on a *compact* manifold V. By a well known result on elliptic differential operators, the vector space $H^0(V, \Theta)$ is of finite dimension. The group of global automorphisms of (V, Γ) is then a Lie group of transformations on V, whose associated Lie algebra is the space $H^0(V, \Theta)$.

2. Nonabelian cohomology of Spencer for transitive continuous pseudogroup

Let Γ be a transitive continuous pseudogroup on a manifold V, so that its linear differential systems A_k are formally completely integrable. If $k \geq m$, the order of Γ, then A_{k+1} is the differential system of prolongation of A_k. By a well-known theorem of Cartan-Kuranishi, the linear differential system A_k of order k in the tangent bundle T is involutive for large k, say $\forall k \geq m_1$ for some integer m_1, which is naturally greater than the order m of Γ, and is named the *stability order* of the pseudogroup Γ. We remark that the nonlinear differential system (Ψ_k, a, V) is also an involutive differential system in the bundle $(V \times V, p, V)$ for all $k \geq m_1$. Hence, the system (Ψ_{k+1}, a, V) is the prolongation differential system (Ψ_k, a, V) of order k in the bundle $(V \times V, p, V)$. In particular, the Lie groupoid Ψ_{k+1} is a Lie subgroupoid of the Lie groupoid $[\Psi_k]_1$, first prolongation of Ψ_k; we can restrict to $\Gamma(\Psi_{k+1})$ the nonlinear Spencer operator $\mathscr{D}$ [Prop. 5.1, Chap. I]:

$$\Gamma(\Psi_{k+1}) \subsetneq \Gamma([\Psi_k]_1) \xrightarrow{\mathscr{D}} N(\Psi_k) \ ,$$

and have the following proposition.

Proposition 2.1. *The nonlinear Spencer operator $\mathscr{D}$ defines an exact sequence of sheaves*

$$1 \longrightarrow \Gamma \xrightarrow{j^{k+1}} \Gamma(\Psi_{k+1}) \xrightarrow{\mathscr{D}} \underline{N(\Psi_k)}$$

in the sense that the morphism j^{k+1} is injective and

(1) *for a section σ of $\Gamma(\Psi_{k+1})$, $\mathscr{D}(\sigma)$ is a neutral section of the group bundle $N(\Psi_k)$, i.e., $\mathscr{D}(\sigma) = 1$, iff σ is integrable, i.e., $\sigma = j^{k+1}s$, with s a section of Γ,*

(2) *σ and σ' being two sections of $\Gamma(\Psi_{k+1})$, we have*

$$\mathscr{D}(\sigma) = \mathscr{D}(\sigma') \qquad \text{iff } \sigma = j^{k+1}s\cdot\sigma' \ .$$

As in Chap. I, § 5, we define next the "nonnaive" Spencer operator. Let us denote by $N_{k+1}(\Gamma)$ the group bundle which is the kernel of the surjective morphism π of group bundles:

$$1 \longrightarrow N_{k+1}(\Gamma) \longrightarrow G(\Psi_{k+1}) \xrightarrow{\pi} G(\Psi_k) \longrightarrow 1 \ .$$

$N_{k+1}(\Gamma)$ is naturally a subgroup bundle of the group bundle $N(\Psi_k)$. $C_k(\Gamma)$ will represent the associated bundle of homogenuous space, whose fiber is the homogenuous space defined by the group fiber of $N(\Psi_k)$ modulo *on the left* the subgroup which is the corresponding fiber of $N_{k+1}(\Gamma)$. For every germ σ of $\Gamma(\Psi_k)$, denote by $\underline{\mathscr{D}}(\sigma)$ the germ of $\underline{C_k(\Gamma)}$

$$\underline{\mathscr{D}}(\sigma) = \mathscr{D}(\zeta) \qquad \text{mod. } N_{k+1}(\Gamma)$$

with ζ any germ of $\Gamma(\Psi_{k+1})$ such that

$$\pi(\zeta) = \sigma \ .$$

We define in this way (see remarks preceding Proposition 5.2, Chap. I) a morphism of sheaves $\underline{\mathscr{D}}$, "nonnaive Spencer" operator, and we have the following proposition with the same meaning and the same proof as Proposition 5.2.

Proposition 2.2. *The "nonnaive" operator of Spencer $\underline{\mathscr{D}}$ defines an exact sequence of sheaves*

$$1 \longrightarrow \Gamma \xrightarrow{j^k} \Gamma(\Psi_k) \xrightarrow{\underline{\mathscr{D}}} C_k(\Gamma) \ .$$

The "nonnaive" Spencer operator $\underline{\mathscr{D}}$ is naturally a differential operator of order 1, and its tangential linear operator along a section σ of $\Gamma(\Psi_k)$ will be denoted by $\underline{D_\sigma}$:

$$\underline{D_\sigma}\colon \underline{\varphi^*(A_k)} \to \underline{F_\sigma} \ ,$$

where $\varphi^*(A_k)$ is the pull-back vector bundle of A_k by the local diffeomorphism $\varphi = b \circ \sigma$, and F_σ is the vector bundle of vertical tangent vectors along the section $\underline{\mathscr{D}}(\sigma)$ of $C_k(\Gamma)$.

Denote by C_k^1 the Spencer bundle associated to the linear differential system A_k, i.e., the quotient bundle of $A_k \otimes T^*$ by the subvector bundle $\partial(g_{k+1})$, with g_{k+1} defined by the exact sequence [12]

$$0 \longrightarrow g_{k+1} \longrightarrow A_{k+1} \xrightarrow{\pi} A_k \longrightarrow 0 \ .$$

Then C_k^1 is isomorphic to the vector bundle of vertical tangent vectors along the section 1 to the bundle $C_k(\Gamma)$, and F_σ is isomorphic to the bundle $(Ad\, j^1\sigma)^{-1}(C_k^1)$, the quotient bundle $(Ad\, j^1\sigma)^{-1}(A_k \otimes T^*)$ by its subbundle $(Ad\, j^1\sigma)^{-1}\partial(g_{k+1})$ (we remind $j^1\sigma$, as a section of $\Gamma([\Psi_k]_1)$, operates naturally by Ad on the Lie-algebra bundle $A_k \otimes T^*$ of the Lie-group bundle $N(\Psi_k)$). We have also the following proposition with a clear meaning (this proposition will be necessary in the proof of the theorem of Malgrange-Newlander-Nirenberg).

Proposition 2.3. *Let δ be a section of A_k*

$$\underline{D}_\sigma[\varphi^*(\delta)] = (Ad\, j^1\sigma)^{-1}\underline{D}(\delta) \ ,$$

where $\underline{D}$ is the "nonnaive" linear operator of Spencer

$$\underline{D}\colon \underline{A_k} \to C_k^1 \ .$$

(See the remark following Proposition 5.3, Chap. I, for the proof.)

With the same argument as in the proof of Proposition 5.4, Chap. I, we prove the following proposition, where we denote again by

$$h(\underline{\mathscr{D}})\colon ([\Psi_k]_{l+1}, a, V) \to J_l[C_k(\Gamma)]$$

the bundle morphism which is naturally defined by the differential operator $\underline{\mathscr{D}}$.

