J. DIFFERENTIAL GEOMETRY
2 (1968) 299-312

HOLOMORPHIC MAPPINGS OF COMPLEX MANIFOLDS

YUNG-CHEN LU

1. Introduction

Schwarz's lemma, as formulated by Pick, can be stated as follows: Every holomorphic map f of the open unit disk D into itself is distance-decreasing with respect to the Poincáre-Bergman metric ds^2, i.e. $f^*(ds^2) \le ds^2$, where the equality holding at one point of D, implies that f is an isometry. Bochner and Martin proved in their book [2] the following generalization of Schwarz's lemma to higher dimensions. Let D_n be the n-dimensional open unit ball. If f is a holomorphic map of D_m into D_n such that $f(0) = 0$, then $f(z) \le z$ for all $z \in D_m$. In other words, every holomorphic map of D_m into D_n is distance-decreasing with respect to the Bergman metric $ds^2_{D_m}$ and $ds^2_{D_n}$ of D_m and D_n respectively. Koranyi proved [9] that if M is a hermitian symmetric space of non-compact type with the Bergman metric ds^2, and f is a holomorphic map of M into itself, then $f^*(ds^2) \le kds^2$, where k denotes the rank of M. This is another generalized Schwarz's lemma. Ahlfors was the first to generalize Schwarz's lemma by essentially considering the curvature; his result can be stated as the following: Let M be a Riemann surface with hermitian metric ds^2_M whose Gaussian curvature is bounded above by a negative constant $-B$, and D the unit disk in C with an invariant metric ds^2_D whose Gaussian curvature is a negative constant $-A$, then every holomorphic map $f: D \to M$ satisfies $f^*(ds^2) \le \frac{A}{B} ds^2_D$. Kobayashi generalized this result to higher dimensional case in his recent paper [6].

Recently Chern [5] has shown that a holomorphic map f of D_n into a n-dimensional hermitian Einstein manifold N with scalar curvature less than or equal to $-2n(n+1)$ is volume-decreasing. Kobayashi [8] generalized the result of Chern to the case of a holomorphic mapping f from a more generalized domain M (of dimension n) into a more general image manifold N (of dimension n). This paper is devoted to a generalization of Schwarz's lemma as well as one of Chern's results in his paper [5] concerning a Laplacian formula for the ratio function of top volume elements between hermitian manifolds.

Received September 30, 1967 and, in revised form, May 24, 1968.

Let M and N be two hermitian manifolds with dimensions m and n respectively. The simplest metrical invariant is the ratio of distance and volume elements. In this case, in which we have different dimensions, the ratio of volume elements of M and N is called the ratio of the intermediate volume elements. This ratio has an important effect on the holomorphic mappings between M and N, and the study of these ratios will be the main topic of this paper.

In this paper we assume some basic knowledge in hermitian geometry, such as complex and hermitian structures on a complex manifold and the notions of connection and curvature of a holomorphic vector bundle, which can be found in [3], [4], or the author's dissertation [7]. In §2 we discuss the Ricci and scalar curvatures in detail, and introduce the Laplacian on an hermitian manifold. §§3 and 4 are devoted to defining some general elementary symmetric functions, such as the ratio function of distances, the ratio function of intermediate volume elements and, in the case of manifolds of the same dimension, the ratio function of top value elements, and to deriving Laplacian formulas for those functions. These formulas are significant because from them we find that the Laplacian of those functions will involve connection form and curvatures only; hence they will give us the desired geometrical result.

From the result of §4 and by using the maximum principle, we will discuss the properties of holomorphic mappings from a compact hermitian manifold into an hermitian manifold with certain curvature restrictions. The main result in the last section will be stated as follows:

Let $F : D_m \to N$ be an holomorphic mapping, where D_m is the unit m-ball in C^m with the standard Kaehler metric, and N is an n-dimensional hermitian manifold with negative constant holomorphic sectional curvature $(= -2m(m+1))$. Then f is distance-decreasing.

2. Foundations of hermitian geometry

From now on we will fix the notations throughout this paper as follows:

$$\begin{gathered} 1 \leq i, j, k, l, i_1, \cdots, i_\mu \leq m \\ 1 \leq \alpha, \beta, \gamma, \eta, \alpha_1, \cdots, \alpha_\mu \leq n \\ m+1 \leq \sigma, \rho, \tau \leq n \\ 1 \leq r, s, t \leq \mu . \end{gathered}$$

In this section we will discuss curvature tensors in detail and derive a Laplacian formula for a real-valued C^∞-function on an hermitian manifold.

