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CONNECTIONS ON TANGENT BUNDLES

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1. Introduction

If $p : E \to X$ is a C^∞ vector bundle, then the tangent space TE has two vector bundle structures, namely the structure $p_* : TE \to TX$ (p_* is the tangent map of p) and the tangent bundle structure $\sigma : TE \to E$. It is known that a connection on $p : E \to X$ canonically induces one on $p_* : TE \to TX$ and also on $\sigma : TE \to E$ if $E = TX$. The (principal bundle analogue of the) first induced connection is due to Kobayashi [4, p. 150] and the second, to Elíasson [3] and Yano and Kobayashi [7].

The object of this paper is to describe the relationship of these two induced connections, and to show their existence in a general setting.

Vector bundles and manifolds are modeled on Banach and Hilbert spaces (the notation of [5] is generally followed). Connections are handled by means of their connection maps (a notion due to Dombrowski [2]); this approach allows nonlinear connections to be included in the results. Only the C^∞ (smooth) case is presented here, although all definitions and results hold in a slightly modified form if less differentiability is assumed.

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2. Connections

Let $p : E \to X$ be a smooth vector bundle over a smooth manifold X. A *smooth connection* on this bundle is a smooth splitting of the (direct) exact sequence

$$(1) \qquad 0 \longrightarrow VE \xrightarrow{J} TE \xrightarrow{p'} p^{-1}TX \longrightarrow 0$$

of vector bundles over the smooth manifold E. Here $p^{-1}TX$ denotes the pull-back bundle of TX via p, p' denotes the map defined by the tangent map $p_* : TE \to TX$, and VE denotes the kernel of p' (or of p_*), with J being the inclusion map. VE is canonically smoothly isomorphic to $p^{-1}E$, so there is a canonical smooth morphism $r : VE \to E$ (over the map p).

Let $V : TE \to VE$ denote the left splitting map of the connection; it is a

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smooth morphism. The morphism $D = rV : TE \to E$ (over p) is the *connection map*. D is fibre preserving for both of the bundle structures on TE. It is continuous linear on the σ fibres, but not in general on the p_* fibres. If D is linear on the p_* fibres, then the connection is a *linear connection*. If it is just 1-homogeneous on these fibres, then the connection is a *homogeneous connection* (also called "nonlinear" by Barthel [1]).

Remark. Let E_0 be an open submanifold of E. A smooth splitting of (1) restricted to E_0 is a *smooth connection on* E_0. (In this case V and D are defined on $TE \mid E_0$.) This added generality is needed for strictly nonlinear connections. Namely, a homogeneous connection is always assumed to be a connection on $E_0 = E - 0$ (otherwise it is linear).

A connection on the tangent bundle $\pi : TX \to X$ is called a *connection on the manifold* X. Let $S : T^2X \to T^2X$ denote the symmetry map of $T^2X = T(TX)$, which is a smooth isomorphism of the two vector bundle structures on T^2X and satisfies $S^{-1} = S$ [4, p. 125]. For a linear connection on X, DS is also the connection map of a linear connection on X; the connection is a *symmetric linear connection* if $D = DS$ (i.e. the torsion map $\mathscr{T} = \frac{1}{2}(D - DS)$ vanishes).

If A is a smooth section of $E \to X$ and u is a smooth section of $TX \to X$ (smooth vector field on X), then the covariant derivative D_uA is defined to be the section DA_*u of E [2]. The analogous definition holds for sections along curves of X (i.e. for curves in E). Namely, for a smooth curve e_t in E with $pe_t = x_t$, D_te_t is defined to be the curve $D\dot{e}_t$. (Here a dot denotes the tangent curve of a smooth curve.) The notions of parallelism and geodesics are then defined as usual via covariant derivatives.