Proposition 2.4. *We have the following exact sequence of bundles on V:*

$$1 \longrightarrow (\Psi_{k+l+1}, a, V) \xrightarrow{i} ([\Psi_k]_{l+1}, a, V) \xrightarrow{h(\underline{\mathscr{D}})} J_l[C_k(\Gamma)],$$

where i is the canonical injection, and exactness means:

(1) *an element X of $[\Psi_k]_{l+1}$ is the l-jet of the section* 1 *of $C_k(\Gamma)$*, i.e., $h(\underline{\mathscr{D}})(X) = 1$, *iff* $X \in \Psi_{k+l+1}$,

(2) *for two elements* X *and* X' *of* $[\varPsi_k]_{l+1}$ *we have*

$$h(\underline{\mathscr{D}})(X) = h(\underline{\mathscr{D}})(X') \quad \textit{iff} \quad X = Y \cdot X' \quad \textit{with} \quad Y \in \varPsi_{k+l+1} .$$

As a consequence of the last two propositions, we have in the same way as for Propositions 5.5 and 5.6, Chap. I:

Proposition 2.5. *For every section* σ *of* $\Gamma(\varPsi_k)$, *the section* $\underline{\mathscr{D}}(\sigma)$ *of* $C_k(\Gamma)$ *is a solution of an involutive differential system* S_1 *of order* 1 *in* $C_k(\Gamma)$, *that is,* S_1 *is the sub-bundle of* $J_1[C_k(\Gamma)]$ *defined as the direct image of the bundle morphism* $h(\underline{\mathscr{D}})$

$$h(\underline{\mathscr{D}})\colon ([\varPsi_k]_2, a, V) \to J_1[C_k(\Gamma)] .$$

Proposition 2.6. *Let* ω *be a section of* $C_k(\Gamma)$, *which is a solution of the differential system* S_1. *Then the set of sections* σ *of* $\Gamma(\varPsi_k)$ *such that*

$$\underline{\mathscr{D}}(\sigma) = \omega$$

is the set of solutions of an involutive differential system $P_1(\omega)$ *of order* 1 *in the bundle* $(\varPsi_k, a, V)$, *namely,* $P_1(\omega)$ *is the sub-bundle of* $([\varPsi_k]_1, a, V)$ *defined as the inverse image of the section* ω *by the bundle morphism* $h(\underline{\mathscr{D}})$:

$$h(\underline{\mathscr{D}})\colon ([\varPsi_k]_1, a, V) \to C_k(\Gamma) .$$

The differential system S_1 is certainly completely integrable. If the pseudogroup Γ is analytic, the Lie groupoid $\varPsi_k$ is analytic, and S_1 is an analytic differential system in the analytic bundle $C_k(\Gamma)$. Let ω be an analytic solution of the differential system S_1; then $P_1(\omega)$ is an analytic sub-bundle of the analytic bundle $([\varPsi_k]_1, a, V)$. In other words, $P_1(\omega)$ is an analytic involutive differential system and hence completely integrable. So we have the following theorem, supposing Γ to be analytic [Example 1.3].

Theorem 2.1 (*D. C. Spencer*). *Denote by* Σ_{an} *the sheaf of analytic solutions of the analytic differential system* S_1. *We have the exact sequence of sheaves of analytic sections*

$$1 \longrightarrow \Gamma_{an} \xrightarrow{j^k} \Gamma_{an}(\varPsi_k) \xrightarrow{\underline{\mathscr{D}}} \Sigma_{an} \longrightarrow 1,$$

where surjectivity of the morphism $\underline{\mathscr{D}}$ *means that for every germ* ω *of* Σ_{an} *given, there is a germ* σ *of* $\Gamma_{an}(\varPsi_k)$ *such that*

$$\underline{\mathscr{D}}(\sigma) = \omega ,$$

and the exactness of the rest of the sequence has the same meaning as stated in Proposition 5.2, *Chap. I.*

The main result of our theory is the following theorem with the same meaning as in Theorem 2.1, but now we only consider the sheaves of differentiable sections.

Theorem 2.2 (*Malgrange-Newlander-Nirenberg*). *If Γ is an elliptic analytic transitive continuous pseudogroup and Σ denotes the sheaf of solutions of the differential system S_1 in the bundle $C_k(\Gamma)$, we have the exact non-linear sequence of sheaves*

$$1 \longrightarrow \Gamma \xrightarrow{j^k} \Gamma(\Psi_k) \xrightarrow{\underline{\mathscr{D}}} \Sigma \longrightarrow 1 .$$

Remark.

1) It is essential for Proposition 2.6 and the last theorem that, as we have supposed, the considered integer k is larger than the stability order m_1 of the transitive continuous pseudogroup Γ.

2) One would like to have in Theorem 2.2 a larger resolution, i.e., to define a differential operator $\underline{\mathscr{D}}$ from $C_k(\Gamma)$ to another bundle E on V with a chosen section 1 such that we have an exact sequence of sheaves

$$1 \longrightarrow \Sigma \longrightarrow \underline{C_k(\Gamma)} \xrightarrow{\mathscr{D}} \underline{E}$$

in the sense that for every germ ω of $\underline{C_k(\Gamma)}$ we have

$$\underline{\mathscr{D}}(\omega) = 1 \qquad \text{iff } \omega \text{ is a germ of } \Sigma .$$

As the differential system S_1 is involutive and completely integrable, it is equivalent to saying that there is a bundle morphism h

$$h\colon J_r[C_k(\Gamma)] \to E$$

for some integer r such that the inverse image of the section 1 in E is the subbundle S_r, $r-1$ prolongation of the differential system S_1. By a remark of B. Mazur, one can see easily that there is not in general such a bundle morphism h with constant rank. Hence, there is not a natural definition of $\underline{\mathscr{D}}$.

3) Let us recall that if Γ is analytic, the differential system S_1 is an involutive differential system which is analytic in the analytic bundle $C_k(\Gamma)$. Hence, if ω is a solution of the differential system S_1, defined in some small neighborhood U_0 of x in V, we can find an analytic solution ω_{an} of S_1, defined in the same neighborhood U_0, provided that U_0 is chosen small enough such that the section ω_{an} is as close as we want to the section ω, say in the topology of the Banach manifold $\mathscr{C}^{2+\alpha}(U, C_k(\Gamma))$ of sections of Hölder class $\mathscr{C}^{2+\alpha}$ on a neighborhood U of x with compact closure $\bar{U}$ contained in U_0.

By this remark and Theorem 2.1 of Spencer, Theorem 2.2 is clearly equivalent to the following:

Theorem 2.2.a. *Let σ_{an} be an analytic section of $\Gamma(\Psi_k)$, defined on some neighborhood U_0 of x in V. If ω is a solution of S_1, defined on the same neighborhood close enough to $\omega_{an} = \underline{\mathscr{D}}(\sigma_{an})$, say in the topology of $\mathscr{C}^{2+\alpha}(U, C_k(\Gamma))$, U being a neighborhood of x with compact closure in U_0,* *then there is a germ of section σ at x of $\Gamma(\Psi_k)$ such that $\underline{\mathscr{D}}(\sigma)$ is the germ at x of ω.*

The proof of this theorem follows straightforwardly from an argument of Malgrange. We shall not give it here and prefer to refer the reader to a forthcoming work of Malgrange.

