J. DIFFERENTIAL GEOMETRY
1 (1967) 1–19

FOCAL SETS OF REGULAR MANIFOLDS M_{n-1} IN E_n

MARSTON MORSE

1. Introduction

The object of this paper is to prove the following theorem and give it a proper setting in differential topology.

Theorem 1.1. *There exists a regular connected $(n-1)$-dimensional manifold M_{n-1} of class C^∞ in a euclidean space E_n such that the focal points of M_{n-1} are everywhere dense in E_n.*

In particular there exists a simple regular curve M_1 of class C^∞ in E_2 whose centers of curvature are everywhere dense in E_2. See §2.

The mainfold M_{n-1} of Theorem 1.1 is without any differentiable singularity in E_n and without self-intersection. However it cannot be compact by virtue of Theorem 1.2.

In Theorem 1.2 we refer to a subset of E_n of *J-content zero.* Given a positive constant e, such a set is characterized by the property that it is included in a finite number of n-rectangles whose total volume is less than e.

Theorem 1.2. *Let M_{n-1} be a regular manifold of class C^m, $m > 1$, in E_n, and let $\hat{M}_{n-1}$ and $\hat{E}_n$ be, respectively, relatively compact open subsets of M_{n-1} and E_n.*

Then the set of focal points of $\hat{M}_{n-1}$ in $\hat{E}_n$ has a J-content zero in E_n, implying that the set of focal points of $\hat{M}_{n-1}$ is nowhere dense in E_n.

Note. It is not affirmed that the set of focal points of $\hat{M}_{n-1}$ in E_n has J-content zero.

Theorem 1.2 admits an *extension* in which M_{n-1} is replaced by M_r where $0 < r < n$. Both Theorem 1.2 and its extension are provable by methods used by the author in his colloquium lectures in treating focal points of extremals "transverse" to a differentiable manifold. We shall establish Theorem 1.2 by non-variational methods later in this section. The extension of Theorem 1.2 can also be established by non-variational methods and this will be done in an introduction to critical point theory in global analysis and differential topology now being written.

Theorem 1.2 implies, but is not implied by, the theorem that the set of focal points of the manifold M_{n-1} in Theorem 1.2 has a Lebesgue measure zero in E_n.

We shall recall some essential definitions.

Received March 14, 1967. Supported by the U.S. Army Research Office-Durham, DA-31-124-AROD-455.

Definition 1.1. *Regular presentations.* With $0 < r < n$ let V be an open subset of E_r and $v^1, \cdots, v^r$ rectangular coordinates of a point $v \in V$. Let $x^1, \cdots, x^n$ be rectangular coordinates of a point $x \in E_n$. A mapping

$$v \to F(v) = (F^1(v), \cdots, F^n(v)) : V \to E_n$$

of class C^m is termed *regular* if the n by r functional matrix of the functions $F^1, \cdots, F^n$ has the rank r at each point $v \in V$. Set $F(V) = X$. If F is regular and a homeomorphism into E_n, F is called a C^m-*embedding* of V in E_n, and a *regular C^m-presentation* $(F : V, X)$ of X in E_n.

Definition 1.2. *Regular C^m-manifolds in E_n.* For $0 < r < n$ let Γ_r be a "topological r-manifold" which is a "subspace" of E_n in the sense of Bourbaki [1]. Suppose that there exists an ensemble of C^m-presentations $(F : V, X)$ of open subsets of Γ_r whose union is Γ_r. Then the set of *all* regular C^m-presentations of open subsets X of Γ_r defines a regular C^m-structure $\mathcal{D}$ on Γ_r. Γ_r taken with such a C^m-structure is called a *regular C^m-manifold M_r in E_n with carrier* $|M_r| = \Gamma_r$ and set of presentations $\mathcal{D}M_r$.

The inverse of a presentation $F \in \mathcal{D}M_r$ is called a *regular chart of* M_r. Given a presentation $(F : V, X) \in \mathcal{D}M_r$ the coordinates $v^1, \cdots, v^r$ of a point $v \in V$ are termed *local coordinates* of the point $F(v)$ in the *coordinate domain* X of M_r.

Focal points of M_{n-1}. Let M_{n-1} be a regular C^m-manifold in E_n, with $m \geq 2$. Let $\mathbf{c} = (c^1, \cdots, c^n)$ be a point in E_n. let ζ be a sensed straight line, meeting $\mathbf{c}$ and a point $q \in M_r$, $\mathbf{c} \neq q$, with ζ normal to M_{n-1} at q. Then $\mathbf{c}$ can be defined as a *focal point of M_n on ζ with base point q* in one of three equivalent ways, termed respectively definition by

I. *critical point characteristics,*

II. *a singular point of the field of normals to M_{n-1},*

III. *a center of principal normal curvature of M_{n-1}.*

I. *Focal points as degenerate critical points of a distance function.* Let p be an arbitrary point on $M_{n-1} - \mathbf{c}$, and $p \to f(p)$ the function on M_{n-1} with values

(1.1)
$$f(p) = ||x - \mathbf{c}|| \qquad (p = x \in M_{n-1} - \mathbf{c}).$$

Then f is of class C^m on $M_{n-1} - \mathbf{c}$, and has a critical point when p is the above point q.

The point $\mathbf{c}$ is called a focal point of M_{n-1} on ζ with base point q, if q is a degenerate critical point of f.

We are assuming that the reader is familiar with the invariant characterization of *degenerate and non-degenerate critical* points of a function f of class C^2 on M_{n-1}.

II. *Focal points as singular points of the field of normals to M_{n-1}.* Let $(F : V, X) \in \mathcal{D}M_{n-1}$ be a presentation of a neighborhood on M_{n-1} of the point q given on M_{n-1}. Set $v_0 = F^{-1}(q)$.

Let λ^0 be the unit vector, normal to M_{n-1} at q with the direction of the given line ζ. The family of unit vectors $\lambda = (\lambda_1, \cdots, \lambda_n)$ orthogonal to M_{n-1} at points $F(v) \in X$, including the vector λ^0 when $v = v_0$, can be represented by C^{m-1} mappings

(1.2)
$$v \to \lambda_i(v) : \hat{V} \to R \qquad (i = 1, \cdots, n)$$

for $\hat{V} \subset V$ a sufficiently small open neighborhood of v_0. Set $F(\hat{V}) = \hat{X}$. The normal to $\hat{X}$ at the point $F(v)$ with the direction $\lambda(v)$ has a representation

(1.3)
$$x = F(v) + s\lambda(v) \qquad (v \in \hat{V})$$

in terms of an algebraic distance coordinate s, measured from the point $F(v)$ on the normal, and taken positive at points of the semi-normal with the direction of $\lambda(v)$.