First we consider curvature tensors. Let M be a hermitian manifold of dimension m, that is, let M be a complex manifold with a hermitian structure on its tangent bundle T. The hermitian metric defines an hermitian inner product in T and a type $(1, 0)$ connection which is invariant under parallel translation. In the following we will discuss the tangent and cotangent bundles. We know that a unitary frame is an ordered set of m tangent vectors $\{s_i\}$ with the same origin such that $\langle s_i, s_j \rangle = \delta_{ij}$. The dual basis to a unitary frame $\{s_i\}$ in the cotangent bundle is a unitary coframe which consists of m complex-valued linear differential forms θ^i of type $(1, 0)$ such that

$$ds_M^2 = \sum_i \theta^i \bar{\theta}^i . \tag{2.1}$$

Let B be the bundle of all unitary frames of M. Then the forms θ^i are forms in B. It is known [6] that there exist connection forms θ_j^i in B such that

$$d\theta^i = \sum_j \theta^j \wedge \theta_j^i + \Theta^i \tag{2.2}$$

with

$$\theta_j^i + \bar{\theta}_i^j = 0 , \tag{2.3}$$

$$\Theta^i = \frac{1}{2} \sum_{j,k} T_{jk}^i \theta^j \wedge \theta^k . \tag{2.4}$$

The θ^i's are called torsion forms, and T_{jk}^i's torsion tensors.

By taking exterior derivative of (2.2) and using (2.2) again, we find ([3] or [4])

$$d\Theta^i = \sum_j \theta^j \wedge \Theta_j^i - \sum_j \Theta^j \wedge \theta_j^i , \tag{2.5}$$

where

$$\Theta_j^i = d\theta_j^i + \sum_k \theta_k^i \wedge \theta_j^k \tag{2.6}$$

are the $(1, 1)$ curvature forms satisfying

$$\Theta_j^i + \Theta_i^j = 0 . \tag{2.7}$$

Those $(1, 1)$ forms Θ_j^i can be written as

$$\Theta_j^i = \frac{1}{2} \sum_{k,l} R_{jkl}^i \theta^k \wedge \bar{\theta}^l . \tag{2.8}$$

In terms of the coefficients R_{jkl}^i, the symmetry properties (2.7) can be written as

$$R_{jkl}^i = \bar{R}^j{}_{ikl} . \tag{2.9}$$

The quantities R^i_{jkl} are functions in B and constitute the curvature tensors of the hermitian metric.

We define the Ricci tensors R_{kl} as

$$R_{kl} = \sum_i R^i_{ikl} = \bar{R}_{lk}\,, \tag{2.10}$$

and the scalar curvature as

$$R = \sum_k R_{kk}\,, \tag{2.11}$$

where R is a real-valued function in M.

Now we derive a Laplacian formula for a real-valued C^∞ function on a hermitian manifold. Let M and N be two arbitrary hermitian manifolds with dimensions m and n respectively. In the previous paragraphs we have already defined

$$\theta^i,\ \theta^i_j,\ \Theta^i,\ \Theta^i_j,\ R^i_{jkl},\ R_{kl},\ R$$

in M. Here we list the corresponding ones for N:

$$\omega^\alpha,\ \omega^\alpha_\beta,\ \Omega^\alpha,\ \Omega^\alpha_\beta,\ S^\alpha_{\beta\gamma\eta},\ S_{\gamma\eta},\ S\,.$$

Naturally all the general formulas discussed above remain valid by inserting these quantities.

In the following we shall make no distinction between θ^i and θ_i, θ^i_j and θ_{ij} etc., because we will select unitary fields and unitary coframe fields.

Let u be any real-valued C^∞ function on M. By choosing a local unitary frame field and restricting ourselves to a neighborhood of M, we let

$$du = \sum_i (u_i\theta_i + \bar{u}_i\bar{\theta}_i)\,. \tag{2.12}$$

Taking the exterior derivative of (2.12) we get

$$\begin{aligned}\sum_i \Big(du_i - \sum_k u_k\theta_{ki}\Big) \wedge \theta_i + \sum_i \Big(d\bar{u}_i - \sum_k \bar{u}_k\bar{\theta}_{ki}\Big) \wedge \bar{\theta}_i \\ + \sum_i u_i\Theta_i + \sum_i \bar{u}_i\bar{\Theta}_i = 0\,.\end{aligned} \tag{2.13}$$

Then let

$$du_i - \sum_k u_k\theta_{ki} = \sum_k (u'_{ik}\theta_k + u_{ik}\bar{\theta}_k)\,. \tag{2.14}$$

By considering the type of forms in (2.13), we have

$$\sum_{i,k} u'_{ik}\theta_k \wedge \theta_i + \sum_i u_i\Theta_i = 0\,, \tag{2.15}$$

or

$$\sum_i d(u_i\theta_i) = \sum_{i,k} u_{ik}\bar{\theta}_k \wedge \theta_i\,. \tag{2.16}$$

From (2.12) we get

$$d'u = \sum_i u_i\theta_i . \tag{2.17}$$

Using (2.16) we have

$$d'd''u = -dd'u = \sum_{i,j} u_{ij}\theta_i \wedge \bar{\theta}_j . \tag{2.18}$$

We define the Laplacian of u to be

$$\Delta u = 4 \sum_i u_{ii} . \tag{2.19}$$

If $u > 0$, we find

$$\Delta \log u = \frac{1}{u}\Delta u - \frac{4}{u^2} \sum_i u_i\bar{u}_k . \tag{2.20}$$

Remark. Here we should emphasize that u_{ik} are the coefficients of $\bar{\theta}_k$ in the equation of du_i, and that u_i are the coefficients of θ_i in the equation of du. This remark will be very useful later.