3. Application to deformation theory

As an immediate corollary of the Malgrange-Newlander-Nirenberg theorem, we have the following proposition, denoting by $H^0(V, \Sigma)$ the set of global sections of $C_k(\Gamma)$ which are solutions of the differential system S_1:

Proposition 3.1. *Let the transitive continuous pseudogroup Γ be analytic elliptic. Then every element ω of $H^0(V, \Sigma)$ defines canonically a Γ-structure $S(\omega)$ on V, which is subordinate to the given structure of differentiable manifold.*

Proof. Given ω, by Theorem 2.2 in the neighborhood of every element x of V we have a section σ of $\Gamma(\Psi_k)$ such that

$$\underline{\mathscr{D}}(\sigma) = \omega .$$

In other words, we have a cover of V by open sets (U_i):

$$V = \bigcup_{i \in I} U_i ,$$

and have a section σ_i of $\Gamma(\Psi_k)$ on each U_i with

$$\underline{\mathscr{D}}(\sigma_i) = \omega \,|\, U_i .$$

For every couple of indices i and j

$$\underline{\mathscr{D}}(\sigma_i) = \underline{\mathscr{D}}(\sigma_j) \quad \text{on} \quad U_i \cap U_j .$$

By Proposition 2.1, there are sections s_j^i of Γ such that

$$\sigma_j = j^k(s_j^i) \cdot \sigma_i \quad \text{on} \quad U_i \cap U_j .$$

If we denote by φ_i the diffeomorphism $b \circ \sigma_i$ of U_i into V, we have

$$\varphi_j = s_j^i \circ \varphi_i \quad \text{on} \quad U_i \cap U_j .$$

Hence the set (U_i, φ_i, s_j^i), $i, j \in I$ defines a Γ-structure on V, which is subordinate to the given structure of differentiable manifold [4]. The reader verifies easily that the Γ-structure $S(\omega)$ defined in this way depends only on the given section ω and not on the choice of $(U_i, \varphi_i)_{i \in I}$. q.e.d.

The Γ-structure $S(\omega)$ will be said to be represented by the section ω.

From now on, the manifold V is supposed to be compact with a transitive continuous pseudogroup Γ. Let S be a Γ-stucture on V, defined as precedently by (U_i, φ_i, s_j^i), $i, j \in I$. S is said to be close to S^0(S^0 is the canonical Γ-structure defined by (V, Id, Id)) iff for every index i the diffeomorphism φ_i on U_i into V is close to the canonical injection Id, say for the $\mathscr{C}^k$-topology,

$$Id\colon U_i \to V$$
$$x \to x\ .$$

We have

Proposition 3.2. *Every Γ-structure S close to S^0 on V can be represented by an element ω of $H^0(V, \Sigma)$.*

Proof. We can suppose the cover (U_i) to be finite:

$$V = \bigcup U_i, \qquad \text{with } i \text{ integer and } 1 \leq i \leq p\ .$$

Moreover, let $U'_i = U_i \cup \varphi_i(U_i)$. Then for every integer i, U_i can be taken small enough such that the Lie groupoid Ψ_k is trivial on U'_i [see Corollary 1.1, Chapter I]:

$$(a, b)^{-1}(U'_i \times U'_i) \simeq U'_i \times G \times U'_i\ ,$$

where G is the Lie group isomorphic to the isotropy group of Ψ_k. Every section σ of $\Gamma(\Psi_k)$ with sourse domain and target domain in U'_i will hence be represented by a couple (φ, g) with φ to be a local diffeomorphism of U'_i into U'_i and g to be a local function on U'_i with values in G; in particular, the "unity" section of $\Gamma(\Psi_k)$ will be represented by (Id, e), where e is the constant function on U_i with value to be the neutral element e of G.

Let us take a refinement of the cover (U_i). Then we can suppose with the same indices:

$$V = \bigcup V_i, \qquad \text{with for every } i,\ \overline{V}_i \subset U_i\ .$$

We are going to construct on each $\overline{V}_i$ a section σ_i of $\Gamma(\Psi_k)$ such that

$$\sigma_j = j^k(s_j^i)\cdot\sigma_i \quad \text{on} \quad \overline{V}_i \cap \overline{V}_j\ .$$

Indeed, let σ_1 be a section defined on $\overline{V}_1$ of $\Gamma(\Psi_k)$, which is represented in the preceding trivialization of Ψ_k on U'_1 by (φ_1, e). On the closed set $\overline{V}_1 \cap \overline{V}_2, j^k(s_2^1)\cdot\sigma_1$ is a differentiable section[7] of (Ψ_k, a, V), and is also a section of $\Gamma(\Psi_k)$, which is represented in the trivialization of Ψ_k on U_2 by (φ_2, g_2). As the Γ-structure S is close to S^0, the section $j^k(s_2^1)\cdot\sigma_1$ is certainly close to the "unity" section. The function g has all its values close to the neutral element

[7] If A is a closed set in a manifold V, a differentiable mapping f of A into a manifold W is by definition the restriction of some differentiable mapping from an open neighborhood of A in V into W.

e of G, and is a differentiable mapping of $\bar{V}_1 \cap \bar{V}_2$ into an absolute retract neighborhood of e in G [11]. Hence, by Borsuk's theorem [11], we can extend g_2 to a differentiable mapping noted again by g_2 from $\bar{V}_2$ into this absolute retract neighborhood. The couple (φ_2, g_2) represents then a section σ_2 on V_2 of $\Gamma(\Psi_k)$. Let us consider now respectively on $\bar{V}_3 \cap \bar{V}_1$ and $\bar{V}_3 \cap \bar{V}_2$ the sections $j^k(s_3^1)\cdot\sigma_2$ of $\Gamma(\Psi_k)$. One verifies that these two sections agree on $\bar{V}_1 \cap \bar{V}_2 \cap \bar{V}_3$, and define a differentiable section of (Ψ_k, a, V) on $\bar{V}_3 \cap (\bar{V}_1 \cup \bar{V}_2)$, which is a section of $\Gamma(\Psi_k)$, represented in the trivialization of Ψ_k on U_3' by (φ_3, g_3). The differentiable mapping g_3 has again by the same argument all its values close to e, and is a differentiable mapping of $\bar{V}_3 \cap (\bar{V}_1 \cup \bar{V}_2)$ into an absolute retract neighborhood of e in G. So we can extend g_3 to a differentiable mapping of $\bar{V}_3$ into G, having all its values close to e. Also, (φ_3, g_3) represents a section σ_3 of $\Gamma(\Psi_k)$ on $\bar{V}_3$. In this way we construct on each V_i a section σ_i of $\Gamma(\Psi_k)$ with the prescribed property. Evidently, there is a section ω of $H^0(V, \Sigma)$ such that

$$\underline{\mathscr{D}}(\sigma_k) = \omega \,|\, V_i \,,$$

and it defines the Γ-structure S as in the proof of Proposition 3.1. q.e.d.