We shall make use of the Jacobian

$$J(s, v) = \frac{D(x^1, \cdots, x^n)}{D(s, v^1, \cdots, v^{n-1})} \qquad (s \in R,\ v \in \hat{V})$$

noting that

(1.4)
$$J(0, v) \neq 0 \qquad (v \in \hat{V}).$$

Our second definition of focal points of M_{n-1} is as follows.

The focal points of M_{n-1} with base points $x = F(v), v \in \hat{V}$, are the points x given by (1.3) *when* $J(s, v) = 0$.

Definition 1.3. *Focal mappings.* A mapping

(1.5)
$$(s, v) \to F(v) + s\lambda(v) \qquad (s \in R,\ v \in \hat{V})$$

conditioned as above, will be called a *focal mapping associated* with the base point q and directed line ζ orthogonal to M_{n-1} at q.

III. *Focal points of M_{n-1} as centers of principal normal curvature of M_{n-1}.* We shall not need this type of definition and so refer to it only to complete the presentation. The interested reader may turn to treatises on differential geometry or to [2], page 403, and in particular to §21 of [2] on "Normals from a point to a manifold".

We state the following theorem.

Theorem 1.3. *The three definitions of focal points are equivalent.*

A proof of this theorem is implied by the analysis in [2].

A corollary of this theorem is that the definition of focal points by means of the vanishing of the Jacobian $J(s, v)$ is independent of the choice of the presentation $(F : V, X) \in \mathcal{D}M_{n-1}$ of a neighborhood X of the given point $q \in M_{n-1}$. One can also establish this independence by dealing directly with the focal mappings (1.3) which are involved.

A proof of Theorem 1.2 recalled. Theorem 1.2 is an almost trivial consequence of Lemma 1.1 below. In Lemma 1.1 we refer to a euclidean m-space U_m of coordinates $u^1, \cdots, u^m$ and to a euclidean m-space E_m of coordinates $x^1, \cdots, x^m$. Given a C^1-mapping $u \to x(u)$ of an open subset W of U_m into E_m, the points $u \in W$ at which the Jacobian

$$\frac{D(x^1, \cdots, x^m)}{D(u^1, \cdots, u^m)} = 0$$

vanish are called the *singular* points of the mapping $u \to x(u)$, and the corresponding points in E_m *singular images.*

In proving Lemma 1.1 we shall refer to *generalized cylinders* in E_m. If r and t are positive integers such that $r + t = m$ a generalized m-cylinder in E_m is the image in E_m under an orthogonal transformation of the product of a mutually orthogonal r-ball and euclidean t-rectangle, given in some auxiliary euclidean m-space. We shall refer to a generalized m-cylinder in E_m as an *elementary volume* V. It is clear that a subset Y of E_n has J-content zero if corresponding to a prescribed constant η, Y can be included in a finite set of "elementary volumes" V whose total volume is less than η.

Lemma 1.1. *Let there be given a* C^1*-mapping*

(1.6)
$$u \to x_i(u) = x_i \qquad (u \in W;\ i = 1, \cdots, m)$$

of an open subset W *of* U_m *into* E_m.

If $\hat{W}$ *is a relatively compact open subset of* W *the image* $x(u)$ *under the mapping* (1.6) *of the singular points* $u \in \hat{W}$ *of the mapping* (1.6), *form a set of* J*-content zero in* E_m.

Proof of Lemma 1.1. The special case of Lemma 1.1 in which the mapping $u \to x(u) =$ grad $f(u)$, where f is a function of class C'' on W, was established and applied by the author frequently between 1926 and 1932. It was first applied in [3] in 1927 and applied several times in the author's colloquium lectures in 1932. It was noted by the author around 1932, too late to put the result into his colloquium lectures, that the proof of Lemma 1.1, in the case in which $x(u) =$ grad $f(u)$, was applicable with at most trivial notational changes, to prove Lemma 1.1 in general.

To make clear the intimate relation between the general proof of Lemma 1.1 and the proof when $x(u) =$ grad $f(u)$, we shall give the proof of Lemma 1.1 by quoting briefly a proof of a lemma on the density of conjugate points, as given on page 625 of [4] in 1930.

It should be noted that it is sufficient to prove Lemma 1.1 for the case in which $\hat{W}$ is the open interior of a closed n-cube $Q \subset W$. In the 1930 quotation the "space (u)" means the space of the points u, and we shall replace the original phrase "conjugate points with $s \leq d$" by the phrase "singular images under (1.6) of points in Q".

The following paragraph is quoted from pp. 625-6 of the 1930 paper, and, properly interpreted, gives a proof of Lemma 1.1.

"Let e now be an arbitrarily small positive constant. Let us break up the space (u) into congruent m-cubes. If the diameter of each of these m-cubes be sufficiently small, then such of the corresponding sets $[x(u)]$ as contain" (singular images under (1.6) of points in Q) "can be enclosed in elementary volumes such as V whose ratios to that of the cubes will be less than e. The sum of these volumes V will be less than e times the total volume of the corresponding cubes. The sum of the elementary volumes will then be arbitrarily small".

Lemma 1.1 follows.

See [5] for additional references.

Lemma 1.1 implies that the set of singular images in E_n of the singular points of the mapping (1.6) has a Lebesgue measure zero in E_n.

This measure theorem is weaker than Lemma 1.1 *because it does not conversely imply Lemma* 1.1.

Lemma 1.1 *applied to prove Theorem* 1.2. Let $(c,\ q,\ \zeta)$ be a set of three elements of which $\boldsymbol{c}$ is a point in $Cl\hat{E}_n, q$ a point in $Cl\hat{M}_{n-1} - c$, and ζ a sensed straight line normal to M_r at q and meeting $\boldsymbol{c}$. With $(c,\ q,\ \zeta)$ there can be *associated* a "focal mapping" of the form (1.5) such that q is a point $F(v_0)$ with $v_0 \in \hat{V}$ and ζ has the direction $\lambda(v_0)$. We can suppose that $\hat{V}$ is a relatively compact open subset of V, where V is given in a presentation $(F{:}V, X) \in \mathcal{D}M_{n-1}$ of a neighborhood X of q.

Let d be a positive constant greater than the distance of an arbitrary point of $Cl\hat{M}_{n-1}$ from an arbitrary point of $Cl\hat{E}_n$. We introduce the interval $\hat{I} = (-d, d)$ and restrict the parameters $(s,\ v)$ of the mapping (1.3) to the relatively compact subset

$$\hat{I} \times \hat{V} \text{ of } R \times V. \tag{1.7}$$

With n parameters $s, v^1, \cdots, v^{n-1}$ so restricted, it follows from Lemma 1.1 that the focal points of M_{n-1} with base points on $\hat{X} = F(\hat{V})$, on normals to $\hat{X}$ with directions $\lambda(v)$ for which $v \in \hat{V}$, and with algebraic distance coordinates $s \in (-d, d)$, form a set of focal points of J-content zero in E_n.

