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VECTOR FORMS AND INTEGRAL FORMULAS FOR HYPERSURFACES IN EUCLIDEAN SPACE

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Introduction

Let Σ be a smooth oriented m-dimensional hypersurface immersed in $(m+1)$-dimensional Euclidean space E^{m+1}. In § 2, we consider some vector form invariants for Σ and their expansions in terms of elementary symmetric functions of pricipal curvatures and certain intrinsic tangent vectors. We use these results in § 3 to obtain integral formulas for Σ assuming that Σ has closed regular boundary. For a compact Σ we have integral formulas of particular interest in Corollary 2 of Theorem 3.1; these are similar to Minkowski formulas and involve gradients of elementary symmetric functions of principal curvatures. Some consequences of these formulas are studied in § 4. In Theorem 3.3 we prove that for a compact hypersurface of constant mean curvature, the surface integral of the gradient of any elementary symmetric function of principal curvatures is identically zero.

1. Preliminaries

Let M be an oriented smooth differentiable manifold of dimension m. Our hypersurface Σ is a mapping $X\colon M \to E^{m+1}$ where the Jacobian matrix has rank m everywhere. Let $n(x)$, $x \in M$, be a unit normal to Σ at $X(x)$. Then choosing an orthonormal frame $e_1, \cdots, e_m$ in the tangent space of Σ at $X(x)$ such that the det $(e_1, \cdots, e_m, n) = 1$, we have

$$(1.1) \qquad dX = \sum_i \sigma_i e_i , \qquad dn = \sum_i \omega_i e_i ,$$

where σ_i and ω_i are differential 1-forms. We express ω_i in terms of the linearly independent σ_i:

$$(1.2) \qquad \omega_i = \sum_j a_{ij}\sigma_j ,$$

where $\|a_{ij}\|$ is symmetric.

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Let $k_1, \cdots, k_m$ denote the principal curvatures at $X(x)$, and $K_1, \cdots, K_m$ the elementary symmetric functions of the principal curvatures, that is,

$$\binom{m}{r} K_r = \sum k_1 \cdots k_r, \qquad 1 \leq r \leq m . \tag{1.3}$$

As usual we assume $K_0 = 1$.

We list below a few formulas for easy reference. For other relevant details we refer to Flanders [2], [3] and Chern [1].

$$[e_1, \cdots, e_m] = n , \tag{1.4}$$

$$[n, \cdots, \hat{e}_j, \cdots, e_m] = (-1)^j e_j , \tag{1.5}$$

where the roof indicates the missing term.

$$[n, \underbrace{dX, \cdots, dX}_{m-1}] = -(m-1)! * dX , \tag{1.6}$$

$$dn \cdot * dX = mK_1 \sigma , \qquad dX \cdot * dX = m\sigma , \tag{1.7}$$

where $\sigma = \sigma_1 \wedge \cdots \wedge \sigma_m$ is the volume element.

$$[\underbrace{dn, \cdots, dn}_{r}, \underbrace{dX, \cdots, dX}_{m-r}] = r!(m-r)! \binom{m}{r} K_r \sigma n . \tag{1.8}$$

By exterior differentiation of (1.6) we have

$$[dn, \underbrace{dX, \cdots, dX}_{m-1}] = -(m-1)! d * dX .$$

But from (1.8) we see that the left hand member is $(m-1)! m K_1 \sigma n$. Hence we get

$$d * dX = -mK_1 \sigma n . \tag{1.9}$$

An immediate consequence of (1.8) is that for a compact hypersurface $\sum$ we have

$$\int_\Sigma K_r \sigma n = 0, \qquad r = 1, \cdots, m , \tag{1.10}$$

that is, the vector surface integral of any elementary symmetric function of principal curvatures is identically zero. The proof of (1.10) is obvious from the fact that

$$[\underbrace{dn, \cdots, dn}_{r}, \underbrace{dX, \cdots, dX}_{m-r}] = d[n, \underbrace{dn, \cdots, dn}_{r-1}, \underbrace{dX, \cdots, dX}_{m-r}] ,$$

where d stands for exterior differentiation.