3. General elementary symmetric functions

We shall now derive a formula for the Laplacian of general elementary symmetric functions of a mapping $f: M \to N$ between two hermitian manifolds. By the latter we mean a symmetric function of the eigenvalues of the linear transformation $B = A\,{}^t\bar{A}$ where

$$A = f^* : T_N^* \to T_M^* \tag{3.1}$$

in the linear map on the cotangent spaces induced by f.

First of all, we will discuss the algebraic situation.

Suppose V, W be two complex vector spaces of dimensions m and n respectively. Let $\{e_1, \cdots, e_m\}$ be a basis for V, and $\{f_1, \cdots, f_n\}$ a basis for W. If $\mu \leq \min\{m, n\}$, then $\{(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\mu}) \mid 1 \leq i_1 < i_2 < \cdots < i_\mu \leq m\}$ forms a basis for $\Lambda^\mu V$, and $\{(f_{\alpha_1} \wedge f_{\alpha_2} \wedge \cdots \wedge f_{\alpha_\mu}) \mid 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_\mu \leq n\}$ forms a basis for $\Lambda^\mu W$, where $\Lambda^\mu V$, $\Lambda^\mu W$ are exterior powers of vector spaces V and W respectively.

Let $A: W \to V$ be a linear transformation such that

$$A(f_{\alpha_r}) = \sum_i a_{\alpha_r i} e_i , \tag{3.2}$$

$${}^t\bar{A}(e_{i_s}) = \sum_\alpha a_{\alpha i_s} f_\alpha . \tag{3.3}$$

We repeat that $1 \leq r, s, t \leq \mu$. Define

$$(3.4)\qquad \Lambda^{\mu}A(f_{\alpha_1}\wedge\cdots\wedge f_{\alpha_\mu})\equiv A(f_{\alpha_1})\wedge\cdots\wedge A(f_{\alpha_\mu})\,.$$

Then

$$(3.5)\qquad \Lambda^{\mu}A(f_{\alpha_1}\wedge\cdots\wedge f_{\alpha_\mu})=\sum_{i_1\wedge\cdots\wedge i_\mu}\det(a_{\alpha_r i_s})e_{i_1}\wedge\cdots\wedge e_{i_\mu}\,.$$

For the purpose of simplification, we adopt the following notations:

Let I denote $(i_1,\cdots,i_\mu)$ with $1\le i_1<i_2<\cdots<i_\mu\le m$, J denote $(\alpha_1,\cdots,\alpha_\mu)$ with $1\le\alpha_1<\alpha_2<\cdots<\alpha_\mu\le n$, and D_I^J denote $\det(a_{\alpha_r i_s})$ where α_r's are components of J and i_s's are components of I. Then

$$(3.6)\qquad \Lambda^{\mu}A(f_{\alpha_1}\wedge\cdots\wedge f_{\alpha_\mu})=\sum_I D_I^J(e_{i_1}\wedge\cdots\wedge e_{i_\mu})\,,$$

where the sum is over all possible I's.

Define

$$(3.7)\qquad B=A\,{}^t\bar{A}\,.$$

Then

$$(3.8)\qquad B(e_{i_r})=\sum_{j_t}\sum_{\alpha_s}a_{\alpha_s j_t}\bar{a}_{\alpha_s i_r}e_{j_t}\,.$$

The elementary symmetric function of f is defined to be the elementary symmetric function of B, in other words, is defined to be the trace of $\Lambda^{\mu}B$ which will be denoted as u. Before going further, we shall state the following lemma in linear algebra; its proof is in [7].

Lemma. *If P is any matrix with n-columns, and Q any matrix with n-rows, then any t-rowed determinant of the matrix PQ is equal to a sum of terms, each of which is the product of a t-row determinant of P and a t-columned determinant of B.*

By using this lemma, we can write

$$(3.9)\qquad u=tr\Lambda^{\mu}B=\sum_I\sum_J|D_I^J|^2\,,$$

where

$$(3.10)\qquad D_I^J=\det(a_{\alpha_r i_s})\,.$$

We shall now discuss some examples of the ratios of distance, volume elements and intermediate volume elements.

Let $\{\theta_i\}$, $\{\omega_\alpha\}$ be the unitary coframe fields of two hermitian manifolds M and N respectively, and $f\colon M\to N$ be a holomorphic mapping such that

$$(3.11)\qquad f^*\omega_\alpha=\sum_i a_{\alpha i}\theta_i\,.$$

Then

$$(3.12)\qquad f^*(\omega_\alpha\wedge\bar{\omega}_\alpha)=\sum_{i,j}a_{\alpha i}\bar{a}_{\alpha j}\theta_i\wedge\bar{\theta}_j\,.$$