The focal points of M_{n-1} are bounded from their base points because of the condition (1.4). Because of this and because of the compactness of the sets $Cl\hat{M}_{n-1}$ and $Cl\hat{E}_n$ each triple $(c,\ q,\ \zeta)$, given and conditioned as above, is "associated" with one of a finite set of focal mappings such as (1.5).

Theorem 1.2 follows.

We turn to the proof of Theorem 1.1, beginning with the plane case $n = 2$. We shall make use of plane involutes and evolutes.

2. Theorem 1.1 in case $n = 2$

Admissible spirals. Let (ρ, θ) be polar coordinates in a plane E_2 of rectangular coordinates $x^{(1)}, x^{(2)}$. A curve Λ with a polar coordinate representation

(2.1)
$$\rho = \omega(\theta) > 0 \qquad (\theta \geq 0)$$

in which the mapping $\theta \to \omega(\theta)$ is unbounded, of class C^∞, and such that

(2.2)
$$\omega(\theta + 2\pi) > \omega(\theta)$$

will be called an *admissible* spiral.

Such a curve is simple because of the condition (2.2). It is regular; if one sets $\theta = t$ and

$$x^{(1)} = \omega(t)\cos t, \qquad x^{(2)} = \omega(t)\sin t,$$

one obtains a "representation" of Λ such that

$$\|\dot{x}(t)\|^2 = \dot{\omega}^2(t) + \omega^2(t) > 0,$$

where differentiation as to t has been indicated by a superimposed dot.

In §6 we shall complete the proof of the following theorem.

Theorem 2.1. *There exists an admissible spiral whose focal points are everywhere dense in E_2.*

Use will be made of the theory of involutes. Graustein [6], p. 74, defines an involute of a regular plane curve h not a straight line, as a curve H which cuts each tangent to h at right angles.

This definition is not adequate for our purposes because such an involute of H of h is not necessarily regular or simple. We shall deal with arcs h and their involutes which are much more restricted. We begin with conditions on h.

Let R_+ be the open positive axis of reals.

Definition 2.1. *Admissible arcs h.* Let h be a simple, sensed regular arc of the form

$$(x^{(1)},\, x^{(2)}) = (\mu^{(1)}(\tau),\, \mu^{(2)}(\tau)) \qquad (0 \leq \tau \leq \tau_0)$$

of class C^∞. We suppose that τ is the arc length on h measured from its initial point, and that the curvature

(2.3)
$$\kappa(\tau) = |\dot{\mu}^{(1)}(\tau)\ddot{\mu}^{(2)}(\tau) - \dot{\mu}^{(2)}(\tau)\ddot{\mu}^{(1)}(\tau)|$$

of h never vanishes. A final condition on h is that the mapping

(2.4)
$$(r, \tau) \to \mu(\tau) + r\dot{\mu}(\tau) = x : R_+ \times [0, \tau_0] \to E_2$$

of the subset $R_+ \times [0, \tau_0]$ of the (r, τ)-plane into E_2 be a homeomorphism into E_2.

Note. To say that the mapping $\tau \to \mu(\tau)$ is of class C^∞ means that it admits an extension of class C^∞ over some open interval containing $[0, \tau_0]$.

Extending μ slightly, the mapping (2.4) has a jacobian in absolute value,

(2.5)
$$\left| \frac{D\left(x^{(1)}, x^{(2)}\right)}{D(r,\tau)} \right| = r\kappa(\tau) > 0 \qquad (r > 0, 0 \leq \tau \leq \tau_0).$$

Since the mapping (2.4) is by hypothesis a homeomorphism into E_2 and has a non-vanishing jacobian, it is a diffeomorphism into E_2. We shall term the mapping (2.4), so conditioned, a *tangent diffeomorphism* $(h : \tau_0)$ into E_2.

Example. A closed subarc of an open plane semi-circle is an admissible arc h.

We state a lemma.

Lemma 2.1. *Given a "tangent diffeomorphism" $(h : \tau_0)$, for each constant $\alpha > \tau_0$ there exists an involute $\boldsymbol{H}^\alpha$ of h, admitting a representation*

(2.6)
$$\tau \to \boldsymbol{H}^\alpha(\tau) = \mu(\tau) + (\alpha - \tau)\dot{\mu}(\tau) \qquad (0 \leq \tau \leq \tau_0)$$

and such that

(a$_1$) *$\boldsymbol{H}^\alpha$ is a simple, regular arc of class C^∞.*

(a$_2$) *$\boldsymbol{H}^\alpha$ is orthogonal at the point $\boldsymbol{H}^\alpha(\tau)$ of $\boldsymbol{H}^\alpha$ to the straight line tangent to h at the point $\mu(\tau)$.*

(a$_3$) *For $0 < \tau < \tau_0$ there is one and only one "focal point" of $\boldsymbol{H}^\alpha$ with "base point" $\boldsymbol{H}^\alpha(\tau) \in \boldsymbol{H}^\alpha$, namely the point $\mu(\tau) \in h$.*

Verification of (a$_1$). The curve $\boldsymbol{H}^\alpha$ in E_2 is the image in E_2 under the diffeomorphism (2.4) of the arc in the (r, τ)-plane of the form

(2.7)
$$r = \alpha - \tau \qquad (0 \leq \tau \leq \tau_0, \ \alpha > \tau_0).$$

$\boldsymbol{H}^\alpha$ accordingly has the properties (a$_1$) since its antecedent (2.7) in the (r, τ)-plane under the diffeomorphism (2.4) exists and has these properties in the (r, τ)-plane.

Verification of (a$_2$). A tangent to $\boldsymbol{H}^\alpha$ at the point represented by τ has direction numbers $\ddot{\mu}^{(1)}(\tau), \ddot{\mu}^{(2)}(\tau)$. These numbers are not both zero since $\kappa(\tau) \neq 0$. Moreover $\|\dot{\mu}(\tau)\| = 1$ identically by hypothesis, from which it follows that

$$\dot{\mu}^{(1)}(\tau)\ddot{\mu}^{(1)}(\tau) + \dot{\mu}^{(2)}\ddot{\mu}^{(2)}(\tau) = 0, \tag{2.8}$$

establishing (a_2).