More generally, let us recall first the following definition of a family (with one parameter) of deformations in the sense of Kodaira-Spencer of (V, Γ), i.e., a manifold V with a pseudogroup Γ[6].

Definition 3.1. A continuous (differentiable) family of deformations of (V, Γ) is to give a cover of V by open sets U_i:

$$V = \bigcup_{i \in I} U_i \,,$$

and to give a continuous differentiable mapping[8] for each index i,

$$\varphi_i \colon U_i \times [0, 1] \to V$$
$$(x, t) \to \varphi_i^t(x)$$

such that:

(1) $\forall i \in I, \forall x \in U_i, \varphi_i^0(x) = x$,

(2) for each fixed t, the set $(U_i, \varphi_i^t), i \in I$, defines a Γ-structure on V, i.e., $\forall i$, φ_i^t is a diffeomorphism of U_i into V, and there are section of Γ, $s_j^{i,t}$, such that for every couple i, j

$$\varphi_j^t = s_j^{i,t} \circ \varphi_j^t \quad \text{on} \quad U_i \cap U_j \,.$$

In the case where V is analytic, we have also the notion of an analytic

[8] φ_i is differentiable (analytic relatively to t) in the sense that it is the restriction to $U_i \times [0, 1]$ of a differentiable mapping (and analytic relatively to t) φ_i defined on $U_i \times J$ with J to be an open interval containing $[0, 1]$.

family of deformations, by requiring φ_i to be analytic relatively to the real parameter t.

Given a family of deformations, we shall denote the Γ-structure $(U_i, \varphi_i^t, s_j^{i,t})$, $i, j \in I$ by S^t and call it a deformation of (V, Γ).

In the case where Γ is a transitive continuous pseudogroup, we also have the following notion:

Definition 3.1.a. A continuous (differentiable) family of sections of $C_k(\Gamma)$ on V is to give a continuous (differentiable) mapping

$$V \times [0, 1] \to C_k(\Gamma)$$
$$(x, t) \to \omega^t(x)$$

such that:

(i) for every fixed t, ω^t is a differentiable section of $C_k(\Gamma)$,

(ii) the section ω^0 is the canonical section 1 of $C_k(\Gamma)$.

This is the corresponding notion of deformations of the canonical section 1 in $C_k(\Gamma)$. Precisely, the manifold V being supposed compact, we have the following theorem:

Theorem 3.1. *Given a continuous (differentiable) family of deformations S^t of (V, Γ), there is a continuous differentiable family of sections ω^t of $C_k(\Gamma)$ such that for every fixed parameter t the section ω^t is an element of $H^0(V, \Sigma)$ and defines the Γ-structure S^t.*

Proof. (We give here the proof for differentiable family, as the continuous case is slightly simpler): Suppose the manifold V to be compact. Then we can find a finite cover

$$V = \cup U_i\,, \qquad \text{with } i \text{ to be integer, } 1 \leq i \leq p\,,$$

and there is (by uniform continuity) a small real number ε such that

(1) $\forall t$, if $t - \varepsilon \leq t' \leq t + \varepsilon$, the diffeomorphism $\varphi_i^{t'}$ is close to the diffeomorphism φ_i^t for every i, say in the $\mathscr{C}^k$-topology,

(2) $\forall t$, the differentiable mapping φ_i maps

$$\varphi_i\colon U_i \times [t - \varepsilon, t + \varepsilon] \to U'_{i,t} \subset V$$

such that for every i, the Lie groupoid Ψ_k is trivial on $U_i \times U'_{i,t}$, i.e., there is a diffeomorphism

$$(a, b)^{-1}(U_i \times U'_{i,t}) \simeq U_i \times G \times U'_{i,t}$$

with G to be a Lie group isomorphic to the isotropy group of Ψ_k. Hence, every section σ of $\Gamma(\Psi_k)$ with source domain and target domain, respectively, in U_i and $U'_{i,t}$ can be represented in this trivialization by a couple (φ, g), where φ is a local diffeomorphism of U_i into $U'_{i,t}$, and g is a local differentiable function on U_i with values in G.

Let us take a refinement of the cover U_i. Then we can suppose with the same indices

$$V = \cup V_i\ , \qquad \text{and for every } i, \text{ the closure } \bar{V}_i \subset U_i .$$

As in the proof of Proposition 4.2, we are going to construct on each $\bar{V}_i$ a differentiable family of sections σ_i^t of (Ψ_k, a, V) such that for fixed t, σ_i^t is a section of $\Gamma(\Psi_k)$ on $\bar{V}_i$ and we have

$$\sigma_j^t = j^k(s_j^{i,t}) \cdot \sigma_i^t \quad \text{on} \quad \bar{V}_i \cap \bar{V}_j\ .$$

(a) We construct first such a family of sections σ_i^t for $t \leq \varepsilon$. Indeed, let us consider on $\bar{V}_1$ the family of sections σ_1^t, represented in the preceding trivialization by the couple (φ_1^t, e). On $\bar{V}_2 \cap \bar{V}_1$, we have the family of sections $j^k(s_2^{1,t}) \cdot \sigma_1^t$ of $\Gamma(\Psi_k)$ which is represented in the corresponding trivialization of Ψ_k by (φ_2^t, g_2^t). The differentiable mapping

$$\begin{aligned} (\bar{V}_2 \cap \bar{V}_1) \times [0, \varepsilon] &\to G \\ (x, t) &\to g_2^t(x) \end{aligned}$$

naturally has all its values close to e, and is a differentiable mapping into an absolute retract neighborhood of e in G. Hence, by Borsuk's theorem we can extend it to a differentiable mapping with all its values close to e:

$$\begin{aligned} \bar{V}_2 \cap [0, \varepsilon] &\to G \\ (x, t) &\to g_2^t(x) \end{aligned}$$

so that the couple (φ_2^t, g_2^t) then represents a family of sections σ_2^t of $\Gamma(\Psi_k)$ on $\bar{V}_2$. Again let us consider on $\bar{V}_3 \cap \bar{V}_1 \times [0, \varepsilon]$ and $(\bar{V}_3 \cap \bar{V}_2) \times [0, \varepsilon]$, respectively, the family of sections $j^k(s_3^{1,t}) \cdot \sigma_1^t$ and $j^k(s_3^{2,t}) \cdot \sigma_2^t$ which agree on $(\bar{V}_3 \cap \bar{V}_1) \cap (\bar{V}_3 \cap \bar{V}_2)$ to define a family of sections of $\Gamma(\Psi_k)$ on $\bar{V}_3 \cap (\bar{V}_1 \cup \bar{V}_2)$ represented in the corresponding trivialization of Ψ_k on $U_3 \times U_{3,0}$ by the couple (φ_3^t, g_3^t). The differentiable mapping

$$\begin{aligned} [\bar{V}_3 \cap (\bar{V}_1 \cup \bar{V}_2)] \times [0, \varepsilon] &\to G \\ (x, t) &\to g_3^t(x) \end{aligned}$$

and the couple (φ_3^t, g_3^t) then represent a family of sections σ_3^t on $\bar{V}_3$ of $\Gamma(\Psi_k)$. So on, we define a family of sections σ_i^t, for $0 \leq t \leq \varepsilon$, on each V_1 with the prescribed property.