For linear connections curvature is defined by the usual formula $R(u, v)A = D_uD_vA - D_vD_uA - D_{[u,v]}A$. For general connections, the curvature form is defined to be the exterior derivative dV of the left splitting map V, which is a 1-form on the manifold E with values in the vector bundle $VE \to E$. (The linear connection on $VE \to E$ used in the definition of the exterior derivative is the Berwald connection induced by the connection on $E \to X$.) dV is horizontal and hence defines a "tensor field" on X, which in the homogeneous case produces the curvature defined in [1, p. 138]. Since only the linear curvature is needed below, the details of the general case are omitted here.

3. The results

The first theorem gives the existence of the induced connection on $p_* : TE \to TX$ of a connection on $E \to X$. It is the vector bundle analogue of a result of Kobayashi about connections on principal bundles [4, p. 150].

Theorem 1. (i) *Each smooth connection on $p : E \to X$ induces a smooth connection on $p_* : TE \to TX$, whose connection map is D_*S and left splitting map is SV_*S.*

(ii) *If the connection on* $E \to X$ *is homogeneous or linear, then so is the induced connection, respectively. (In the homogeneous case, the induced connection is on* $p_* : TE \mid E - 0 \to TX$.)

Theorem 2 results from Theorem 1 by means of the following lemma, which is a restatement in terms of connection maps and splitting maps of a well-known fact of the theory of connections.

Let $q : F \to X$ be another smooth vector bundle and let $\phi : E \to F$ be a smooth isomorphism.

Lemma. (i) *For each smooth connection on* $E \to X$, ϕ *defines a smooth connection on* $F \to X$ *with connection map* $\phi D \phi_*^{-1}$ *and left splitting map* $\phi_* V \phi_*^{-1}$.

(ii) *If the connection on* $E \to X$ *is homogeneous or linear, then so is the connection on* $F \to X$, *respectively.*

The symmetry map $S : T^2X \to T^2X$ is a smooth isomorphism of the bundle $\pi_* : T^2X \to TX$ onto the tangent bundle $\sigma : T^2X \to TX$. For a connection on X, Theorem 1 gives a connection on $\pi_* : T^2X \to TX$. Hence the Lemma can be applied, with $\phi = S = S^{-1}$, to get a connection on $\sigma : T^2X \to TX$, i.e. on the manifold TX. This result is summarized as Theorem 2; the existence portion generalizes results of Elíasson [3] and of Yano and Kobayashi [7].

Theorem 2. (i) *For a smooth connection on* X, *the symmetry map* S *maps the induced connection on* $\pi_* : T^2X \to TX$ *into a smooth connection on the manifold* TX *with connection map* SD_*SS_* *and left splitting map* $(SS_*)^{-1}V_*(SS_*)$.

(ii) *If the connection on* X *is homogeneous, linear, or symmetric linear, then so is the connection on* TX, *respectively. (In the homogeneous case the connection is on* $\sigma : T^2X \mid TX - 0 \to TX$.)

The following theorem is given here for the sake of completeness; its first part has been proved by Elíasson [3], and both parts have been proved by Yano and Kobayashi [7] in the finite dimensional case.

Theorem 3. (i) *If the connection on* X *is symmetric and linear, then the geodesics of the induced connection on* TX *are Jacobi fields along geodesics of* X.

(ii) *If the connection on* X *is the canonical connection of a positive definite metric* g *on* X, *then the induced connection on* TX *is the canonical connection of the indefinite metric* L *on* TX *defined by* $L(A,B) = g(\pi_*A,DB) + g(DA,\pi_*B)$ *for* $A, B \in T^2X$ *with* $\sigma A = \sigma B$.

Remark 1. In [7] Yano and Kobayashi describe the induced connection on TX via complete lifts of vector fields. The same idea actually works for the connection on $p_* : TE \to TX$ as well.