Verification of (a$_3$). The open subarc of $\boldsymbol{H}^\alpha$ on which $0 < \tau < \tau_0$ may be considered as a 1-dimensional manifold M_1. A "focal mapping" based on M_1 exists in the form (*cf.* (1.5))

(2.9)
$$(s, \tau) \to \boldsymbol{H}^\alpha(\tau) + s\dot{\mu}(\tau) = x \qquad (0 < \tau < \tau_0).$$

The normal to $\boldsymbol{H}^\alpha$ at the point $\boldsymbol{H}^\alpha(\tau)$ meets the point $\mu(\tau)$ on h when $s = \tau - \alpha$, as one verifies using (2.6). A simple calculation shows that under (2.9)

$$\left|\frac{D\left(x^{(1)}, x^{(2)}\right)}{D(s,\tau)}\right| = |(s-\tau+\alpha)\kappa(\tau)|, \tag{2.10}$$

where $\kappa(\tau)$ is the curvature of h. Hence the jacobian (2.10) vanishes when $s = \tau - \alpha$, that is at the point s representing the point $\mu(\tau) \in h$ on the normal to $\boldsymbol{H}^\alpha$ at $\boldsymbol{H}^\alpha(\tau)$.

This establishes (a_3) and completes the proof of Lemma 2.1.

Definition 2.2. *Mated evolute h and involute $\boldsymbol{H}^\alpha$.* The arc h is admissible in the sense of Definition 2.1. We term such an arc h and the above involute $\boldsymbol{H}^\alpha$ of h a *mated evolute and involute.*

We shall restrict ourselves to mated evolutes and involutes.

3. Method of proof of Theorem 2.1

The only spirals of which we shall have need in proving Theorem 2.1 are of a limited type which we now characterize in a series of definitions.

The annulus $A(n', n'')$. Given two integers n' and n'' such that $n'' > n' > 0$, by the *annulus* $A(n', n'')$ is meant the set of points $x \in E_2$ such that $n' \leq \|x\| \leq n''$.

The interval $I(i)$. Given a positive integer $i, I(i)$ shall denote the interval $[(2i-2)\pi, 2i\pi]$.

By an arc ξ *spanning an annulus $A(n', n'')$ over an interval $I(i)$* is meant a curve ξ in $A(n', n'')$ with the properties (a_1), (a_2), (a_3).

(a_1) ξ shall have a polar coordinate representation $\rho = \rho(\theta) > 0$ where the mapping $\theta \to \rho(\theta)$ is defined and of class C^∞ over $I(i)$.

(a_2) $$n' < \rho(\theta) < n'' \qquad (\theta \in \mathring{I}(i)),$$

where $\mathring{I}(i)$ is the open interior of I(i).

(a_3) The mapping $\theta \to \rho(\theta)$ shall admit an *extension* of class C^∞ over the θ-axis such that

$$\begin{aligned} \rho(\theta) &= n' \qquad && (\theta \leq (2i-2)\pi) \\ \rho(\theta) &= n'' && (\theta \geq 2i\pi). \end{aligned} \tag{3.0}$$

Definition 3.1. *Special spirals Λ.* To define such a spiral there is given an increasing sequence

$$n_0 < n_1 < n_2 < \cdots \qquad (n_0 = 1) \tag{3.1}$$

of integers and for each positive integer i an arc

$$\xi_i : \rho = \omega_i(\theta) > 0 \qquad (\theta \in I(i)) \tag{3.2}$$

in polar coordinates, "spanning" the annulus $A(n_{i-1}, n_i)$ over the interval $I(i)$. A spiral Λ, admissible in the sense of §2, is defined by the sequence $\xi_1, \ \xi_2, \ \xi_3, \cdots$ of the above arcs. Otherwise expressed Λ shall have a polar coordinate representation $\rho = \omega(\theta) > 0$ for $\theta \geq 0$ such that for each i

(3.3)
$$\omega(\theta) = \omega_i(\theta) \qquad (\theta \in I(i)).$$

To indicate how such a spiral Λ, if suitably chosen, will satisfy Theorem 2.1 a convention and definition are needed.

A convention. An arc γ with a representation

(3.4)
$$\rho = R(\theta) > 0 \qquad (\theta' \leq \theta \leq \theta'')$$

in polar coordinates, will be regarded as real *analytic* if and only if the mapping $\theta \to R(\theta)$ is analytic and admits a real analytic extension over an open interval which includes the interval $[\theta', \theta'']$.

Definition 3.2. A real analytic arc γ of form (3.4) will be said to *belong* to an annulus $A(n', n'')$ and interval $I(i)$ if γ is included in the open interior of $A(n', n'')$ and if $[\theta', \theta'']$ is included in the open interior $\overset{\circ}{I}(i)$ of $I(i)$.

The following lemma is readily proved.

Lemma 3.1. *An analytic arc γ of form* (3.4) *which "belongs" to an annulus $A(n', n'')$ and interval $I(i)$, admits an extension in polar coordinate form which "spans" the annulus $A(n', n'')$ over $I(i)$.*

In §5 we shall study "mated" evolutes h and involutes H in which h is a subarc of a quarter circle so oriented and placed in E_2 that H has the form (3.4) and "belongs" to an annulus $A(n', n'')$ over an interval $I(i)$. The circular arcs h admitted are such that there are infinitely many choices of an annulus $A(n', n'')$ and interval $I(i)$ to which some mated involute H of h belongs. In §6 we shall take advantage of this freedom in the choice of the annuli $A(n', n'')$ and intervals $I(i)$ to prove the following lemma.

Lemma 3.2. *In the class K (cf.* §4) *of circular arcs h to be defined in* §4, *there exists a countable sequence $h_1, \ h_2, \ h_3, \ \cdots$ whose point set union*

$$X = h_1 \cup h_2 \cup h_3 \cup \cdots \tag{3.5}$$

is everywhere dense in E_2 and can be associated with a "special" spiral Λ in the following way.

For each integer $i > 0$ the circular arc h_i is the evolute of an involute H_i "mated to h_i" which is an analytic arc of the form (3.4), *"belongs" to the annulus $A(n_i, n_{i+1})$ and interval $I(i)$ of Λ, and has an extension which is the subarc ξ_i of Λ.*

The spiral Λ of Lemma 3.2 has a focal set which includes the set X of (3.5) and so satisfies Theorem 2.1.

In §4 we characterize a class K of circular arcs h which, in retrospect (§6), will appear to be an adequate class of arcs to be used as evolutes in proving Theorem 2.1. In §4 we show that there is a countable subset of arcs $h \in K$ which is everywhere dense in K.