From now on we shall omit f^* without fear of confusion. It is clear that the $m \times m$ matrix

$$(b_{ij}) = (\sum_\alpha a_{\alpha i}\bar{a}_{\alpha j}) \tag{3.13}$$

is self-adjoint and positive semi-definite. Let $\lambda_1, \cdots, \lambda_m$ be its eigenvalues. Then

$$\lambda_i \geq 0\,, \quad \text{for } 1 \leq i \leq m\,, \tag{3.14}$$

$$\begin{aligned} ds_N^2 = \sum_\alpha \omega_\alpha \bar{\omega}_\alpha &= \lambda_1\theta_1\bar{\theta}_1 + \cdots + \lambda_m\theta_m\bar{\theta}_m \\ &\leq (\lambda_1 + \cdots + \lambda_m)(\theta_1\bar{\theta}_1 + \cdots + \theta_m\bar{\theta}_m)\,. \end{aligned} \tag{3.15}$$

Let

$$u = \sum_i \lambda_i = \sum_{\alpha,i} a_{\alpha i}\bar{a}_{\alpha i}\,. \tag{3.16}$$

Then

$$ds_N^2 < u ds_M^2\,, \tag{3.17}$$

where u is a well defined function on M, and gives, by looking at (3.17), a good control of the ratio between the distances of M and N.

By raising (3.12) to the m^{th} power, let

$$v = \frac{f^*(\sum_\alpha \omega_\alpha \wedge \bar{\omega}_\alpha)^m}{(\sum_\alpha \theta_i \wedge \bar{\theta}_i)^m} = \sum_J |D^J|^2\,, \quad \text{for } m \leq n\,, \tag{3.18}$$

where

$$D^J = \det(a_{\alpha_i j})\,. \tag{3.19}$$

Then v is geometrically the ratio of intermediate volume elements.

For the special case in which $m = n$, v is the ratio of volume elements and

$$v = D\bar{D}\,, \tag{3.20}$$

where

$$D = \det(a_{\alpha\beta})\,. \tag{3.21}$$

4. Holomorphic mappings of hermitian manifolds

As before, let $f: M \to N$ be a holomorphic map such that

$$f^*\omega_\alpha = \sum_i a_{\alpha i}\theta_i\,,$$

or in short,

$$\omega_\alpha = \sum_i a_{\alpha i}\theta_i . \tag{4.1}$$

By taking exterior derivatives

$$d\omega_\alpha = \sum_i (da_{\alpha i} \wedge \theta_i + a_{\alpha i} d\theta_i) , \tag{4.2}$$

and making use of (2.2) and its corresponding formulas for $d\theta_i$, we get

$$\sum_i (da_{\alpha i} - \sum_j a_{\alpha j}\theta_{ji} + \sum_\beta a_{\beta i}\omega_{\alpha\beta}) \wedge \theta_i + \sum_i a_{\alpha i}\Theta_i - \Omega_\alpha = 0 . \tag{4.3}$$

Since the torsion terms are of type (2, 0), it allows us to put

$$da_{\alpha i} - \sum_j a_{\alpha j}\theta_{ji} + \sum_\beta a_{\beta i}\omega_{\alpha\beta} = \sum_k a_{\alpha ik}\theta_k , \tag{4.4}$$

where $a_{\alpha ik}$ satisfies the relations

$$\sum_{i,k} a_{\alpha ik}\theta_k \wedge \theta_i + \sum_i a_{\alpha i}\Theta_i - \Omega_\alpha = 0 . \tag{4.5}$$

Similarly, we take the exterior derivative of (4.4) and obtain

$$\begin{aligned} - \sum_j da_{\alpha j} \wedge \theta_{ji} - \sum_j a_{\alpha j} d\theta_{ji} + \sum_\beta da_{\beta i} \wedge \omega_{\alpha\beta} + \sum_\beta a_{\beta i} d\omega_{\alpha\beta} \\ = \sum_k da_{\alpha ik} \wedge \theta_k + \sum_k a_{\alpha ik} d\theta_k . \end{aligned} \tag{4.6}$$

Using (4.4), (2.2), (2.5) and simplifying we get

$$\begin{aligned} \sum_k (da_{\alpha ik} - \sum_j a_{\alpha ij}\theta_{jk} - \sum_j a_{\alpha jk}\theta_{ji} + \sum a_{\beta ik}\omega_{\alpha\beta}) \wedge \theta_k \\ = - \sum_k a_{\alpha ik}\Theta_k + \sum_j a_{\alpha j}\Theta_{ji} - \sum_\beta a_{\beta i}\Omega_{\alpha\beta} . \end{aligned} \tag{4.7}$$

Suppose that

$$- \sum_k a_{\alpha ik}\Theta_k = \sum_{k,l} a_{\alpha ikl}\theta_k \wedge \theta_l , \tag{4.8}$$

$$\sum_j a_{\alpha j}\Theta_{ji} - \sum_\beta a_{\beta i}\Omega_{\alpha\beta} = \sum_{k,l} b_{\alpha ikl}\theta_k \wedge \bar{\theta}_l , \tag{4.9}$$

where

$$b_{\alpha ikl} = \frac{1}{2}(\sum_j a_{\alpha j}R_{jikl} - \sum_{\beta,\gamma,\eta} a_{\beta i}a_{\gamma k}\bar{a}_{\eta k}S_{\alpha\beta\gamma\eta}) . \tag{4.10}$$