Let f be a smooth function defined on $\sum$. By grad$f$ or ∇f we mean $\nabla f = \sum_i f_i e_i$, where f_i are given by $df = \sum_i f_i \sigma_i$. We have

$$df \wedge *dX = (\nabla f)\sigma . \tag{1.11}$$

We consider a formula for the divergence of a tangent vector $\boldsymbol{a}$ in the tangent space of $\sum$ at $X(x)$.

Let $\boldsymbol{a} = \sum a_i e_i$, where a_i are smooth functions. Then

$$d\boldsymbol{a} = \sum_j \left(da_j + \sum_i a_i \omega_{ij} \right) e_j - \left(\sum_i a_i \omega_i \right) \boldsymbol{n} ,$$

where ω_{ij} and ω_i are 1-forms. (For details see Flanders [2].) We write

$$\omega_{ij} = \sum_k \Gamma_i{}^j{}_k \sigma_k , \qquad da_j = \sum_l (a_j)_l \sigma_l .$$

Then

$$\begin{aligned} d\boldsymbol{a} \cdot *dX &= \sum_j \left\{ \sum_l (a_j)_l \sigma_l \wedge *\sigma_j + \sum_i \sum_k a_i \Gamma_i{}^j{}_k \sigma_k \wedge *\sigma_j \right\} \\ &= \sum_j \left\{ (a_j)_j + \sum_i a_i \Gamma_i{}^j{}_j \right\} \sigma \\ &= (\mathrm{div}\, \boldsymbol{a})\sigma . \end{aligned}$$

Thus

$$d\boldsymbol{a} \cdot *dX = (\mathrm{div}\, \boldsymbol{a})\sigma . \tag{1.12}$$

Since

$$\begin{aligned} d(\boldsymbol{a} \cdot *dX) &= d\boldsymbol{a} \cdot *dX - \boldsymbol{a} \cdot mK_1 \sigma \boldsymbol{n} \\ &= (\mathrm{div}\, \boldsymbol{a})\sigma , \end{aligned}$$

it follows that for a compact hypersurface $\sum$ and tangent vector field $\boldsymbol{a}$

$$\int_\Sigma (\mathrm{div}\, \boldsymbol{a})\sigma = 0 . \tag{1.13}$$

Finally we consider an algebraic identity for the elementary symmetric functions of the principal curvatures.

Definition 1.1. Let C_r denote the rth elementary symmetric function of the principal curvatures, that is, let $C_r = \binom{m}{r} K_r$. For a fixed integer i, $1 \leq i \leq m$, and any integer j such that $1 \leq j \leq m$, we define

$$C_j^i = \sum k_1 \cdots k_j$$

where in each product, the j curvatures are chosen from the $m-1$ curvatures $k_1, \cdots, k_{i-1}, k_{i+1}, \cdots, k_m$. It is convenient to define $C_0^i = 1$.

Lemma 1.1.

$$C_r^i = \sum_{j=0}^{r} \binom{m}{r-j} (-1)^j K_{r-j} (k_i)^j . \tag{1.14}$$

Proof. We have the recursive relations:

$$\begin{aligned} C_r^i &= C_r - k_i C_{r-1}^i , \\ C_{r-1}^i &= C_{r-1} - k_i C_{r-2}^i , \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ C_1^i &= C_1 - k_i C_0^i = C_1 - k_i . \end{aligned}$$

Hence

$$\begin{aligned} C_r^i &= C_r - k_i(C_{r-1} - k_i C_{r-2}^i) \\ &= C_r - k_i C_{r-1} + k_i^2 C_{r-2}^i \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &= C_r - k_i C_{r-1} + k_i^2 C_{r-2} - \cdots + (-1)^r k_i^r . \end{aligned}$$

The Lemma follows from the fact that $C_r, C_{r-1}, \cdots, C_1$ are respectively the rth, $(r-1)$th, $\cdots$, 1st elementary symmetric functions of the principal curvatures.