4. The class K of circular arcs

The circular arc g. The class K shall contain that closed subarc g of the circle

$$(x^{(1)}+1)^2+(x^{(2)})^2=1 \tag{4.1}$$

which meets the origin, is symmetric with respect to the $x^{(1)}$-axis and has a length $b=1/2$. The arc g has a real analytic representation

$$x=g(t)=(g^{(1)}(t),g^{(2)}(t)) \qquad (0\leq t\leq b=2a) \tag{4.2}$$

such that $g^{(1)}(t)=\cos(t-a)-1$, $g^{(2)}(t)=\sin(t-a)$. The parameter t is the arc length on g measured from g's initial point $(\cos a-1,-\sin a)$. The terminal point of g is $(\cos a-1,\sin a)$. The arc g meets the origin when $t=a$ and is there tangent to the $x^{(2)}$-axis.

We note that g is in the open disc

$$D_a=(x\in E_2|\ \|x\|<a) \qquad (2a=b). \tag{4.3}$$

The set K of circular arcs. We shall define K by means of linear transformations operating on g. With a as in (4.2) and $c\in(-a,a)$ let T_c be the translation of E_2 such that

$$T_c(x^{(1)},x^{(2)})=(x^{(1)}+c,x^{(2)}). \tag{4.4}$$

Let N be the ensemble of positive integers. For $p\in N$ let T^p be the radial expansion of E_2 such that

$$T^p(x^{(1)},x^{(2)})=(px^{(1)},px^{(2)}). \tag{4.5}$$

Set $T^p\circ T_c=T^p_c$.

Let K_1 be the ensemble of images h of g under T_c as c ranges on the interval $(-a,a)$. Set $K_p=T^pK_1$ and $K=\underset{p\in N}{\mathrm{Union}}\ K_p$. Finally let $|K_p|$ be the point set covered by the arcs in K_p. We see that $|K_1|$ is a neighborhood of the origin and is included in the open disc D_{2a} with radius $2a$ and center at the origin.

We shall verify the following lemma.

Lemma 4.1. *There is a countable subset $\{h_i\}$ of the arcs h in K whose point set union is everywhere dense in E_2.*

To prove Lemma 4.1 let $e_1,\ e_2,\ e_3,\ \cdots$ be a decreasing sequence of positive numbers e_i such that e_i tends to zero as a limit as $i\uparrow\infty$. For each $p\in N$ let $(h)_p$ be a *finite* subset of the arcs h in K_p whose point set union contains points within a distance e_p of a prescribed point of $|K_p|$. The set of arcs

$$\{h_i\} = \underset{p \in N}{\text{Union}}\,(h)_p$$

is countable and satisfies the lemma.

In §5 we shall study a class of admissible involutes of arcs $h \in K$.

5. Involutes of the arcs $T_c^p g$

Recall that the length of the circular arc g introduced in (4.2) is $b = 1/2$. We have set $b = 2a$.

Involutes G_k of g. For each $k > b$ an involute G_k of g, "mated" to g, is defined by the mapping (*cf.* (2.6))

(5.1)
$$t \to \boldsymbol{G}_k(t) = \boldsymbol{g}(t) + (k-t)\dot{\boldsymbol{g}}(t) \qquad (0 \leq t \leq b).$$

This mapping is analytic, and according to Lemma 2.1, simple and regular. It is extendable as a simple, regular, analytic mapping over an open interval containing [0, b].

We shall verify the following.

(λ) *For $k > b$ the arc G_k is included in the half-plane of E_2 on which $x^{(2)} > 0$.*

Statement (λ) follows on noting that the minimum value of $x^{(2)}$ on G_k is attained when $t = b$ and is $(k - b)\cos a + \sin a > 0$.

The transformations T_c^p, introduced in §4, are linear, conformal homeomorphisms of E_2 onto E_2. They carry circles into circles and simple, regular, analytic arcs γ into such arcs. If y is a focal point of γ with a base point x on γ, $T_c^p y$ is a focal point of $T_c^p\gamma$ with the base point $T_c^p x$ on $T_c^p\gamma$. Given g and a mated involute G_k of g, with $k > b = 2a$, we infer that the arcs

$$h = T_c^p g, \quad H = T_c^p G_k \tag{5.2}$$

are mated evolute and involute.

An arc $h = T_c^p g$ has the representation

(5.3)
$$t \to \nu(t) = p(\boldsymbol{g}(t) + c) \qquad (0 \leq t \leq b).$$

Its length parameter is pt and total length pb. An involute $H = T_c^p \boldsymbol{G}_k$, $k > b$ of h has a representation

(5.4)
$$t \to p(\boldsymbol{G}_k(t) + c) = \nu(t) + (k-t)\dot{\nu}(t) \qquad (0 \leq t \leq b).$$

We shall verify the following properties of the circular arc $h = T_c^p g$ and its mated involute H, as given by (5.4).

Subject to the condition that $k > 3b$ the following is true:

(a$_1$) *The arc H is simple, regular and analytic, with an open analytic extension. On $H, x^{(2)} > 0$.*

(a$_2$) *The arc H is included in the open annulus*

$$p(k-2b) < \|x\| < p(k+2b). \tag{5.5}$$

(a$_3$) *No arc H is tangent to a ray from the origin.*

(a$_4$) *An analytic polar coordinate representation*

$$\rho = R(t) > 0, \quad \theta = \Theta(t), \qquad (0 \leq t \leq b) \tag{5.6}$$

of H exists in which $R(t), \Theta(t)$ are polar coordinates of the points $T_c^p \boldsymbol{G}_k(t)$ of H. In such a representation $\dot{\Theta}(t) > 0$.

Verification of (a$_1$). That H is simple, regular and analytic with an open, analytic extension follows from the fact that G_k has these properties. Since the point $\boldsymbol{G}_k(t)$ is in the open upper half-plane by (λ), the corresponding point $p(\boldsymbol{G}_k(t) + c)$ of H is in the open upper half-plane.

Verification of (a$_2$). We first examine the case in which $p = 1$ and $c = 0$. In this case $T_c^p G_k = G_k$.

Let u and v be respectively points on g and G_k such that v is on the ray tangent to g at u. Let q be the origin. If $x = v$ then $\|x\| = d(q, v)$. Consideration of the triangle with the vertices u, v, q shows that

$$d(u,v) - d(q,u) \leq \|x\| \leq d(u,v) + d(q,u). \tag{5.7}$$

Since $d(q,u) < a$ and $d(u,v) = k - t \geq k - 2a$ for points $v \in G_k$ in accord with (5.1), it follows from (5.7) that

$$k - 3a < \|x\| < k + 3a. \tag{5.8}$$

For points $x \in T_c G_k$ it follows from (5.8) that

$$k - 2b < \|x\| < k + 2b \tag{5.9}$$

since $-a < c < a$ and $T_c \boldsymbol{G}_k(t) = \boldsymbol{G}_k(t) + c$. Since $T_c^p = T^p \circ T_c$, the inequalities (5.5) follow from (5.9) for $x \in T_c^p G_k$.