Then

$$\begin{aligned} da_{\alpha ik} - \sum_j a_{\alpha ij}\theta_{jk} - \sum_j a_{\alpha jk}\theta_{ji} + \sum_\beta a_{\beta ik}\omega_{\alpha\beta} \\ = \sum_l a_{\alpha ikl}\theta_l + \sum_l b_{\alpha ikl}\bar{\theta}_l , \end{aligned} \tag{4.11}$$

which will be useful in proving

Theorem 4.1. *Let $f: M \to N$ be a holomorphic mapping as in* (4.1), *and u the function on M defined in* (3.16). *Then*

$$\frac{1}{2}\Delta u = 2\sum_{\alpha,i,k}|a_{\alpha ik}|^2 + \sum_{\alpha,i,j} a_{\alpha i}\bar{a}_{\alpha j}R_{ji} - \sum_{i,k}\sum_{\alpha,\beta,\gamma,\eta} \bar{a}_{\alpha i}a_{\beta i}a_{\gamma k}\bar{a}_{\eta k}S_{\alpha\beta\gamma\eta}, \tag{4.12}$$

or

$$\frac{1}{2}\Delta \log u = \frac{2}{u}\Big(\sum_{\alpha,i,j}\bar{a}_{\alpha i}a_{\alpha j}R_{ji} - \sum_{i,k}\sum_{\alpha,\beta,\gamma,\eta}\bar{a}_{\alpha i}a_{\beta i}a_{\gamma k}\bar{a}_{\eta k}S_{\alpha\beta\gamma\eta}\Big). \tag{4.13}$$

Proof. By taking exterior derivative of (3.16) and using (4.4),

$$\begin{aligned} du = &\sum_{\alpha,i}\bar{a}_{\alpha i}\Big(\sum_j a_{\alpha ij}\theta_j + \sum_j a_{\alpha j}\theta_{ji} - \sum_\beta a_{\beta i}\omega_{\alpha\beta}\Big) \\ &+ \sum_{\alpha,i}a_{\alpha i}\Big(\sum_j \bar{a}_{\alpha ij}\bar{\theta}_j + \sum_j \bar{a}_{\alpha j}\bar{\theta}_{ji} - \sum_\beta \bar{a}_{\beta i}\bar{\omega}_{\alpha\beta}\Big). \end{aligned} \tag{4.14}$$

Since we select unitary coframes, we have

$$\begin{aligned} \theta_{ij} + \bar{\theta}_{ji} &= 0, \\ \omega_{\alpha\beta} + \bar{\omega}_{\alpha\beta} &= 0. \end{aligned} \tag{4.15}$$

By interchanging appropriate indices, i into j and α into β, and using (4.15) we get

$$du = \sum_{\alpha,i,j}\bar{a}_{\alpha i}a_{\alpha ij}\theta_j + \sum_{\alpha,i,j}a_{\alpha i}\bar{a}_{\alpha ij}\bar{\theta}_j. \tag{4.16}$$

Comparing (4.16) with (2.20) we have

$$u_j = \sum_{\alpha,i}\bar{a}_{\alpha i}a_{\alpha ij}. \tag{4.17}$$

Remark. We should realize that (4.17) is obvious even without using (4.15), since u and u_j are functions on M, and θ_{ij}, $\omega_{\alpha\beta}$, etc. are pull-back forms of the principal bundle, which should cancel out each other automatically. Then by considering the remark at the end of §2 and comparing (4.16) with (2.2), (4.17) can be written out in one step.

Now making use of the remark at the end of §2, and (4.10), (4.11), and applying the above technique to

$$du_j = \sum_{\alpha,i}\bar{a}_{\alpha i}da_{\alpha ij} + \sum_{\alpha,i}a_{\alpha ij}d\bar{a}_{\alpha i}, \tag{4.18}$$

we get

$$u_{jk} = \sum_{\alpha,i}a_{\alpha ij}\bar{a}_{\alpha ik} + \sum_{\alpha,i}a_{\alpha i}b_{\alpha ijk}. \tag{4.19}$$

It follows that

$$\begin{aligned} u_{kk} &= \sum_{\alpha,i} |a_{\alpha ik}|^2 + \sum_{\alpha,i} a_{\alpha i} b_{\alpha ikk} \\ &= \sum_{\alpha,i} |a_{\alpha ik}|^2 + \frac{1}{2} \sum_{\alpha,i,j} a_{\alpha i} a_{\alpha j} R_{jikk} \\ &\quad - \frac{1}{2} \sum_{i,k,\gamma,\eta} \bar{a}_{\alpha i} a_{\beta i} a_{\gamma k} \bar{a}_{\eta k} S_{\alpha\beta\gamma\eta} . \end{aligned} \tag{4.20}$$

From (2.27) and (2.28) we have (4.12) and (4.13), since

$$\begin{aligned} \sum_k R_{jikk} &= R_{ji} , \\ u \sum_{\alpha,i,k} |a_{\alpha ik}|^2 - \sum_k u_k \bar{u}_k &= 0 . \end{aligned} \tag{4.21}$$

In general, we can prove

Theorem 4.2. *Let $f: M \to N$ be a holomorphic mapping as in* (4.1), *and u the general elementary symmetric function of f as in* (3.10). *Then*

$$\tfrac{1}{2}\Delta u = E + R(\xi, \xi) - S(\xi, \xi) , \tag{4.22}$$

where E is a non-negative quantity, $R(\xi, \xi)$ and $S(\xi, \xi)$ are sums of Ricci tensors of M and curvature tensors of N respectively as given by (4.32) *and* (4.33).