Let A and u be smooth sections of $p : E \to X$ and $\pi : TX \to X$, respectively. Define their complete lifts as $A^b = A_*$ and $u^c = Su_*$; they are smooth sections of $p_* : TE \to TX$ and $\sigma : T^2X \to TX$, respectively (u^c is due to Sasaki [6, p. 341]). Let the connection maps of the induced connections given by Theorems

1 and 2 be ${}_1D$ and ${}_2D$, respectively. Then a straightforward calculation (using the relation $SA_{**} = A_{**}S$ and the definition $D_uA = DA_*u$) gives

$$ {}_1D_{u^c}A^b = (D_uA)^b\,, \qquad {}_2D_{u^c}v^c = (D_uv)^c\,. $$

Remark 2. If the hypothesis of Theorem 3 (ii) holds, then the manifold TX also has the positive definite Sasaki metric [6] defined by $G(A, B) = g(DA, DB) + g(\pi_*A, \pi_*B)$ for all $A, B \in T^2X$ with $\sigma A = \sigma B$ $(=u)$. Let ${}_GD$ be its connection map. Then it follows from Theorem 2 (i) and results in [6, p. 352], that the bilinear difference form is

$$ \begin{aligned} ({}_GD - {}_2D)_u(A, B) = {} & \tfrac{1}{2}(R(DA, u)\pi_*B + R(DB, u)\pi_*A)^H \\ & + \tfrac{1}{2}(R(u, \pi_*A)\pi_*B + R(u, \pi_*B)\pi_*A)^V, \end{aligned} $$

where R denotes the curvature of D, and H, V denote the horizontal and vertical lifts, respectively [2]. It follows that ${}_2D = {}_GD$ iff $\mathrm{R} = 0$.

4. Local components

The above Lemma, together with Theorems 1, 2, and 3, will be proved in the following sections by local calculations. In this prefactory section, the local components of a connection are defined. Then necessary and sufficient local conditions for a map to be the connection map of a smooth connection are established, together with a characterization of homogeneous, linear, and symmetric linear connections.

Let U be the domain of a smooth local chart on X, and identify it with its homeomorphic image in the model space $\boldsymbol{B}$ of X. Suppose there is a smooth bundle chart $U \times \boldsymbol{E} \approx E \mid U$, where $\boldsymbol{E}$ is the model fibre of E. (In the case $E = TX$ it is assumed that this chart is the tangent map of the given X-chart with domain U.) Then the tangent map defines a smooth chart $U \times \boldsymbol{E} \times \boldsymbol{B} \times \boldsymbol{E} \approx TE \mid (E \mid U)$, and the sequence (1) restricted to $E \mid U$ becomes the sequence

$$ (2) \qquad 0 \longrightarrow U \times \boldsymbol{E} \times 0 \times \boldsymbol{E} \xrightarrow{J} U \times \boldsymbol{E} \times \boldsymbol{B} \times \boldsymbol{E} \xrightarrow{p'} U \times \boldsymbol{E} \times \boldsymbol{B} \longrightarrow 0 $$

of bundles over $U \times \boldsymbol{E}$. J is the inclusion map and $p'(x, a, \lambda, b) = (x, a, \lambda)$. The canonical epimorphism $r : VE \to E$ is locally $r(x, a, 0, b) = (x, b)$.

Consider a smooth connection on $p : E \to X$. Its connection map $D : TE \to E$ is defined by $D = rV$, with V the left splitting map of (1). For D the equation $VJ = I$ means locally that $D(x, a, 0, b) = (x, b)$. Since D is continuous linear on the fibres, there is a local map $\omega : U \times \boldsymbol{E} \to L(\boldsymbol{B}, \boldsymbol{E})$ given by $D(x, a, \lambda, 0) = (x, \omega(x, a)\lambda)$. ω is the *local component* of the connection for the given smooth local charts. Hence D is locally given by

$$ (3) \qquad D(x, a, \lambda, b) = (x, b + \omega(x, a)\lambda)\,. $$

Lemma 1. *A map $D : TE \to E$ is the connection map of a smooth connection on $p : E \to X$ iff for each smooth local chart D is given by* (3), *where $\omega : U \times E \to L(B, E)$ is smooth.*

Proof. Assume D is the connection map of a smooth connection, i.e. $D = rV$ where V is the left splitting map. Then D is locally given by (3) as was just shown above. D is a smooth morphism since r and V are. By definition, this means that the maps $(x, a) \mapsto D(x, a, \_, \_) : U \times E \to L(B \times E, E)$ are smooth for each smooth local chart. But under the topological isomorphism $L(B \times E, E) \approx L(B, E) \times L(E, E)$, $D(x, a, \_, \_)$ corresponds to $(\omega(x, a), I)$. Hence D is a smooth morphism iff each ω is smooth.