Verification of (a$_3$). We begin by verifying (a$_3$) for the case of an involute $T_c G_k$ of $T_c g$. Since $k > 3b$ by hypothesis it follows from (5.9) that $T_c G_k$ does not meet the disc D_b. On the other hand $T_c g$ is included in D_b.

Suppose (a$_3$) false for the case of $T_c G_k$. There then exists a ray ζ from the origin, tangent to $T_c G_k$ at a point v. Such a ray ζ would be orthogonal at v to a line τ tangent to $T_c g$. Thus $\tau \cap \zeta = v$. But the point $\tau \cap \zeta$ must be the point on τ nearest the origin. Since τ meets D_b the point $\tau \cap \zeta = v$ must be in D_b, contrary to the fact that no point of $T_c G_k$ is in D_b.

Since T^p is a radial expansion with center at the origin, statement (a$_3$) is true for $T_c^p G_k$, since it is true for $T_c G_k$.

Verification of (a$_4$). A simple, closed, regular, analytic arc H on which $x^{(2)} > 0$, admits infinitely many analytic polar coordinate representations (5.6). In (5.6) Θ is uniquely determined up to a function whose values are an integral multiple of 2π. For no value of $t \in [0, b]$ is $\dot{\Theta}(t) = 0$, since at a point x_0 of H at which $\dot{\Theta}(t_0) = 0$ the ray from the origin meeting x_0 would then be tangent to H at x_0. This would be contrary to (a_3).

It remains to show that $\dot{\Theta}(t) > 0$.

Since T^p is directly conformal and no arc $T_c G_k$ meets the origin it is sufficient to prove that $\dot{\Theta}(t) > 0$ in a representation (5.6) of $T_c G_k$. Since $\dot{\Theta}(t) \neq 0$ it is sufficient to verify that $\dot{\Theta}(a) > 0$, making use of the relation

$$T_c \boldsymbol{G}_k(t) = \boldsymbol{g}(t) + (k-t)\dot{\boldsymbol{g}}(t) + c \qquad (cf.\ (5.1))$$

and the formula for $\boldsymbol{g}(t)$. If $T_c G_k$ has the representation $x^{(1)} = \varphi(t)$, $x^{(2)} = \phi(t)$ one finds that

$$\dot{\phi}(a)\varphi(a) - \dot{\varphi}(a)\phi(a) = (k-a)^2 > 0$$

from which it follows that $\dot{\Theta}(a) > 0$.

Thus the properties (a_1) to (a_4) of H hold as stated.

6. Proof of Theorem 2.1

The following lemma is a consequence of the properties of the mated evolutes and involutes

(6.1)
$$h = T_c^p g, \qquad H = T_c^p G_k \qquad (k > 3b)$$

as recorded in §5.

Lemma 6.1. *Let $h = T_c^p g$ be prescribed in K, and positive integers n' and i given. Then for a sufficiently large $k > 3b$ the involute $H = T_c^p G_k$ of h is included in the interior of the annulus $A(n', n'')$ for some $n'' > n'$, and H has a unique analytic polar coordinate representation*

(6.2)
$$\rho = \eta(\theta) > 0 \qquad (\theta' \leq \theta \leq \theta'')$$

such that

(6.3)
$$[\theta', \theta''] \subset \mathring{I}(i).$$

The choice of k. Given h as in Lemma 6.1, whatever the choice of $k > 3b$, the corresponding involute $T_c^p G_k$ of h satisfies (5.5). Let $k > 3b$ be chosen so that $p(k - 2b) > n'$. Then $H = T_c^p G_k$ is included in the open interior of an annulus $A(n', n'')$ for some choice of $n'' > n'$.

The choice of $\eta(\theta)$ in (6.2). Let (5.6) be a representation of H so chosen (as is possible) that

(6.4)
$$(2i-2)\pi < \Theta(0) < \Theta(b) < (2i-1)\pi$$

and set $\theta' = \Theta(0), \theta'' = \Theta(b)$. Since $\dot{\Theta}(t) > 0$ in a representation (5.6), a unique analytic representation of H exists of the form (6.2) with (6.3) holding.

This completes the proof of Lemma 6.1.

To prove Theorem 2.1 it is sufficient to prove Lemma 3.2.

Proof of Lemma 3.2. According to Lemma 4.1 there exists a sequence of circular arcs

(6.5)
$$h_i = T_{c_i}^{p_i} g \in K \qquad (i = 1, 2, \cdots)$$

whose point set union is everywhere dense in E_2.

According to Lemma 6.1, if k_1 is sufficiently large the arc

$$H_1 = T_{c_1}^{p_1} G_{k_1}$$

is an involute mated to h_1 as evolute, and admits an analytic representation (6.2) "belonging" to $I(1)$ and to an annulus $A(n_0, n_1)$ in which $n_0 = 1$.

Proceeding inductively, we assume that for some integer $r > 0$ there exist integers

(6.6)
$$n_0 < n_1 < n_2 < \cdots < n_r \qquad (n_0 = 1)$$

and involutes

(6.7)
$$H_i = T_{c_i}^{p_i} G_{k_i} \qquad (i = 1, \cdots, r)$$

"mated" to the respective arcs h_i as evolutes, and admitting analytic representations of the form (6.2), "belonging" to the respective annuli $A(n_{i-1}, n_i)$ over the corresponding intervals $I(i)$.

Corresponding to the given arc h_{r+1}, Lemma 6.1 implies the existence of a constant $k_{r+1} > 3b$ so large that the involute

$$H_{r+1} = T_{c_{r+1}}^{p_{r+1}} G_{k_{r+1}}$$

"mated" to h_{r+1} is included in the annulus $A(n_r, n_{r+1})$ for some choice of $n_{r+1} > n_r$. According to Lemma 6.1 H_{r+1} has a unique analytic polar coordinate representation of form (6.2) "belonging" to $A(n_r, n_{r+1})$ over $I(r+1)$. Thus there exist integers

(6.8)
$$n_0 < n_1 < n_2 < \cdots \qquad (n_0 = 1)$$

and involutes $H_i = T_{c_i}^{p_i} G_{k_i}$ mated to the respective arcs h_i as evolutes, and "belonging" to the respective annuli $A(n_{i-1}, n_i)$ over the corresponding intervals $I(i), i > 0$.