(b) Suppose now that we have constructed in this way a family of sections σ_i^t of $\Gamma(\Psi_k)$ on each V_i, for $t \leq t_0$. We are going to show that by modifying the family near t_0, we can extend the family σ_i^t to be defined for $t \leq t_0 + \varepsilon$. Indeed, let t_1 be chosen such that $t_0 - \varepsilon < t_1 < t_0$. The given family of sections

σ_1^t, for $t_0 - \varepsilon \leq t \leq t_0$, will be represented by a family of couples (φ_1^t, g_1^t) in the trivialization of Ψ_k on $U_1 \times U'_{1,t_0}$.

The differentiable mapping

$$\begin{aligned} \bar{V}_1 \times [t_0 - \varepsilon, t_0] &\to G \\ (x, t) &\to g_1^t(x) \end{aligned}$$

has all its values, as we can suppose by the preceding construction, close to $g_1^{t_0}(x')$ with x' to be any element in $\bar{V}_1$; so it is a differentiable mapping into an absolute retract neighborhood in G. Again by Borsuk's theorem, we can modify g_1^t in the neighborhood of t_0 such that we can extend it to a differentiable mapping

$$\bar{V}_1 \times [t_0 - \varepsilon, t_0 + \varepsilon] \to G$$

with the same restriction to $\bar{V}_1 \times [t_0 - \varepsilon, t_1]$ as g_1^t. So the couple (φ_1^t, g_1^t), where we denote by the same letter g_1^t the new function defined on $\bar{V}_1 \times [t_0 - \varepsilon, t_0 + \varepsilon]$ with values in G, represents a family of sections σ_1^t on $\bar{V}_1$. Hence, the family of sections σ_1^t is defined now for every $t \leq t_0 + \varepsilon$. On $\bar{V}_2 \times [t_0 - \varepsilon, t_1] \cup (\bar{V}_2 \cap \bar{V}_1) \times [t_0 - \varepsilon, t_0 + \varepsilon]$, we have the family of sections $j^k(s_2^{1,t}) \cdot \sigma_1^t$, which is represented by (φ_2^t, g_2^t) in the trivialization of Ψ_k on $U_2 \times U'_{2,t_0}$. The mapping g_2^t has again all its values contained in an absolute retract neighborhood of G, so we can extend it to a differentiable mapping

$$\bar{V}_2 \times [t - \varepsilon, t_0 + \varepsilon] \to G ,$$

and the couple (φ_2^t, g_2^t) represents a family of sections σ_2^t of $\Gamma(\Psi_k)$ on V_2, which extends the given family σ_2^t to define a family of sections for $t \leq t + \varepsilon$, as these two families are the same for t between $t_0 - \varepsilon$ and t_1. The rest of the argument will be carried in the same way to define steps by steps the family of sections σ_i^t on each $\bar{V}_i$ of $\Gamma(\Psi_k)$ for $t \leq t + \varepsilon$.

Hence, we can conclude that there is a family of sections σ_i^t on each V_i of $\Gamma(\Psi_k)$ for all t in the interval [0, 1], and naturally, there is a family of sections ω^t of $C_k(\Gamma)$ such that

$$\underline{\mathscr{D}}(\sigma_i^t) = \omega^t \,|\, V_i ,$$

which defines the family of Γ-structure S^t. q.e.d.

In the case where Γ is an analytic transitive continuous pseudogroup (hence V is an analytic manifold), the bundle $C_k(\Gamma)$ is analytic on V. Then we have also the notion of an analytic family of sections ω^t of $C_k(\Gamma)$: given a differentiable mapping, analytic relative to the real parameter t,

$$\begin{aligned} V \times [0, 1] &\to C_k(\Gamma) \\ (x, t) &\to \omega^t(x) \end{aligned}$$

such that for every fixed parameter t, ω^t is a section of $C_k(\Gamma)$ on V.

Theorem 3.1.a. *Given any analytic family of deformations S^t of (V, Γ) with analytic pseudogroup Γ, there is a germ of analytic family of sections ω^t of $C_k(\Gamma)$, i.e., the family is only defined for t in $[0, \varepsilon]$ with some small ε such that for a fixed t the section ω^t is an element of $H^0(V, \Sigma)$ and defines canonically the Γ-structure S^t.*

The proof follows from the same argument as in part (a) of the proof of Theorem 4.1; the reader is asked to verify that in our construction of the family of sections σ_i^t on V_i of (Ψ_k, a, V) with t in $[0, \varepsilon]$, we can require that these families depend now analytically on t.

Remark. We do not know if every Γ-structure on V can be represented by an element of $H^0(V, \Sigma)$. By Theorem 3.1, it is only true for a Γ-structure, which is close to a deformation S^t of (V, Γ) (see also Proposition 3.2).

On the universal space of deformations. Let V be compact, and Γ an analytic elliptic transitive continuous pseudogroup on V. The space $H^0(V, \Sigma)$ has a natural topological structure defined by the uniform convergence of sections in the fiber bundle $C_k(\Gamma)$ on V. As an immediate improvement of Proposition 4.1 one can prove that every continuous mapping of $[0, 1]$ into $H^0(V, \Sigma)$ defines canonically a continuous family of deformations of (V, Γ). Hence by Theorem 4.1, the topological space $H^0(V, \Sigma)$ has the following universal propriety:

A continuous family of deformations of (V, Γ) is equivalent to a continuous mapping of $[0, 1]$ into $H^0(V, \Sigma)$.

The topological space $H^0(V, \Sigma)$ is in this sense an universal space of deformations; it is complete, but evidently not effective [6]. Moreover, the continuous mapping of $[0, 1]$ into $H^0(V, \Sigma)$, associated to a given continuous family of deformations, is not uniquely defined.

Let us then consider in the space $H^0(V, \Sigma)$ the following equivalence relation:

Two elements ω and ω' of $H^0(V, \Sigma)$ are equivalent iff there is a global section η of $\Gamma(\Psi_k)$ such that

(1) $b \circ \eta$ is a diffeomorphism of V into itself,

(2) for any germ σ of $\Gamma(\Psi_k)$, which is a solution of ω, i.e. $\underline{\mathscr{D}}(\sigma) = \omega$, the corresponding germ $\sigma \cdot \eta$ of $\Gamma(\Psi_k)$ is a solution of ω'.

We shall denote by $H^1(V, \Gamma)$ the quotient space of $H^0(V, \Sigma)$ by this equivalence relation. $H^1(V, \Gamma)$, provided with the quotient topology, can be regarded as a Teichmüller space of (V, Γ), namely,

(1) Every continuous family of deformations of (V, Γ) defines uniquely a continuous mapping of $[0, 1]$ into $H^1(V, \Gamma)$. In particular, to every deformation S^t of (V, Γ) one associates a representative class of S^t into $H^1(V, \Gamma)$.