Now suppose $D : TE \to E$ is a map locally defined by (3). It is clear from (3) that D is fibre preserving (over p) and is continuous linear on the fibres. Then smoothness of ω implies that D is a smooth bundle morphism. Hence (by definition of r) D factors uniquely into $D = rV$, with $V : TE \to VE$ a smooth morphism, locally given by $V(x, a, \lambda, b) = (x, a, 0, b + \omega(x, a)\lambda)$. Substituting $\lambda = 0$ gives $VJ = I$, which means that V is the left splitting map of a smooth splitting of (1).

Lemma 2. *Let D be the connection map of a smooth connection on $p : E \to X$. A map $V : TE \to VE$ is the corresponding left splitting map iff it is fibre preserving and satisfies $D = rV$.*

Proof. Obvious from local equations for r and D.

Lemma 3. *A connection is linear or homogeneous iff each local component is linear or homogeneous in its second variable, respectively.*

Proof. The p_* fibres of TE are locally the spaces $(x) \times E \times \lambda \times E \approx E \times E$. Hence D is homogeneous or linear on these fibres iff the maps $(a, b) \mapsto b + \omega(x, a)\lambda : E \times E \to E$ are homogeneous or linear, respectively.

Remark 1. For a homogeneous connection, smoothness means D is a smooth morphism on $TE \mid E - 0$, i.e. each ω is smooth on $U \times (E - 0)$. Otherwise, $\partial_2\omega(x, 0)(a) = \omega(x, a)$ implies the connection is linear. (∂_2 denotes the first partial derivative with respect to the second variable.)

Remark 2. Suppose the connection on $E \to X$ is linear. Then the continuity of ω implies that for each $x \in U$, $\omega(x, \_) \in L(E, L(B, E))$, to which there corresponds a $\Gamma(x) \in L^2(E, B; E)$ by the topological isomorphism between these spaces [5, p. 5]. $\Gamma : U \to L^2(E, B; E)$ is the local Christoffel component of the linear connection in the given local chart; it satisfies $\Gamma(x)(a, \lambda) = \omega(x, a)\lambda$. It is easy to see that smoothness of ω implies that Γ is smooth. Furthermore, the connection is symmetric iff each $\Gamma(x)$ is symmetric.

5. Proof of Theorem 1

The theorem will be proved by finding the local expression for $D_*S : T^2E \to TE$ and showing that it satisfies the conditions of Lemma 1 of §4.

First, these conditions for the bundle $p_* : TE \to TX$ will be examined.