Proof. In this case $u = \sum_I \sum_J |D_I^J|^2$. Define

$$u_I^J = D_I^J \bar{D}_I^J , \tag{4.23}$$

$$D_{I,k}^J = \sum_{s=1}^{\mu} \sum_B \varepsilon_{\alpha_1 \cdots \alpha_\mu}^{\beta_1 \cdots \beta_\mu} a_{\beta_1 i_1} \cdots (a_{\beta_s i_s k}) \cdots a_{\beta_\mu i_\mu} , \tag{4.24}$$

$$D_{I,kl}^J = \sum_{s=1}^{\mu} \sum_B \varepsilon_{\alpha_1 \cdots \alpha_\mu}^{\beta_1 \cdots \beta_\mu} a_{\beta_1 i_1} \cdots (b_{\beta_s i_s kl}) \cdots a_{\beta_\mu i_\mu} , \tag{4.25}$$

where $a_{\beta_s i_s k}$ and $b_{\beta_s i_s kl}$ are defined in (7.4) and (7.10). Further define

$$[D_I^J]_j^{i_s} = \sum_B \varepsilon_{\alpha_1 \cdots \alpha_\mu}^{\beta_1 \cdots \beta_\mu} a_{\beta_1 i_1} \cdots (a_{\beta_s j}) \cdots a_{\beta_s i_s} , \tag{4.26}$$

$$[D_I^J]_\alpha^{\beta_s} = \sum_B \varepsilon_{\alpha_1 \cdots \alpha_\mu}^{\beta_1 \cdots \beta_\mu} a_{\beta_1 i_1} \cdots (a_{\alpha i_s}) \cdots a_{\beta_s i_s} . \tag{4.27}$$

Using the same technique as in Theorem 4.1, it is easy to see

$$\begin{aligned} \frac{1}{2}\Delta u_I^J &= 2 \sum_{k=1}^{m} |D_{I,k}^J|^2 + \bar{D}_I^J \sum_{s=1}^{\mu} \sum_j [D_I^J]_j^{i_s} R_{ji_s} \\ &\quad - D_I^J \sum_k \sum_{\gamma,\eta} \{ \sum_{s,\alpha} [D_I^J]_\alpha^{\beta_s} a_{\gamma k} \bar{a}_{\eta k} S_{\alpha\beta_s\gamma\eta} \} , \end{aligned} \tag{4.28}$$

or

$$\begin{aligned} \frac{1}{2}\Delta u &= 2 \sum_{I,J} \sum_k |D_{I,k}^J|^2 + \sum_{I,J} \{ \bar{D}_I^J \sum_{s,j} [D_I^J]_j^{i_s} R_{ji_s} \\ &\quad - \bar{D}_I^J \sum_{k,\gamma,\eta} (\sum_{s,\alpha} [D_I^J]_\alpha^{\beta_s} a_{\gamma k} \bar{a}_{\eta k} S_{\alpha\beta_s\gamma\eta}) . \end{aligned} \tag{4.29}$$

Letting

$$E = 2 \sum_{I,J} \sum_{k} |D^J_{I,k}|^2 , \tag{4.30}$$

$$R(\xi, \xi) = \sum_{I,J} \{\bar{D}^J_I \sum_{s,j} [D^J_I]^{i_s}_j R_{ji_s}\} , \tag{4.31}$$

$$S(\xi, \xi) = \sum_{I,J} \{\bar{D}^J_I \sum_{k,\gamma,\eta} (\sum_{s,\alpha} [D^J_I]^{\beta_s}_\alpha a_{\gamma k}\bar{a}_{\eta k} S_{\alpha\beta_s\gamma\eta})\} , \tag{4.32}$$

we can then see that (4.29) gives exactly what we need. q.e.d.

If we define

$$D^J{}_{,k} = \sum_{i=1} \sum_{B} \varepsilon^{\beta_1\cdots\beta_m}_{\alpha_1\cdots\alpha_m} a_{\beta_1 1} \cdots (a_{\beta_i ik}) \cdots a_{\beta_m m} , \tag{4.33}$$

$$[D^J]^i_j = \sum_{B} \varepsilon^{\beta_1\cdots\beta_m}_{\alpha_1\cdots\alpha_m} a_{\beta_1 1} \cdots (a_{\beta_i j}) \cdots a_{\beta_m m} , \tag{4.34}$$

$$[D^J]^{\beta_i}_\alpha = \sum_{B} \varepsilon^{\beta_1\cdots\beta_m}_{\alpha_1\cdots\alpha_m} a_{\beta_1 1} \cdots (a_{\alpha j}) \cdots a_{\beta_m m} , \tag{4.35}$$

then we have

Corollary 4.3. *For the function v defined in* (3.18),

$$\tfrac{1}{2}\Delta v = \sum_{k} \sum_{J} |D^J{}_{,k}|^2 + vR - \sum_{J} \bar{D}^J\{\sum_{k,\gamma,\eta} (\sum_{\alpha,i} [D^J]^{\beta_i}_\alpha S_{\alpha\beta_i\gamma\eta}) a_{\gamma k}\bar{a}_{\eta k}\} . \tag{4.36}$$