(2) Let S^t and $S^{t'}$ be two deformations of (V, Γ). If there is a diffeomorphism φ of V into itself isotopic to the identity such that the Γ-structure $S^{t'}$ is the transposed $\varphi^*(S^t)$, then the representative classes of S^t

and $S^{t'}$ are identical.

(3) Every element of $H^1(V, \Gamma)$ defines canonically a class of Γ-structures S on V such that two Γ-structures S of the same class are transposed from each other by a diffeomorphism of V into itself.

It should be remarked that as Kodaira and Spencer have pointed out by an explicit example (Chapter VI, § 15 [6]) the topological space $H^1(V, \Gamma)$ is not in general separated.

4. Locally universal space of deformations

It is well known that M. Kuranishi [7] has proved the existence of a locally universal space of deformations for complex structures. In this section we want to reformulate the argument of Kuranishi in our formalism and to prove the existence theorem for general analytic elliptic transitive continuous pseudogroup structure on a compact manifold V.

Let V be a compact manifold, and Γ an analytic elliptic transitive continuous pseudogroup on V. Then we have the following theorem:

Theorem 4.1 (*M. Kuranishi*) [7]. *Given* (V, Γ), *there is a real analytic space* K *of finite dimension with base-point* S^0 *with the following universal proprieties*:

(1) *Any analytic* (*differentiable or continuous*) *mapping of* $[0, 1]$ *into* K, *with* 0 *sent to the base point* S^0, *defines canonically an analytic* (*respectively differentiable, continuous*) *family of deformations of* (V, Γ).

(2) *Given an analytic* (*differentiable or continuous*) *family of deformations of* (V, Γ), *there is a germ of analytic* (*respectively differentiable, continuous*) *mapping of* $[0, 1]$ *into* K, *defined in the neighborhood of* 0 *and with* 0 *sent to the base point* S^0, *defining an equivalent germ of deformations of* (V, Γ) *by* (1).

Remarks.

(1) The analytic space K will be simply the set of "zeros" of an analytic mapping from same open set of R^p into R^q. An analytic (differentiable or continuous) mapping of $[0, 1]$ to K is an analytic (respectively differentiable, continuous) mapping of $[0, 1]$ to R^p with values in K.

(2) Two germs of deformations S_1^t and S_2^t are said to be equivalent iff for every fixed small t the Γ-structures S_1^t and S_2^t on V are transposed from each other by a diffeomorphism of V.

(3) Following Douady, one should say the space K is a locally "versal" space of deformations. Indeed, given a family of deformations of (V, Γ) the corresponding germ of mapping of $[0, 1]$ into K is not uniquely defined; it is uniquely defined, as we shall see only in the case where

$$H^0(V, \Theta) = 0 \,, \qquad \Theta \text{ being the sheaf of } \Gamma\text{-vector fields,}$$

or, in other words, the group of automorphisms of (V, Γ) is a discrete Lie group.

(4) It should be noted that in the case, where V is a compact complex manifold and Γ a pseudogroup of biholomorphic mappings, the space K is an analytic complex space. Furthermore, applying essentially the complex Frobenius theorem with parameters, one can see that there is a structure of complex analytic fibered space on $K \times V$, by the first projection of $K \times V$ on K, such that the induced complex structure on each fiber $\{t\} \times V$ is the complex structure of V "parametrized" by t in K.

Proof of the Kuranishi theorem. The proof will be broken down into several steps. **a.** A_k denotes as previously (Definition 1.2) the involutive linear differential system defining the sheaf of Γ-vector fields. As Γ is analytic elliptic, by theorem of Quillen and Spencer, its "non-naive" linear Spencer sequence

$$A_k \xrightarrow{D} C_k^1 \xrightarrow{D} C_k^2 \xrightarrow{D} C_k^3 \xrightarrow{D} \cdots$$

is an exact elliptic complex of analytic differential operators of order 1 on the compact manifold V[12].

A differential operator $\underline{\mathscr{D}}$ will be an analytic "local" differential operator of order 1 from $C_k(\Gamma)$ into $\underline{C}_k^2$, defined in the neighborhood of the canonical section 1 of the bundle $C_k(\Gamma)$ of homogenuous spaces, if $\underline{\mathscr{D}}$ is defined by an analytic bundle morphism h

$$h\colon W \to C_k^2$$

with W being a bundle neighborhood of the canonical section $j^1 1$ in the bundle $J_1[C_k(\Gamma)]$.

The following proposition gives a local resolution of Σ, certainly not canonical (see Remark 2.2).

Proposition 4.1. *There is an analytic "local" differential operator $\underline{\mathscr{D}}$ of order 1 from $C_k(\Gamma)$ into the vector bundle C_k^2 defined in the neighborhood of the section 1 in $C_k(\Gamma)$ such that*

(i) $\underline{\mathscr{D}}(\omega) = 0$, *the 0 section of C_k^2, iff ω is a germ of Σ,*

(ii) *its tangential linear operator along the section 1 in $C_k(\Gamma)$ is the Spencer operator*

$$\underline{D}\colon \underline{C}_k^1 \to \underline{C}_k^2\ .$$

Proof. Indeed, the bundle C_k^1 is canonically isomorphic to the vector bundle of vertical tangent vectors along the section 1 to $C_k(\Gamma)$ (Proposition 2.3), and the bundle kernel of the vector bundle morphism

$$h(\underline{D})\colon J_1[C_k^1] \to C_k^2$$

is isomorphic to the vector bundle of vertical tangent vectors along the section $j^1 1$ to the sub-bundle S_1 in $J_1[C_k(\Gamma)]$, S_1 being the differential system defining Σ.

By a standard argument [10], there is a bundle neighborhood W of the section $j^1 1$ in $J_1[C_k(\Gamma)]$ and an analytic bundle morphism

$$h: W \to C_k^2$$

such that its tangential vector bundle morphism along the section $j^1 1$ is the vector bundle morphism $h(\underline{D})$, and the inverse image of the 0-section in C_k^2 is the sub-bundle $S_1 \cap W$. The "local" differential operator $\underline{\mathscr{D}}$, defined by the bundle morphism h, is clearly the suitable operator.

b. Let ζ and ζ' be two germs of $\Gamma(\Psi_{k+1})$ such that $\zeta' \cdot \zeta^{-1}$ is defined. We have

$$\mathscr{D}(\zeta' \cdot \zeta^{-1}) = \Omega , \qquad \text{a germ of } \underline{N(\Psi_k)} .$$

By Proposition 5.1, Chapter I,

$$\mathscr{D}(\zeta') = [Ad(\zeta)^{-1}\Omega] \cdot \mathscr{D}(\zeta) .$$

Lemma 4.1. *The germ $\mathscr{F}(\omega, \sigma)$ of $\underline{C_k(\Gamma)}$,*

$$\mathscr{F}(\omega, \sigma) = [Ad(\zeta)^{-1}\Omega] \cdot \mathscr{D}(\zeta) \qquad \operatorname{mod} N_{k+1}(\Gamma) ,$$

depends only on the class germ

$$\omega = \Omega \qquad \operatorname{mod} N_{k+1}(\Gamma)$$

and on the germ

$$\sigma = \pi(\zeta) \quad \textit{of} \quad \Gamma(\Psi_k) .$$

Proof. Let $\zeta_1 = \zeta \cdot \eta$ and $\Omega_1 = \eta' \cdot \Omega$, where η and η' are two germs of $N_{k+1}(\Gamma)$. Then we have

$$[Ad(\zeta_1)^{-1}\Omega_1] \cdot \mathscr{D}(\zeta_1) = \eta^{-1} \cdot [Ad(\zeta)^{-1}\eta'] \cdot [Ad(\zeta)^{-1}\Omega] \cdot \mathscr{D}(\zeta)$$

with $\eta^{-1} \cdot [Ad(\zeta)^{-1}\eta']$ being a germ of $N_{k+1}(\Gamma)$. q.e.d.