As a corollary to Lemma 1.1, it is possible to deduce the following identity of Newton for the elementary symmetric functions:

$$\begin{aligned} r\binom{m}{r} K_r = m\binom{m}{r-1} K_{r-1} K_1 - \binom{m}{r-2} K_{r-2} \sum_{i=1}^{m} k_i^2 \\ + \cdots + (-1)^{r-1} \sum_{i=1}^{m} k_i^r . \end{aligned} \tag{1.15}$$

2. Differential formulas

A self adjoint linear transformation A of the tangent space of $\sum$ at $X(x)$ into itself is defined by (see Flanders [2])

$$A e_i = \sum_j a_{ij} e_j , \tag{2.1}$$

where the symmetric matrix $\|a_{ij}\|$ is given by (1.2). It follows that

$$(2.2) \qquad AdX = A \sum_i \sigma_i e_i = \sum_i \sigma_i A e_i = \sum_{i,j} \sigma_i a_{ij} e_j = \sum_i \omega_i e_i = dn .$$

We look for other intrinsic tangent vectors which are obtained as the result of repeated application of the transformation A to dX. Let $A^{(j)}dX$ denote the intrinsic tangent vector obtained from dX by applying A repeatedly j times. For convenience we write

$$(2.3) \qquad U_0 = dX , \quad U_j = A^{(j)}dX , \quad 1 \leq j \leq m .$$

Definition 2.1. An orthonormal frame $e_1, \cdots, e_m$ will be called a principal frame if each e_i is tangent to a principal direction.

Since the tangent vectors U_j are intrinsic, we can use any admissible frame locally to describe their components. If $X(x)$ is a non-umbilic point we have a well defined principal frame at $X(x)$. With reference to this frame we have

$$(2.4) \qquad \omega_i = \sigma_i k_i \qquad (i \text{ not summed}), i = 1, \cdots, m .$$

The components of U_j assume a simple form and are given by

$$(2.5) \qquad U_i = \sum_j (k_j)^i \sigma_j e_j .$$

Lemma 2.1. *Let*

$$\Delta_r = [n, \underbrace{dn, \cdots, dn}_{r}, \underbrace{dX, \cdots, dX}_{m-r-1}] .$$

Then we have

$$(2.6) \qquad \Delta_r = -r!(m-r-1)! \sum_{i=0}^{r} (-1)^i \binom{m}{r-i} K_{r-i}{*}U_i ,$$

where U_i are the vectors defined in (2.3).

Proof. Since we are concerned with proving a local result, we can choose the principal frame for computational purpose. We do this and use (2.4) to get

$$\begin{aligned} \Delta_r &= [n, \sum k_{i_1}\sigma_{i_1}e_{i_1}, \cdots, \sum k_{i_r}\sigma_{i_r}e_{i_r}, \sum \sigma_{j_1}e_{j_1}, \cdots, \sum \sigma_{j_{m-r-1}}e_{j_{m-r-1}}] \\ &= \sum_j B_j[n, e_1, \cdots, \hat{e}_j, \cdots, e_m] , \end{aligned}$$

where B_j is a $(m-1)$th order determinant given by

$$B_j = \begin{vmatrix} k_1\sigma_1 & \cdots & k_{j-1}\sigma_{j-1} & k_{j+1}\sigma_{j+1} & \cdots & k_m\sigma_m \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ k_1\sigma_1 & \cdots & k_{j-1}\sigma_{j-1} & k_{j+1}\sigma_{j+1} & \cdots & k_m\sigma_m \\ \sigma_1 & \cdots & \sigma_{j-1} & \sigma_{j+1} & \cdots & \sigma_m \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_1 & \cdots & \sigma_{j-1} & \sigma_{j+1} & \cdots & \sigma_m \end{vmatrix} .$$

In B_j, the first r rows are identical and so are the last $m - r - 1$ rows. In the expansion of B_j the multiplication of differential forms is in the sense of exterior multiplication.

Use of (1.5) yields

$$\Delta_r = \sum_j (-1)^j B_j e_j . \tag{2.7}$$

In expanding B_j we use Laplace's method of expansion by complimentary minors. Let $H = (h_1, \cdots, h_r)$, $L = (l_1, \cdots, l_{m-r-1})$, where

$$1 \leq h_1 < \cdots < h_r \leq m ,$$
$$1 \leq l_1 < \cdots < l_{m-r-1} \leq m ,$$

and the range of each h_i and each l_i is $(1, \cdots, j-1, j+1, \cdots m)$. Let $(k\sigma)_H$ denote an $r \times r$ minor of B_j, each row of which is $k_{h_1}\sigma_{h_1} \cdots k_{h_r}\sigma_{h_r}$. Then

$$(k\sigma)_H = r!(k_{h_1} \cdots k_{h_r})\sigma_{h_1} \wedge \cdots \wedge \sigma_{h_r} .$$