Proof. By looking at (4.29) we only have to explain the second term on the right hand side. In this case we can easily see that

$$\sum_{J} \bar{D}^J \sum_{i,j} [D^J]^i_j R_{ji} = \sum_{J} \bar{D}^J D^J R = vR . \tag{4.37}$$

By substituting (4.37) in (4.29) and changing D^J_I's into D^J's, (4.36) is then obvious.

Corollary 4.4. *For the function v, defined in* (3.19), *which is the ratio of volume elements of M and N in the case $m = n$,*

$$\frac{1}{4}\Delta v = \sum_{k} D_k \bar{D}_k + \frac{v}{2}(R - \sum_{\alpha,\beta,k} a_{\alpha k}\bar{a}_{\beta k} S_{\alpha\beta}) , \tag{4.38}$$

where

$$D_k = \sum_{B} \varepsilon^{i_1\cdots i_n}_{1,\cdots,n} a_{i_1 1} \cdots (a_{i_j jk}) \cdots a_{i_n n} . \tag{4.39}$$

Proof. By the same reason as that for deriving (4.37),

$$\begin{aligned} \bar{D} \sum_{k,\gamma,\eta} \sum_{\alpha,\beta} [D]^\beta_\alpha S_{\alpha\beta\gamma\eta} a_{\gamma k}\bar{a}_{\eta k} &= \sum_{\gamma,\eta,k} \bar{D}DS_{\gamma\eta} a_{\gamma k}\bar{a}_{\eta k} \\ &= v \sum_{\gamma,\eta,k} S_{\gamma\eta} a_{\gamma k}\bar{a}_{\eta k} . \end{aligned} \tag{4.40}$$

Then (4.38) is obvious, and is the formula proved by Chern in [5].

5. Applications

So far what we have derived is the general Laplacian formula for symmetric functions. In this section we use (4.36) and (4.12) to prove some geometrical conclusions. The first conclusion will be a generalization of a theorem in Chern's paper [5], and another one is a theorem, in a different form from that appearing in Kobayashi's recent paper [6], concerning distance-decreasing mappings.

Now we assume $f: M \to N$ is a holomorphic mapping between hermitian manifolds with dimensions m and n respectively.

Definition 5.1. f is degenerate at a point p in M if u vanishes at p, where $u = \sum_J |D^J|^2$ is the ratio function of intermediate volume elements as in (3.18).

Geometrically, f degenerate means that the induced linear map f_* on the tangent space at p is not univalent.

Definition 5.2. f is totally degenerate if u vanishes identically.

Definition 5.3. An Einstein manifold M of dimension m has K-exhaustion property if M is exhausted by a sequence of open submanifolds

$$M_1 \subset M_2 \subset \cdots \subset M, \tag{5.1}$$

whose closures $\bar{M}_a$ are compact and such that

(1) to each $a = 1, 2, \cdots$ there is a C^∞ function $v_a \geq 0$, defined in M_a, satisfying the inequality

$$\frac{1}{2}\Delta v_a \leq \frac{R}{m} + K \exp v_a, \tag{5.2}$$

where K is a given positive constant;

(2) $v_a(p_b) \to \infty$ if p_b is a divergent sequence of points in M_a (an infinite sequence of points p_b in M_a is called divergent, if every compact open set of M_a contains only a finite number of points of the sequence).

This kind of manifold is not uncommon because all bounded symmetric domains and especially the m-dimensional unit ball have this property.

Now we proceed to establish the main theorems of this section.

Theorem 5.1. *For compact M, let R be the scalar curvature of M, and $S_{\alpha\beta\gamma\eta}$ the curvature tensor of N.*

(1) *If $R > 0$ and the curvature transformations are non-positive on all kinds of tensors, then f is totally degenerate.*

(2) *If $R < 0$ and the curvature transformations are non-negative on all kinds of tensors, then f is degenerate at some point of M.*

Proof. (1) Since M is compact, u attains its maximum at a point p_0 in M. If u is not identically zero, $u(p_0) > 0$. Since D^J is a function on M, covariant differentiation is the same as the usual one and hence $D^J{}_{,k}$ is zero at a maximum point, i.e. $D^J{}_{,k}(p_0) = 0$. With our hypothesis and formula (4.36) it is clear that

(5.3) $$(\Delta u)_{p_0} > 0 .$$

However, since p_0 is the maximum point of u, we have

(5.4) $$(\Delta u)_{p_0} \leq 0 ,$$

which is a contradiction.

Assertion (2) is proved similarly by the consideration of the minimum of u.