In other words, we have defined a morphism of sheaves

$$\begin{aligned} \mathscr{F}\omega \colon \Gamma(\Psi_k) &\to \underline{C_k(\Gamma)} \\ \sigma &\to \mathscr{F}(\omega, \sigma) \end{aligned}$$

for every given global section ω of $C_k(\Gamma)$. This morphism is clearly a differential operator of order 1 from the bundle (Ψ_k, a, V) into the bundle $C_k(\Gamma)$. And for $\omega = 1$, the canonical section of $C_k(\Gamma)$, its tangential linear differential operator is the Spencer linear operator

$$\underline{D} \colon \underline{A_k} \to \underline{C_k^1} .$$

Proposition 4.2. *If ω is a solution of the differential system S_1 in $C_k(\Gamma)$, the same is the section $\mathscr{F}(\omega, \sigma)$ for every section σ of $\Gamma(\Psi_k)$.*

Proof. Recall that the differential system S_1 is the direct image of the bundle morphism

$$h(\underline{\mathscr{D}})\colon ([\Psi_k]_2, a, V) \to J_1[C_k(\Gamma)] .$$

If X is an element of $[\Psi_k]_2$, by Proposition 2.4 we have

$$h(\underline{\mathscr{D}})(X) = Y^{-1}\cdot X \qquad \operatorname{mod} J_1[N_{k+1}(\Gamma)]$$

with Y to be any element of Ψ_{k+2} such that by the canonical mapping π on Ψ_k we have

$$\pi(Y) = \pi(X) .$$

Hence, a section ω of $C_k(\Gamma)$ is a solution of S_1 iff for every given x of V,

$$\begin{aligned} j^1_x\omega &= h(\underline{\mathscr{D}})(X) , && \text{for some } X \text{ in } [\Psi_k]_2 \\ &= Y^{-1}\cdot X && \operatorname{mod} J_1[N_{k+1}(\Gamma)] . \end{aligned}$$

Let σ be a section $\Gamma(\Psi_k)$, and y an element of V such that

$$b \circ \sigma(y) = x .$$

Let X' be the element of $[\Psi_k]_2$:

$$X' = j^2_y\sigma .$$

We have by an immediate computation

$$\begin{aligned} j^1_x\mathscr{F}(\omega, \sigma) &= (Y'^{-1}\cdot Y^{-1}\cdot X\cdot X') \qquad \operatorname{mod} J_1[N_{k+1}(\Gamma)] \\ &= h(\underline{\mathscr{D}})(X\cdot X') . \end{aligned}$$

Remark. In particular, if σ is a section of $\Gamma(\Psi_k)$ such that $b \circ \sigma$ is a diffeomorphism of V into itself, ω and $\mathscr{F}(\omega, \sigma)$, being two sections of Σ, define canonically by Proposition 3.1 two equivalent Γ-structures on V.

c. Let $\underline{D}^*$ be a formal adjoint of the Spencer operator $\underline{D}$:

$$\underline{D}^*\colon \underline{C^1_k} \to \underline{A_k} .$$

As C^1_k is isomorphic to the vector bundle of vertical tangent vectors along the canonical section 1 in $C_k(\Gamma)$, there is a bundle neighborhood W of the section 1 in $C_k(\Gamma)$ isomorphic to a bundle neighborhood of the 0-section in C^1_k [10]. The formal adjoint $\underline{D}^*$, composed with this bundle isomorphism, defines a "local" differential operator $\underline{\mathscr{D}}^*$ in the neighborhood of the section 1 in $C_k(\Gamma)$:

$$\underline{\mathscr{D}}^*\colon \underline{C_k(\Gamma)} \to \underline{A_k} .$$

Proposition 4.3. *Given an analytic (respectively differentiable, continuous) family* $\omega(t)$ *of sections of* $C_k(\Gamma)$, *there is a germ of analytic (respectively differentiable, continuous) family*[9] $\sigma(t)$ *of sections of* $\Gamma(\Psi_k)$ *such that*:

i) $\sigma(0)$ *is the unity-section of* (Ψ_k, a, V),

ii) *for every* t *in the domain of definition of* $\sigma(t)$,

$$\underline{\mathscr{D}}^*[\mathscr{F}(\omega(t), \sigma(t))] = 0 .$$

The germ of family $\sigma(t)$ *is uniquely defined if*

$$H^0(V, \Theta) = 0 .$$

Proof. For every bundle E on V, $\mathscr{C}^{p+\alpha}(E)$ denotes the Banach manifolds of sections of E, of Hölder class $\mathscr{C}^{p+\alpha}$[3]. In the particular case of (Ψ_k, a, V), we have the open submanifold $\mathscr{C}^{p+\alpha}[\Gamma(\Psi_k)]$ of $\mathscr{C}^{p+\alpha}(\Psi_k, a, V)$, the set of sections σ in $\mathscr{C}^{p+\alpha}(\Psi_k, a, V)$ such that composed with the target map b, $b \circ \sigma$ is a submersion of V into itself.

It is easy to see that the morphism $\mathscr{F}$ extends to be an analytic mapping of Banach manifold:

$$\begin{aligned} \mathscr{F}\colon \mathscr{C}^{2+\alpha}[C_k(\Gamma)] \times \mathscr{C}^{2+\alpha}[\Gamma(\Psi_k)] &\to \mathscr{C}^{1+2}[C_k(\Gamma)] \\ (\omega, \sigma) &\to \mathscr{F}(\omega, \sigma) . \end{aligned}$$

So a germ of analytic mapping

$$\underline{\mathscr{D}}^* \circ \mathscr{F}\colon \mathscr{C}^{2+\alpha}[C_k(\Gamma)] \times \mathscr{C}^{2+\alpha}[\Gamma(\Psi_k)] \to \mathscr{C}^{\alpha}[A_k]$$

defines in the neighborhood of the element $(1, 1)$, respectively, the canonical section of $C_k(\Gamma)$ and the unity-section of (Ψ_k, a, V) with

$$\underline{\mathscr{D}}^* \circ \mathscr{F}(1, 1) = 0 , \qquad \text{the 0-section of } A_k .$$

The partial tangential mapping of $\underline{\mathscr{D}}^* \circ \mathscr{F}$ relative to the space $\mathscr{C}^{2+\alpha}[\Gamma(\Psi_k)]$, at the element $(1, 1)$, is the mapping

$$\square = \underline{D}^* \circ \underline{D}\colon \mathscr{C}^{2+\alpha}(A_k) \to \mathscr{C}^{\alpha}(A_k) .$$

By the Hodge-Kodaira theory, we have the split exact sequence of Banach vector spaces

$$0 \longrightarrow H^0(V, \Theta) \xrightarrow{\ i\ } \mathscr{C}^{2+\alpha}(A_k) \xrightarrow{\ \square\ } \mathscr{C}^{\alpha}(A_k) \xrightarrow{\ H\ } H^0(V, \Theta) \longrightarrow 0 ,$$

where i is the canonical injection, and H the orthogonal projection in the sense of L^2 on the "harmonic" space.