Locally p_* is the map $U \times \boldsymbol{E} \times \boldsymbol{B} \times \boldsymbol{E} \to U \times \boldsymbol{E}$ defined by $p_*(s, a, \lambda, b) = (x, \lambda)$. Hence the p_* fibres in TE are $(x) \times \boldsymbol{E} \times \lambda \times \boldsymbol{E} \approx \boldsymbol{E}^2$. Locally $T^2E \approx (U \times \boldsymbol{E} \times \boldsymbol{B} \times \boldsymbol{E}) \times \boldsymbol{B} \times \boldsymbol{E} \times \boldsymbol{B} \times \boldsymbol{E}$. The tangent fibres are $(x, a, \lambda, b) \times \boldsymbol{B} \times \boldsymbol{E} \times \boldsymbol{B} \times \boldsymbol{E}$, whereas since $p_{**}(x, a, \lambda, b; \mu, c, \nu, d) = (x, \lambda, \mu, \nu)$, the p_{**} fibres are $(x) \times \boldsymbol{E} \times \lambda \times \boldsymbol{E} \times \mu \times \boldsymbol{E} \times \nu \times \boldsymbol{E} \approx \boldsymbol{E}^4$. Hence x in (3) corresponds to (x, λ) here, a to (a, b), λ to (μ, ν), and b to (c, d). Thus for a connection on $p_*: TE \to TX$, the local component is a smooth map $\Omega = (\Omega_1, \Omega_2): U \times \boldsymbol{B} \times \boldsymbol{E}^2 \to L(\boldsymbol{B}^2, \boldsymbol{E}^2) \approx L(\boldsymbol{B}^2, \boldsymbol{E}) \times L(\boldsymbol{B}^2, \boldsymbol{E})$, and a connection map ${}_1D: T^2E \to TE$ is locally given by ${}_1D(x, a, \lambda, b; \mu, c, \nu, d) = (x, c + \Omega_1((x, \lambda), (a, b))(\mu, \nu), \lambda, d + \Omega_2((x, \lambda), (a, b))(\mu, \nu))$.

Now D_*S shall be calculated locally and shown to be a map of this type. $D_*S(x, a, \lambda, b; \mu, c, \nu, d) = D_*(x, a, \mu, c; \lambda, b, \nu, d) =$ the tangent vector at $t = 0$ to the curve

$$\begin{aligned} &D(x + t\lambda, a + tb, \mu + t\nu, c + td) \\ &\quad = (x + t\lambda, c + td + \omega(x + t\lambda, a + tb)(\mu + t\nu)) . \end{aligned}$$

Hence (with primes denoting derivatives)

$$\begin{aligned} &D_*S(x, a, \lambda, b; \mu, c, \nu, d) \\ &\quad = (x, c + \omega(x, a)\mu, \lambda, d + \omega(x, a)\nu + \omega'(x, a)(\lambda, b)\mu) . \end{aligned} \tag{4}$$

Define $\Omega((x, \lambda), (a, b))(\mu, \nu) = (\omega(x, a)\mu, \omega(x, a)\nu + \omega'(x, a)(\lambda, b)\mu)$. It is clear from the properties of ω given in Lemma 1, that Ω is a smooth map $U \times \boldsymbol{B} \times \boldsymbol{E}^2 \to L(\boldsymbol{B}^2, \boldsymbol{E}) \times L(\boldsymbol{B}^2, \boldsymbol{E})$. Hence the preceding observation shows (via Lemma 1) that D_*S is the connection map of a smooth connection.

To show SV_*S is the left splitting map, observe that (1) is in this case the sequence

$$0 \longrightarrow V(TE) \xrightarrow{J_1} T^2E \xrightarrow{p'_*} p_*^{-1}T^2X \longrightarrow 0$$

of bundles over TE, with J_1 the inclusion and $V(TE) =$ kernel $p_{**} =$ kernel p'_*. Provided the local charts are defined by taking tangent maps of charts on X, the symmetry map on T^2X is locally $S(x, a, b, c) = (x, b, a, c)$, i.e. it switches the middle coordinates [4, p. 125]. Then easy local calculations show that on T^2E, S defines a diffeomorphism $T(VE) \approx V(TE)$, and that the canonical epimorphism $r_1: V(TE) \to TE$ is defined by $r_1 = r_*S$. Hence if $V_1 = SV_*S$, $r_1V_1 = D_*S$. But locally $r_1(x, a, \lambda, b; 0, c, 0, d) = (x, c, \lambda, d)$, whence by (4)

$$\begin{aligned} &V_1(x, a, \lambda, b; \mu, c, \nu, d) \\ &\quad = (x, a, \lambda, b; 0, c + \omega(x, a)\mu, 0, d + \omega(x, a)\nu + \omega'(x, a)(\lambda, b)\mu) . \end{aligned} \tag{5}$$

This shows V_1 to be a fibre preserving map $T^2E \to V(TE)$, whence Lemma 2 gives the conclusion.