According to Lemma 3.1, for $i > 0$, the involute H_i of h_i admits an analytic extension ξ_i in polar coordinate form "spanning" the annulus $A(n_{i-1}, n_i)$ over $I(i)$. The sequence of these extensions ξ_i defines a spiral Λ admissible in the sense of §2.

This completes the proof of Lemma 3.2.

Proof of Theorem 2.1. The set of focal points of the spiral Λ of Lemma 3.2 includes the set

$$X = h_1 \cup h_2 \cup h_3 \cup \cdots$$

and so is everywhere dense in E_2.

This establishes Theorem 2.1.

7. Proof of Theorem 1.1 and comments

Let E_2 be a 2-plane of coordinates x^1, x^2 serving as a coordinate 2-plane in a euclidean n-space $E_n, n > 2$, of coordinates $x^1, \cdots, x^n$. Let π be the projection of E_n onto E_2 under which

$$\pi(x^1, \cdots, x^n) = (x^1, x^2).$$

Let

(7.1)
$$x^1 = \varphi^1(t), \quad x^2 = \varphi^2(t) \qquad (0 \le t < \infty)$$

be a simple regular spiral Λ in E_2 whose centers of curvature in E_2 are everywhere dense. Such a curve exists by virtue of Theorem 2.1. We suppose that t is the arc length on the spiral Λ, measured from the initial point of Λ.

The manifold M_1. Let M_1 be the regular manifold in E_2 obtained by restricting the representation (7.1) to the open interval $0 < t < \infty$. Concerning M_1 we shall prove the following lemma.

Lemma 7.1. *The antecedent Γ_{n-1} in E_n of M_1 under the projection π is representable as a regular $(n-1)$-manifold M_{n-1} in E_n of class C^∞. The focal points of M_{n-1} are everywhere dense in E_n.*

The set Γ_{n-1}, with a topology induced by that of E_n, is the image X of a single "regular presentation" $(F : V, X)$ in E_n in which V is the open subset of the euclidean space E_{n-1} of coordinates $v^1, \cdots, v^{n-1}$ on which $v^1 > 0$. We define the presentation $v \to F(v)$: $V \to X$ by setting

(7.2)
$$\begin{aligned} F^i(v) &= \varphi^i(\mathrm{v}^1) && (i = 1, 2) \\ F^j(v) &= v^{j-1} && (j = 3, 4, \cdots, \mathrm{n}). \end{aligned}$$

The resulting presentation of X is regular and of class C^∞.

A "focal mapping" based on M_{n-1} and representing *all* normals to M_{n-1} exists, with $R \times V$ the domain of the parameters (s, v), and with the form

(7.3)
$$\begin{aligned} &(s, v) \to F^1(v) + s\dot{\varphi}^2(v^1) = x^1 \\ &(s, v) \to F^2(v) - s\dot{\varphi}^1(v^1) = x^2 && ((s, v) \in R \times V) \\ &(s, v) \to F^j(v) = x^j, \end{aligned}$$

where j has the range $3, \cdots, n$. Setting $v^1 = t$, the Jacobian

$$(7.4) \qquad J(s,v) = \frac{D(x^1, \cdots, x^n)}{D(s, v^1, \cdots, v^{n-1})} = \begin{vmatrix} \dot\varphi^2(t), \dot\varphi^1(t) + s\ddot\varphi^2(t) \\ -\dot\varphi^1(t), \dot\varphi^2(t) - s\ddot\varphi^1(t) \end{vmatrix}$$

evaluated for arbitary $(s,v) \in R \times V$, reduces to $1 + sk(t)$ where

$$k(t) = \dot\varphi^1(t)\ddot\varphi^2(t) - \dot\varphi^2(t)\ddot\varphi^1(t).$$

The relation $J(s,v) = 1 + sk(t)$ and the form of (7.3) show that the focal points of M_{n-1} are the antecedents under π of the centers of curvature of M_1 in E_2 and so are everywhere dense in E_n.

This establishes Lemma 7.1 as well as Theorem 1.1.

Comments on the role of non-degenerate (*ND*) functions. Focal points of a regular connected C^∞-manifold M_{n-1} in E_n are significant topologically largely because of their relation to ND functions on M_{n-1}. It follows from Theorem 1.3 that if $\mathbf{c}$ is not a focal point of M_{n-1}, nor on M_{n-1}, the restriction to points $x \in M_{n-1}$ of the distance $\|x - \mathbf{c}\|$ between $x \in E_n$ and $\mathbf{c}$ gives the values of a ND function f on M_{n-1}. Moreover the index of a critical point q of f is the number of centers of principal normal curvature (counted with their multiplicities) (*cf.* [2]) of M_{n-1}, based on q, and on the normal from q to $\mathbf{c}$ between q and $\mathbf{c}$.

This theorem and the following extension are special types of "index theorems" for a "critical extremal" of an integral in the variational theory under admissible boundary conditions. In the book which the author is now writing the exposition of these results will be in detail and independent of the variational theory.

The preceding theorem has the following extension.

For $0 < r < n$ let M_r be a regular C^∞-manifold in E_n and let $\mathbf{c} = (c_1, \cdots, c_n)$ be a point in E_n. Let q be a point in $M_r - \mathbf{c}$ such that the directed line $\zeta = \vec{q\mathbf{c}}$ is orthogonal to M_r at q. Let $p \to f(p)$ be the C^∞-function on $M_n - \mathbf{c}$ obtained by restricting the mapping $x \to \|x - \mathbf{c}\|$ to $M_r - \mathbf{c}$. If q is a non-degenerate critical point of f, its *index* k can be evaluated as follows.

Let P_{r+1} be the $(r+1)$-plane determined by $\mathbf{c}$ and the r-plane tangent to M_r at q. Let π be the orthogonal projection of E_n onto P_{r+1}. The projection under π of a sufficiently small open neighborhood of q, relative to M_r, will be a regular C^∞-manifold $\hat{M}_r$ in P_{r+1}.

The index k of the critical point q of f is the number of centers of principal normal curvature in P_{r+1} of $\hat{M}_r$ on ζ between q and $\mathbf{c}$ and based on q.

Frankel and Andreotti [7] have made notable use of such properties in proving a fundamental theorem of Lefschetz on affine algebraic manifolds [8].

A lacunary theorem. One of the most useful theorems on ND functions on a compact differentiable manifold M_r is as follows. If f is a ND function on M_r which has $m \geq 0$ critical points of index k, but no critical points of index $k-1$ or $k+1$, then the k-th connectivity of M_r is m. This follows from the author's inequalities between the connectivities of M_r and the numbers m_k of critical points of index k. This result is well-illustrated in [7] and in Milnor's elegant derivation of the homology groups of a complex projective space [9].