Similarly, if σ_L denotes $(m - r - 1) \times (m - r - 1)$ minor of B_j, each row of which is $\sigma_{l_1} \cdots \sigma_{l_{m-r-1}}$, then

$$\sigma_L = (m - r - 1)!\sigma_{l_1} \wedge \cdots \wedge \sigma_{l_{m-r-1}} ,$$

and

$$B_j = \sum_{H,L} \varepsilon^{H,L}(k\sigma)_H \wedge \sigma_L ,$$

where

$$\varepsilon^{H,L} = \operatorname{sgn}\begin{pmatrix} 1 \cdots j-1 & j+1 \cdots m \\ h_1 \cdots h_r & \cdots l_1 \cdots l_{m-r-1} \end{pmatrix}.$$

Hence

$$B_j = r!(m - r - 1)!\sigma_1 \wedge \cdots \wedge \hat{\sigma}_j \wedge \cdots \wedge \sigma_m C_r^j ,$$

where C_r^j is a function of the principal curvatures (see Definition 1.1). Substituting the expression for C_r^j from (1.14) we get

$$B_j = r!(m-r-1)!\sigma_1 \wedge \cdots \wedge \hat{\sigma}_j \wedge \cdots \\ \cdots \wedge \sigma_m \sum_{i=0}^{r} (-1)^i \binom{m}{r-i} K_{r-i}(k_j)^i .$$

Hence

$$(-1)^j B_j e_j = -r!(m-r-1)! \sum_{i=0}^{r} (-1)^i \binom{m}{r-i} K_{r-i}(k_j)^i * \sigma_j e_j ,$$

where

$$*\sigma_j = (-1)^{j-1}\sigma_1 \wedge \cdots \wedge \hat{\sigma}_j \wedge \cdots \wedge \sigma_m .$$

Thus finally using (2.5) we have, from (2.7),

$$\begin{aligned} \varDelta_r &= -r!(m-r-1)! \sum_{i=0}^{r} (-1)^i \binom{m}{r-i} K_{r-i} \left\{ \sum_{j=1}^{m} (k_j)^i * \sigma_j e_j \right\} \\ &= -r!(m-r-1)! \sum_{i=0}^{r} (-1)^i \binom{m}{r-i} K_{r-i} * U_i . \end{aligned}$$

Remark. From (2.2) we have $AdX = dn$, and from (2.3) it follows that $A^{(i)} * dX = * U_i$. Hence (2.6) may also be expressed in the form

$$\begin{aligned} \varDelta_r &= [n, \underbrace{AdX, \cdots, AdX}_{r}, \underbrace{dX, \cdots, dX}_{m-r-1}] \\ &= -r!(m-r-1)! \sum_{i=0}^{r} (-1)^i \binom{m}{r-i} K_{r-i} A^{(i)} * dX . \end{aligned} \tag{2.8}$$

Corollaries.

1. Let $r = 0$. Then from (2.6) we get the known formula (1.6).
2. Let $r = m - 1$. Then

$$\varDelta_{m-1} = [n, \underbrace{dn, \cdots, dn}_{m-1}] = -(m-1)! \star dn ,$$

where $\star$ is the star operator on the m-sphere which is the Gauss map of $\sum$. From (2.6) we get

$$\star dn = \sum_{i=0}^{m-1} (-1)^i \binom{m}{m-1-i} K_{r-i} * U_i . \tag{2.9}$$

3. In Chern's notations [1],

$$A_{m-r-1} = X \cdot \varDelta_r .$$

Lemma 2.2. *Let* $X = v + pn$, *where* $v = \sum p_i e_i$ *is the component of* X *tangential to the hypersurface* $\sum$, *and* p *is the support function. Then*

$$(2.10) \qquad [X, \underbrace{dX, \cdots, dX}_{m-1}] = (m-1)!((v \cdot *dX)n - p*dX) ,$$

$$(2.11) \qquad \operatorname{div} v = m(1 - pK_1) , \qquad \nabla p = \sum_i p_i k_i e_i .$$