Theorem 5.2. *Let $f: M \to N$ be a holomorphic mapping, where M is an Einstein manifold of dimension m with K-exhaustion property, and N is an hermitian manifold of dimension n with negative constant holomorphic sectional curvatures ($= -K$, for some positive constant K). Then*

(5.5) $$v = \log u \leq v_a ,$$

where $u = \sum_{\alpha,i} a_{\alpha i}\bar{a}_{\alpha i}$ as in (3.16).

Proof. Let E be the open set defined by $E = \{x \in M \mid v > v_a\}$. In E we have $u = \sum_{\alpha,i} a_{\alpha i}\bar{a}_{\alpha i} \geq 0$. From (4.31) we have

(5.6) $$\frac{1}{2}\Delta \log u = \frac{1}{u}\Big[\sum_{\alpha,i,j} \bar{a}_{\alpha i}a_{\alpha j}R_{ji} - \sum_{i,k} \sum_{\alpha,\beta,\gamma,\eta} \bar{a}_{\alpha i}a_{\beta i}a_{\gamma k}\bar{a}_{\eta k}S_{\alpha\beta\gamma\eta}\Big] .$$

The condition for M to be Einsteinian is

(5.7) $$R_{ji} = \frac{R}{m}\delta_{ji} ,$$

in which case (5.6) reduces to

(5.8) $$\frac{1}{2}\Delta v = \frac{R}{m} - \frac{1}{n} \sum_{i,k} \sum_{\alpha,\beta,\gamma,\eta} \bar{a}_{\alpha i}a_{\beta i}a_{\gamma k}\bar{a}_{\eta k}S_{\alpha\beta\gamma\eta} .$$

The condition for N to have negative constant holomorphic sectional curvature ($= -K$) is

$$\sum_{i,k} \sum_{\alpha,\beta,\gamma,\eta} \bar{a}_{\alpha i}a_{\beta i}a_{\gamma k}\bar{a}_{\eta k}S_{\alpha\beta\gamma\eta} = -K \sum_{i,k} \Big(\sum_{\alpha} \bar{a}_{\alpha i}a_{\alpha k}\Big)\Big(\sum_{\beta} a_{\beta i}\bar{a}_{\beta k}\Big) < -Ku^2 ,$$

which reduces (5.8) to

(5.9) $$\frac{1}{2}\Delta v \geq \frac{R}{m} + K \exp v .$$

From (5.2) and (5.9) we have

(5.10) $$\frac{1}{2}\Delta(v - v_a) \geq K(\exp v - \exp v_a) .$$

Since the exponential function is increasing, $\Delta(v - v_a) > 0$ in E, and $v - v_a$ cannot have a maximum in E. Hence $v - v_a$ must approach its least upper bound on a sequence of points $p_1, p_2, \cdots$ tending to the boundary of E. This sequence cannot have a limit point p_0 in M_a, for otherwise at the point p_0, $v - v_a > 0$, and p_0 would belong to E and be a maximum for $v - v_a$. It cannot be divergent either, for otherwise $v_a \to \infty$ and v is bounded. These show that E is empty, and hence $v \leq v_a$. q.e.d.

In Chern's paper [5] we know that the unit ball D_m is an Einstein manifold with $2n(n+1)$-exhaustion property such that

$$v_\rho = \log\left(\frac{1 - r^2}{\rho^2 - r^2}\right)^{m+1}, \tag{5.11}$$

where $0 < r < \rho < 1$. As $\rho \to 1$, $v_\rho \to 0$ giving us

Theorem 5.3. *Let $f: D_m \to N$ be a holomorphic mapping, where D_m is the unit m-ball in C^m with the standard Kaehler metric, and N is a n-dimensional hermitian manifold with negative constant holomorphic sectional curvature $(= -2m(m+1))$. Then f is distance-decreasing.*

Bibliography

[1] L. V. Ahlfors, *An extension of Schwarz's lemma,* Tran. Amer. Math. Soc. **43** (1938) 359–364.

[2] S. Bochner & W. T. Martin, *Several complex variables,* Princeton University Press, Princeton, 1948.

[3] S. S. Chern, *Complex manifolds,* mimeographed notes, University of Chicago.

[4] ——, *Complex manifolds without potential theory,* Mathematical Studies, No. 15, Van Nostrand, Princeton, 1967.

[5] ——, *On holomorphic mappings of hermitian manifolds of the same dimension,* to appear in Proc. Summer Inst. on Entire Functions, 1968.

[6] P. Griffiths, *The extension problem in complex analysis* II; *embeddings with positive normal bundle,* Amer. J. Math. **88** (1966) 366–446.

[7] Y. C. Lu, *On holomorphic mappings of hermitian manifold,* Ph. D. dissertation at University of California, Berkeley.

[8] S. Kobayashi, *Invariant distances on complex manifolds and holomorphic mappings,* to appear in J. Math. Soc. Japan.

[9] ——, *Volume elements, holomorphic mappings and Schwarz's lemma,* to appear in Proc. Summer Inst. on Entire Functions, 1968.

[10] A. Koranyi, *A Schwarz lemma for bounded symmetric domains,* Proc. Amer. Math. Soc. **17** (1966) 210–213.

UNIVERSITY OF CALIFORNIA AT BERKELEY
BOWLING GREEN STATE UNIVERSITY