To prove part (ii), observe that $\Omega((x, \lambda), (a, b))(\mu, \nu)$ is always continuous linear in the variable b, and is homogeneous or linear in a iff $\omega(x, a)$ is, respectively. Lemma 3 then gives the desired conclusion. Note that in the homogeneous case Ω is defined and smooth for $a \neq 0$ only, i.e. the connection is on $TE \mid E - 0$.

6. Proofs of the Lemma and Theorem 2

To prove the Lemma, observe that ϕ is locally the map $U \times \boldsymbol{E} \to U \times \boldsymbol{F}$ given by $\phi(x, a) = (x, f(x)a)$, where $f(x) \in \mathrm{Isom}\,(\boldsymbol{E}, \boldsymbol{F})$ (which is an open subset of $L(\boldsymbol{E}, \boldsymbol{F})$ since $\boldsymbol{E} \approx \boldsymbol{F}$) and f is a smooth map. Likewise $\phi^{-1}: F \to E$ is locally $\phi^{-1}(x, a') = (x, f^{-1}(x)a')$, where $f^{-1}(x) = f(x)^{-1}$ and f^{-1} is smooth. Furthermore $\phi_*^{-1} = (\phi^{-1})_* : TF \to TE$ is locally the map $U \times \boldsymbol{F} \times \boldsymbol{B} \times \boldsymbol{F} \to U \times \boldsymbol{E} \times \boldsymbol{B} \times \boldsymbol{E}$ given by

$$\phi_*^{-1}(x, a', \lambda, b') = (x, f(x)^{-1}a', \lambda, f(x)^{-1}b' + (f^{-1})'(x)(\lambda)a') \,.$$

Hence

$$\begin{aligned} D'(x, a', \lambda, b') &= \phi D \phi_*^{-1}(x, a', \lambda, b') \\ &= \phi(x, f(x)^{-1}b' + (f^{-1})'(x)(\lambda)a' + \omega(x, f(x)^{-1}a')\lambda) \\ &= (x, b' + f(x)((f^{-1})'(x)(\lambda)a' + \omega(x, f^{-1}(x)a')\lambda)) \\ &= (x, b' + \eta(x, a')\lambda) \,. \end{aligned}$$

Now the smoothness of f, f^{-1}, and ω implies that η is a smooth map $U \times \boldsymbol{F} \to L(\boldsymbol{B}, \boldsymbol{F})$. Therefore Lemma 1 shows D' to be a smooth connection on $q : F \to X$. An easy calculation shows its splitting map to be $\phi_* V \phi_*^{-1}$, which proves (i). Part (ii) follows by Lemma 3 from the equation for $\eta(x, a')$.

The proof of Theorem 2 (i) was already indicated in §3. The assertions about homogeneity and linearity in part (ii) also follow directly from Theorem 1 and the Lemma. To complete the proof, assume $D = DS$. Observe $SS_*S = S_*SS_*$ (local calculation), whence for ${}_2D = SD_*SS_*$, ${}_2DS = SD_*S_*SS_* = S(DS)_*SS_* = {}_2D$, so that ${}_2D$ is symmetric.

7. Proof of Theorem 3

Consider the sequence (1) for the case $E = TX$, $p = \pi$. Then $V(TX) \approx \pi^{-1}TX$ canonically. Hence the direct sum decomposition of T^2X given by the left and right splitting maps of the connection on X is a smooth isomorphism $T^2X \approx \pi^{-1}TX \oplus \pi^{-1}TX$. Fibre-wise it is given as $(\pi u = x)$

$$\begin{aligned} T^2X(u) &\approx TX(x) \times TX(x) \\ A &\mapsto (DA, \pi_*A) \,. \end{aligned} \tag{6}$$

A smooth curve u_t on TX is by definition a geodesic iff ${}_2D\dot{u}_t = 0$ for all t. From the direct sum decomposition (6) it follows (by setting $A = {}_2D\dot{u}_t$) that this happens iff $D_2D\dot{u}_t = 0$ and $\pi_{*2}D\dot{u}_t = 0$ for all t.