Proof. By the linearity of the vector form we have

$$[X, dX, \cdots, dX] = [v, dX, \cdots, dX] + p[n, dX, \cdots, dX] .$$

It follows from (1.6) that the last term on the right side is $-(m-1)!p*dX$. Let $\Delta = [v, dX, \cdots, dX]$. Then

$$\Delta = [\sum p_{i_1} e_{i_1}, \sum \sigma_{i_2} e_{i_2}, \cdots, \sum \sigma_{i_m} e_{i_m}]$$
$$= \begin{vmatrix} p_1 & p_2 \cdots & p_m \\ \sigma_1 & \sigma_2 \cdots & \sigma_m \\ \cdot & \cdot \quad \cdot & \cdot \\ \sigma_1 & \sigma_2 \cdots & \sigma_m \end{vmatrix} [e_1, e_2, \cdots, e_m] ,$$

where the last $m - 1$ rows of the determinant are identical. Using (1.4) and observing that the cofactor of p_i is $(m-1)!*\sigma_i$ we get

$$\Delta = (m-1)!(\sum p_i * \sigma_i)n = (m-1)!(\nu \cdot *dX)n .$$

Now exterior differentiation of (2.10) and use of (1.8) give

$$m!\sigma n = (m-1)![(dv \cdot *dX + v \cdot d*dX)n + dn \wedge (v \cdot *dX) \\ - dp \wedge *dX - pd*dX] .$$

Using (1.9) and (1.12) and observing that v is a tangent vector we have

$$(2.12) \qquad m\sigma n = (\operatorname{div} v)\sigma n + \sum p_i k_i e_i \sigma - \nabla p \sigma + mpK_1 \sigma n .$$

Equating the tangential and normal components in (2.12) we get (2.11).

Corollary 1. *From* (2.11) *we get the known result* [3]:

$$(2.13) \qquad dp = \sum p_i \omega_i .$$

Proof. $dp = \nabla p \cdot dX = \sum \sigma_i k_i p_i = \sum \omega_i p_i$.

Corollary 2. *If* $\sum$ *is a minimal hypersurface, then* $K_1 = 0$, *and* (2.11) *shows that* $\operatorname{div} v =$ *constant at each point of* $\sum$.

3. Integral formulas

Theorem 3.1. *For a smooth and oriented m-dimensional hypersurface $\sum$ with closed regular boundary,*

$$\binom{m}{r}\left[\int_{\Sigma} X\cdot\nabla K_r\sigma - m\int_{\Sigma}(K_1K_r - K_{r+1})p\sigma\right] = r\binom{m}{r}\int_{\Sigma}(K_{r+1}p - K_r)\sigma - \sum_{i=1}^{r}(-1)^i\binom{m}{r-i}\int_{\partial\Sigma}K_{r-i}X\cdot * U_i\ , \tag{3.1}$$

$$r = 0, 1, \cdots, m-1\ ,$$

where $p = X\cdot n$ is the support function, and the vectors U_i are given by (2.3).

Proof. We have, from (2.6),

$$\Delta_r = -r!(m-r-1)!\left\{\binom{m}{r}K_r * dX + \sum_{i=1}^{r}(-1)^i\binom{m}{r-i}K_{r-i} * U_i\right\}.$$

By exterior differentiation and using (1.8), (1.9) and (1.11) we obtain

$$(r+1)\binom{m}{r+1}K_{r+1}\sigma n = -\left\{\binom{m}{r}\nabla K_r\sigma - m\binom{m}{r}K_1K_r\sigma n + \sum_{i=1}^{r}(-1)^i\binom{m}{r-i}d(K_{r-i} * U_i)\right\}.$$

Taking scalar product with X we have

$$(r+1)\binom{m}{r+1}K_{r+1}\sigma p = -\binom{m}{r}\{X\cdot\nabla K_r\sigma - mK_1K_r\sigma p\} - \sum_{i=1}^{r}(-1)^i\binom{m}{r-i}X\cdot d(K_{r-i} * U_i)\ .$$