[9] The family $\sigma(t)$ is analytic, differentiable or continuous in the sense of a family of sections in the analytic bundle (Ψ_k, a, V).

Hence, we can apply the implicit function theorem to define a germ of analytic mapping in the neighborhood of the section 1 in $C_k(\Gamma)$:

$$\mathscr{C}^{2+\alpha}[C_k(\Gamma)] \to \mathscr{C}^{2+\alpha}[\Gamma(\Psi_k)]$$
$$\omega \to \sigma[\omega]$$

such that

(1) $\sigma(1) = 1$, the unity-section of (Ψ_k, a, V),

(2) $\underline{\mathscr{D}}^* \circ \mathscr{F}(\omega, \sigma[\omega]) = 0$.

The germ of analytic mapping $\sigma[\omega]$ is uniquely defined if $H^0(V, \Theta) = 0$. Clearly, if ω^t is an analytic, differentiable or continuous family of sections, we have a germ of respectively analytic, differentiable or continuous family of sections σ^t:

$$\sigma^t = \sigma[\omega^t] .$$

The proposition is then proved if given a section ω of class $\mathscr{C}^\infty$ in $C_k(\Gamma)$, the section $\sigma[\omega]$, which is by construction of Hölder class $\mathscr{C}^{2+\alpha}$, is in fact differentiable of class $\mathscr{C}^\infty$ in (Ψ_k, a, V). Indeed, as ω is a differentiable section of class $\mathscr{C}^\infty$, the "local" operator $\underline{\mathscr{D}}^* \circ \mathscr{F}_\omega$ is a ($\mathscr{C}^\infty$-differentiable) differential operator:

$$\underline{\mathscr{D}}^* \circ \mathscr{F}_\omega \colon \Gamma(\Psi_k) \to \underline{A_k}$$
$$\sigma \to \underline{\mathscr{D}}^* \circ \mathscr{F}(\omega, \sigma) .$$

Moreover, if ω is close to the canonical section 1 of $C_k(\Gamma)$, this nonlinear differential operator is easily seen to be elliptic in the sense of Morrey [8], i.e., its tangential linear operator along the section 1 of (Ψ_k, a, V) is an elliptic differential operator of A_k into itself. Hence, by the regularity theorem of Morrey, the solution section $\sigma[\omega]$, which is itself close to the unity-section 1 of (Ψ_k, a, V),

$$\underline{\mathscr{D}}^* \circ \mathscr{F}_\omega(\sigma[\omega]) = 0$$

is $\mathscr{C}^\infty$-differentiable. It should be pointed out that Morrey deals with strongly elliptic differential operators, and the tangential linear differential operator of $\underline{\mathscr{D}}^* \circ \mathscr{F}_\omega$ is not necessarily strongly elliptic, but the regularity theorem of Morrey is still relevant in the case of elliptic operators.

d. We construct now the Kuranishi analytic space K and prove its universal propriety. The "local" differential operators $\underline{\mathscr{D}}$ and $\underline{\mathscr{D}}^*$ of $C_k(\Gamma)$ respectively to C_k^2 and A_k induce a germ of analytic mapping [10] of Banach manifolds

$$\underline{\mathscr{D}} \times \underline{\mathscr{D}}^* \colon \mathscr{C}^{1+\alpha}[C_k(\Gamma)] \to \mathscr{C}^\alpha[C_k^2] \times \mathscr{C}^\alpha[A_k] .$$

This germ is defined in the neighborhood of the canonical section 1 and sends this section to the section (0, 0) of $C_k^2 \times A_k$. By Proposition 4.1 and definition of the operator $\underline{\mathscr{D}}^*$, its tangential linear mapping is the linear operator

$$\underline{D} \times \underline{D}^*: \mathscr{C}^{1+\alpha}[C_k^1] \to \mathscr{C}^\alpha[C_k^2] \times \mathscr{C}^\alpha[A_k] .$$

Applying the Hodge-Kodaira theory to the exact elliptic complex, which is the "nonnaive" Spencer sequence (see Part **a**), we see that this linear operator is a split homomorphism of Banach vector spaces, and has as kernel the finite dimensional "harmonic" vector space, isomorphic to $H^1(V, \Theta)$ (the Spencer sequence defines indeed a fine resolution of the sheaf of Γ-vector fields [12]).

Let K be the set of "zeros" of the germ of analytic mapping $\underline{\mathscr{D}} \times \underline{\mathscr{D}}^*$, i.e., the set of ω in $\mathscr{C}^{1+\alpha}[C_k(\Gamma)]$ such that

$$\underline{\mathscr{D}} \times \underline{\mathscr{D}}^*(\omega) = (0, 0) .$$

Let G be the orthogonal supplementary (in the sense of L^2) to the "harmonic" space in $\mathscr{C}^{1+\alpha}[C_k^1]$, and B the orthogonal supplementary to the close range of $\underline{D} \times \underline{D}^*$ in $\mathscr{C}^\alpha[C_k^2] \times \mathscr{C}^\alpha[A_k]$. By the implicit function theorem and modulo a germ of analytic isomorphism of $\mathscr{C}^{1+\alpha}[C_k(\Gamma)]$ with $H^1(V, \Theta) \times G$, the set K is the set of "zeros" of a germ of an analytic mapping f:

$$f: H^1(V, \Theta) \to B .$$

Hence, by a theorem of Douady (Proposition 7, Chapter 7 [2]), the set K is an analytic space of finite dimension (see Remark (1)).

The "local" differential operator

$$\underline{\mathscr{D}} \times \underline{\mathscr{D}}^*: \underline{C_k(\Gamma)} \to \underline{C_k^2} \times \underline{A_k}$$

is an analytic elliptic differential operator in the sense of Morrey. By the regularity theorem of Morrey, every element in K is in fact an analytic section of $C_k(\Gamma)$, which is a solution of the differential system S_1. The space K has then clearly the first universal propriety of the Kuranishi theorem. By Propositions 4.2 and 4.3, one sees also easily that the analytic space K verifies the second universal propiety of the Kuranishi theorem.

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