A homotopy theorem. The existence of a non-degenerate function f of class C^∞ on a compact connected C^∞ manifold M_r sharply conditions homotopy relations on M_r in a way which we shall recall.

We can suppose that f has been so modified that it has just one critical point of index 0 and one of index r (*cf.* [12]) and that its critical values are all distinct. Let $a < b$ be two critical values of f between which there are no critical values of f. Suppose that α and β are the critical points of f with the respective critical values a and b. Let h and k be the indices of α and β respectively. Set

$$\begin{aligned} A &= (p \in M_r \,|\, f(p) \leq a) \\ B &= (p \in M_r \,|\, f(p) \leq b). \end{aligned} \tag{7.5}$$

There is then a *deformation* D of $B - \beta$ on itself onto A leaving A pointwise fixed and with especially simple properties which we shall describe.

A family F of f-arcs. The deformation parameter under D is t and increases from 0 to 1. When $t = 0$ each point is in its original position, and when $t = 1$ in its final position.

By an *f-arc* γ on M_r we mean a simple arc on which the value of f at a point $p \in \gamma$ is the parameter τ of p on γ. Set

$$H = (p \in M_r \,|\, a \leq f(p) \leq b). \tag{7.6}$$

There is a continuous family F of sensed f-arcs γ, with at least one arc meeting each point $p \in H$, and only one such arc, except for the points α and β of H. An f-arc in F has an initial point at the f-level b and a terminal point at the f-level a. On each f-arc of F, f decreases from b to a.

The *family* F can be so defined that it has the following characteristics.

Let c be a value such that $a < c < b$, and set

$$f^c = (p \in M_r \,|\, f(p) = c).$$

We can make the arcs $\gamma \in F$ correspond to the points $q \in f^c$ in a 1-1-manner, with q corresponding to that unique arc $\gamma \in F$ which meets q. F can then be so defined that there is a continuous map onto H

$$(q, \tau) \to F^*(q, \tau) : f^c \times \ [a, b] \to H \tag{7.7}$$

with $\gamma \in F$ defined by the "partial map" in which the parameter q of γ is fixed in (7.7). Moreover the family F can be so defined that the arcs of F which meet β and on which the parameter $f = \tau$ increases from $\boldsymbol{c}$ to b are represented by rays of a topological k-disc, with all rays distinct except for their intersection at the topological center β when $\tau = b$. Similarly the arcs of F which meet the point α and on which the parameter $s = \tau$ decreases from $\boldsymbol{c}$ to a can be represented by rays of a topological $(r - h)$-disc, with all rays distinct except that they meet in α as a topological center when $\tau = a$.

Lemma 6.1 *of* [12]. The proof of the existence of the family F follows the methods of [12], in particular, one uses the fundamental Lemma

6.1 of [12]. A Riemannian metric is thereby assigned to M_r with the following important special property. Near each critical point z of f there is a special set of local coordinates $u_1, \cdots, u_r$ of M_r in terms of which the trajectories which are orthogonal in the *Riemannian* sense to the level manifolds of f near z have representations defined by the trajectories, orthogonal in the *euclidean* sense to quadric level manifolds of a form

$$-u_1^2 - \cdots - u_s^2 + u_{s+1}^2 + \cdots + u_r^2,$$

where the origin represents z and s is the index of z. It should be clear that the global definition in [12] of a Riemannian metric on M_r is a *global* extension of the author's earlier *local* euclidean "reduction theorem".

Definition of the deformation D. Let L be the non-singular linear transformation of the real axis onto itself such that $L(b) = 0$ and $L(a) = 1$. Under D a point $p \in B - \beta$ shall remain invariant for $0 \leq t \leq L(f(p))$. If $f(p) \leq a$, p shall remain invariant for all $t \in [0,1]$. If $f(p) > a$ and $p \in B - \beta$ let T_p be the unique arc of F meeting p. For each t on the interval $1 \geq t \geq L(f(p))\, p$ shall be replaced under D at the time t by the unique point p_t on T_p such that $L(f(p_t)) = t$.

Homotopy Theorem 7.1. *The resultant deformation D is then a continuous deformation of $B - \beta$ on itself onto A, leaving A pointwise fixed, and having the special properties implied by the nature of the family F.*

In particular there is no essential difference between homotopy relations on A and on $B - \beta$. The *main problem* is how does the addition of the point β to $B - \beta$ make $B - \beta$ differ homotopically from B.

Thus Lemma 6.1 of [12] permits one to establish a fundamental basis for a study of homotopy relations on M_r. This is without use of the Whitehead theory of "homotopy equivalences".

In [5] the names of a few mathematicians are given who have contributed significantly to the critical point theory and its applications in recent years. The contribution [10] has just come to my attention. This paper bears on the Lie theory as does the work of R. Bott. The work of the Russian mathematician Vladimir Arnold [11] is of interest in this connection.

References

[1] N. Bourbaki, *Topologie général*, Hermann, Paris, 1961; Chapitre 1.

[2] M. Morse, *The foundations of the calculus of variations in the large in* m-*space* (first paper), Trans. Amer. Math. Soc. **31** (1929) 379–404.

[3] ——, *The analysis and analysis situs of regular* n-*spreads in* (n + r)-*space*, Proc. Nat. Acad. Sci. U.S.A. **13** (1927) 813–814.

[4] ——, *The foundations of a theory of the calculus of variations in the large in* m-*space* (second paper), Trans. Amer. Math. Soc. **32** (1930) 599–631.

[5] ——, *Non-degenerate functions on abstract differentiable manifolds* M_n, J. Analyse Math. (1967)

[6] W. Graustein, *Differential geometry*, The Macmillan Co., New York, 1935.

[7] A. Andreotti & T. Frankel, *The Lefschetz theorem on hyperplane sections,* Ann. of Math. **69** (1959) 713–717.

[8] S. Lefschetz, *L'analysis situs et la géométrie algébrique*, Gauthier-Villars, Paris, 1924.

[9] J. Milnor, *Morse theory*, Ann. of Math. Studies, Princeton University Press, Princeton, 1963.

[10] M. Takeuchi, *Cell decompositions and Morse equalities on certain symmetric spaces*, Proceedings of the United States-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965.

[11] V. Arnold, *Sur la courbure de Riemann des groupes de difféomorphisms,* C.R. Acad. Sci., Paris **260** (1965) 5668–5671.

[12] M. Morse, *The existence of polar non-degenerate functions on differentiable manifolds*, Ann. of Math. **71** (1960) 352–383.

INSTITUTE FOR ADVANCED STUDY