Since

$$d(K_{r-i}X\cdot * U_i) = K_{r-i}dX\cdot * U_i + X\cdot d(K_{r-i} * U_i) = K_{r-i}\sum_j (k_j)^i\sigma + X\cdot d(K_{r-i} * U_i)\ ,$$

using (2.5), we have

$$(r+1)\binom{m}{r+1}K_{r+1}p\sigma = -\binom{m}{r}\{X\cdot\nabla K_r\sigma - mK_1K_rp\sigma\} - \sum_{i=1}^{r}(-1)^i\binom{m}{r-i}\{d(K_{r-i}X\cdot * U_i) - K_{r-i}\sum_j (k_j)^i\sigma\}\ . \tag{3.2}$$

But

$$\sum_{i=1}^{r}(-1)^{i-1}\binom{m}{r-i}K_{r-i}\sum_j (k_j)^i = r\binom{m}{r}K_r\ ,$$

by Newton's formula for symmetric functions (see (1.15)). Substituting this value in (3.2) and integrating we get, by Stokes' theorem,

$$(r+1)\binom{m}{r+1}\int_\Sigma K_{r+1}p\sigma = \binom{m}{r}\left[-\int_\Sigma X\cdot\nabla K_r\sigma + m\int_\Sigma K_1K_rp\sigma - r\int_\Sigma K_r\sigma\right]$$
$$-\sum_{i=1}^{r}(-1)^i\binom{m}{r-i}\int_{\partial\Sigma}K_{r-i}X\cdot *U_i\ .$$

Observing that $(r+1)\binom{m}{r+1} = (m-r)\binom{m}{r}$ and rearranging we get (3.1).

Corollary 1. *For a hypersurface* $\sum$ *with the same properties as in Theorem* 3.1 *we have*

$$(3.3)\qquad (m-r)\binom{m}{r}\int_\Sigma (K_{r+1}p - K_r)\sigma = -\sum_{i=0}^{r}(-1)^i\binom{m}{r-i}\int_{\partial\Sigma}K_{r-i}X\cdot *U_i\ ,$$

and if $\sum$ *is compact, then we have the Minkowski equations*

$$(3.4)\qquad \int_\Sigma K_{r+1}p\sigma = \int_\Sigma K_r\sigma\ , \qquad r = 0, 1, \cdots, m-1\ .$$

Proof. We have

$$d(K_r * dX) = \nabla K_r\sigma - mK_1K_r\sigma n\ .$$

Scalar product with X gives

$$X\cdot d(K_r * dX) = X\cdot\nabla K_r\sigma - mK_1K_r\sigma p\ .$$

But

$$\begin{aligned} d(K_rX\cdot * dX) &= K_r dX\cdot * dX + X\cdot d(K_r * dX) \\ &= mK_r\sigma + X\cdot\nabla K_r\sigma - mK_1K_r\sigma p\ . \end{aligned}$$

Substituting (3.5) in (3.1) we get (3.3).

If $\sum$ is compact the right side member of (3.3) drops out and we get (3.4).

Corollary 2. *If* $\sum$ *is compact and oriented, then*

$$(3.6)\qquad \int_\Sigma X\cdot\nabla K_r\sigma = m\int_\Sigma (K_1K_r - K_{r+1})p\sigma, \qquad r = 0, 1, \cdots, m-1\ .$$

Proof. The result follows from (3.1) and the Minkowski equations (3.4).

Remark 1. For a hypersurface $\sum$ satisfying the conditions of Theorem 3.1, from (3.5) we have

$$\int_{\partial\Sigma} K_r X\cdot *dX = \int_{\Sigma} X\cdot \nabla K_r \sigma - m\int_{\Sigma}(K_1K_r p - K_r)\sigma\,, \tag{3.7}$$
$$r = 0, 1, \cdots, m-1\,.$$

And if $\sum$ is compact, using (3.4) we get equations (3.6).

Remark 2. Equations (3.6) can also be expressed in the form

$$\int_{\Sigma} X\cdot \nabla K_r\sigma = m\int_{\Sigma} K_1K_r p\sigma - m\int_{\Sigma} K_r\sigma\,. \tag{3.8}$$

Remark 3. Formulas similar to (3.6) and (3.8) are known for a closed curve C in E^3.