Now ${}_2D$ satisfies $\pi_{*2}D = D\pi_{**}$ (this can be seen for example by calculating the local expression for ${}_2D$ from (6)). On the other hand, from $D = DS$ it follows that

$$D_2D = DD_*SS_* = DD_* - (DD_* - DD_*SS_*) = DD_* - \mathscr{R}SS_* ,$$

where $\mathscr{R} = DD_*S - DD_*$. Let τ denote the tangent bundle projection on T^3X. Then it can be verified that $\mathscr{R}$ satisfies $\mathscr{R}\mathscr{A} = R(\pi_*\tau\mathscr{A}, \pi_*\tau\mathscr{A})\sigma\tau\mathscr{A}$ for $\mathscr{A} \in T^3X$.

Putting $\mathscr{A} = \dot{u}_t$ and $\pi u_t = x_t$, one has

$$D\pi_{**}\dot{u}_t = D\dot{x}_t = D_t\dot{x}_t,\ DD_*\dot{u}_t - \mathscr{R}SS_*\dot{u}_t = D_tD_tu_t - R(\dot{x}_t, u_t)\dot{x}_t .$$

Hence u_t is a geodesic in TX iff for all t

$$D_t\dot{x}_t = 0 , \qquad D_tD_tu_t + R(u_t, \dot{x}_t)\dot{x}_t = 0 .$$

But these are the classical equations stating that x_t is a geodesic and u_t is a Jacobi field along x_t.

To prove part (ii) of Theorem 3, recall that g-invariance of the connection on X means

$$\frac{d}{dt}g(u_t, v_t) = g(D_tu_t, v_t) + g(u_t, D_tv_t) \tag{7}$$

for all smooth curves u_t, v_t in TX above the curve x_t in X.

By part (i) the induced connection on TX is symmetric, so that only L-invariance must be shown, i.e. that

$$\frac{d}{dt}L(A_t, B_t) = L({}_2D_tA_t, B_t) + L(A_t, {}_2D_tB_t) \tag{8}$$

for all smooth curves A_t and B_t in T^2X above the curve u_t in TX. By (7) the left side of (8) is

$$g(D_tDA_t, \pi_*B_t) + g(DA_t, D_t\pi_*B_t) + g(D_t\pi_*A_t, DB_t) + g(\pi_*A_t, D_tDB_t) .$$

To calculate the right side, observe

$$\pi_{*2}D_tA_t = \pi_{*2}D\dot{A}_t = D_t\pi_*A_t,$$
$$D_2D_tA_t = D_2D\dot{A}_t = DD_*\dot{A}_t - \mathscr{R}SS_*\dot{A}_t = D_tDA_t - R(u_t, \pi_*\dot{u}_t)\pi_*A_t .$$

Therefore the right side of (8) equals the left side plus

$$-g(\pi_* A_t, R(u_t, \pi_* \dot{u}_t)\pi_* B_t) - g(\pi_* B_t, R(u_t, \pi_* \dot{u}_t)\pi_* A_t) .$$

But this extra term is zero, due to a classical identity in Riemannian geometry $g(a, R(b, c)d) = - g(d, R(b, c)a)$. Hence ${}_2D$ is the canonical connection of L.

At each $u \in TX$, the isomorphism (6) transfers L and the Sasaki metric G onto the bilinear forms on $TX(x) \times TX(x)$ given by

$$G((u, v), (w, z)) = g(u, w) + g(v, z) ,$$
$$L((u, v), (w, z)) = g(u, z) + g(v, w) = G(P(u, v), (w, z)) ,$$

where $P(u, v) = (v, u)$ is the symmetry map of $TX(x) \times TX(x)$. It has eigenvalues $+1$ and -1 with corresponding eigenspaces being the positive and negative diagonal, respectively. Since P is a topological isomorphism, L is nondegenerate.

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