Let $C\colon X = X(s)$ be a smooth curve in E^3, k the curvature and t the unit tangent vector at $X(s)$. Then

$$d(X\cdot kt) = dX\cdot kt + X\cdot(dk)t + X\cdot kdt\,.$$

But

$$dX = (ds)t\,,\quad dk = (ds)k'\,,\quad dt = knds\,,$$

where n is the principal normal. Hence

$$\oint (X\cdot\nabla k)ds = \oint k^2pds - \oint kds\,, \tag{3.9}$$

where $p = X\cdot n$, n is considered along the outward normal, and $\nabla k = k't$.

Similarly, by considering $d(X\cdot\tau t)$ where τ is the torsion of C at $X(s)$, we obtain

$$\oint (X\cdot\nabla\tau)ds = \oint k\tau pds - \oint \tau ds\,. \tag{3.10}$$

Remark 4. From (2.11), for a hypersurface $\sum$ with the properties of Theorem 3.1 we get

$$\int_{\Sigma}\operatorname{div} v\sigma = m\int_{\Sigma}(1 - pK_1)\sigma\,. \tag{3.11}$$

But

$$(\operatorname{div} v)\sigma = dv\cdot *dX = d(v\cdot *dX) = d(X\cdot *dX)\,,$$

since v and $*dX$ are tangent vectors, and $d{*dX} = -mK_1\sigma n$. Hence from (3.11) we get

$$\int_{\partial\Sigma} X\cdot *dX = m\int_{\Sigma}(1 - pK_1)\sigma\,,$$

which is precisely the equation we get from (3.3) by putting $r = 0$.

Theorem 3.2. *For a compact smooth oriented hypersurface $\sum$ of constant mean curvature,*

$$\int_{\Sigma} \nabla K_r \sigma = 0 , \qquad r = 1, \cdots, m . \tag{3.12}$$

Proof. From Theorem 2 of [3] we have

$$\int_{\Sigma} \nabla f \sigma = m \int_{\Sigma} f K_1 \sigma \boldsymbol{n} ,$$

where f is a smooth function on $\sum$. Since all the elementary symmetric functions of the principal curvatures are smooth functions on $\sum$ we have

$$\int_{\Sigma} \nabla K_r \sigma = m \int_{\Sigma} K_r K_1 \sigma \boldsymbol{n}, \qquad r = 1, \cdots, m.$$

Since $\sum$ is assumed to be of constant mean curvature we get

$$\int_{\Sigma} \nabla K_r \sigma = m K_1 \int_{\Sigma} K_r \sigma \boldsymbol{n} .$$

But from (1.10) it follows that $\int_{\Sigma} K_r \sigma \boldsymbol{n} = 0$, $r = 1, \cdots, m$. Hence we get equations (3.12).

4. Some consequences

For a compact and oriented hypersurface $\sum$, C. C. Hsiung [4] has shown that if $K_i > 0$, $i = 1, \cdots, s$, $1 \leq s \leq n$, $K_s =$ constant and p keeps the same sign at all points of $\sum$, then $\sum$ is a hypersphere. This result follows as an immediate consequence of Corollary 2 of Theorem 3.1.

A variation of the above result is obtained, if instead of requiring p to keep the same sign at all points of $\sum$ we assume that the mean curvature K_1 of $\sum$ is constant. To this end we have

Theorem 4.1. *Let $\sum$ be a compact and oriented hypersurface. If $K_1 =$ constant, $K_i > 0$, $i = 1, \cdots, s$, $2 \leq s \leq n$, and $K_s =$ constant, then $\sum$ is a hypersphere.*

Proof. Under the hypothesis of the theorem, we have

$$K_1 K_{s-1} \geq K_s . \tag{4.1}$$

Since $K_1 =$ constant, from (3.6) we have

$$\int_{\Sigma} X \cdot \nabla K_r \sigma = mK_1 \int_{\Sigma} K_r p\sigma - \int_{\Sigma} K_{r+1} p\sigma$$
$$= m \int_{\Sigma} (K_1 K_{r-1} - K_r)\sigma$$

using Minkowski equations.

Further, if K_s = constant, we have

$$0 = \int_{\Sigma} (K_1 K_{s-1} - K_s)\sigma ,$$

which together with (4.1) implies that the equality $K_1 K_{s-1} = K_s$ should hold. The equality in its turn implies that $\sum$ is a hypersphere.

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