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LEFSCHETZ PENCILS ON SYMPLECTIC MANIFOLDS

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This paper is a sequel to [3], in which techniques from complex geometry were adapted to prove a general existence theorem for symplectic submanifolds of compact symplectic manifolds. These submanifolds were obtained as the zero-sets of suitable sections of complex line bundles. In the present paper we take the ideas further, developing the symplectic analogue of "pencils", generated by a pair of sections of a line bundle. Our main results are a general existence theorem for topological Lefschetz pencils (Theorem 2 below), together with an asymptotic uniqueness statement (Theorem 20). These results, along with recent work of R. Gompf ([4, Theorem 10.2.18]), go some way towards giving a topological characterisation of symplectic manifolds; more generally they, along with various further extensions, give a means of translating many questions in symplectic topology into questions about the "monodromy" of the pencil. We will leave the discussion of these further topics for future papers, and concentrate here on the proofs of the main existence theorems.

The techniques of [3] were developed further by D. Auroux [1], [2]. In particular, Auroux proved the asymptotic uniqueness of the submanifolds constructed in [3], and there are many areas of overlap between Auroux' work and that in this paper. Our approach to the proofs is to throw the main burden of work onto a transversality result (Theorem 12) for holomorphic maps, extending the corresponding result in [3]. With this result to hand, the rest of the proof is a fairly straightforward extension of the argument in [3]. It is probably possible to arrange the

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proof differently, avoiding the use of the local Theorem 12 at the cost of a more complicated global construction, but the author hopes that the local result may have interest in its own right. There are a number of natural further generalisations of these ideas which one may contemplate; the symplectic analogues of other "linear systems" in complex geometry, and it is possible that the local transversality result in this paper could be useful in treating these in a systematic way.

1. The existence theorem

Let (V, ω) be a compact symplectic manifold, of dimension $2n$. We say that a system of local complex coordinates $(z_1, \ldots, z_n)$ centred at a point x of V is compatible with ω if the symplectic form at the origin of the coordinates is a positive form of type (1,1).

Definition 1. A topological Lefschetz pencil on V consists of the following data,

1. a codimension-4 submanifold $A \subset V$,
2. a finite set of points $\{b_\lambda\} \subset V \setminus A$,
3. a smooth map $f : V \setminus A \to S^2$ whose restriction to $V \setminus A \setminus \{b_\lambda\}$ is a submersion, and with $f(b_\lambda) \neq f(b_\mu)$ for $\lambda \neq \mu$.

This data is required to conform to the following standard local models.

At a point $a \in A$ there are compatible local complex co-ordinates such that A is given by $z_1 = z_2 = 0$ and, on the complement of A in a neighbourhood of a in V, f is given by

$$(z_1, \ldots, z_n) \mapsto z_1/z_2 \in \mathbf{CP}^1 \cong S^2.$$

At a point b_λ there are compatible local complex co-ordinates in which f is represented by the non-degenerate quadratic form

$$(z_1, \ldots, z_n) \mapsto f(b_\lambda) + z_1^2 + \cdots + z_n^2.$$

Suppose we have such a topological Lefschetz pencil (V, A, f). For a point $c \in S^2$ the fibre $f^{-1}(c)$ is, strictly, a subset of $V \setminus A$, but from the local model around a point of A it follows that the closure of the fibre in V is a manifold in a neighbourhood of A. For simplicity we will just call these closures the fibres of f. Then by the definition the fibre over c is a compact submanifold of V if b does not lie in the finite set of critical values $c_\lambda = f(b_\lambda)$. If c is equal to a critical value c_λ then the fibre has precisely one singular point, modelled on the singular complex hypersurface $\sum z_\alpha^2 = 0$ in $\mathbf{C}^n$.

The main result of this paper is

Theorem 2. *Suppose that $[\omega] \in H^2(V;\mathbf{R})$ is the reduction of an integral class h. For a sufficiently large integer k there is a topological Lefschetz pencil on V whose fibres are symplectic subvarieties, homologous to k times the Poincaré dual of h.*

(Here the statement that the fibres are symplectic subvarieties can be taken to mean that the restriction of the symplectic form to the set of smooth points of each fibre is non-degenerate.)

Of course the paradigm for this result is that case where (V,ω) is a complex Kähler manifold. In this case it is a standard fact that V admits a (holomorphic) Lefschetz pencil—a meromorphic function with nondegenerate critical points. The purpose of Definition 1 is to abstract the main topological content of this classical set-up, and the overall strategy of the proof of Theorem 2 is to adapt the standard arguments from complex geometry to the framework of *almost-complex* structures.

2. Approximately holomorphic sections, and transversality

We adopt the same set-up as in [3], which we will now review. Choose a fixed Hermitian line bundle $L \to V$ with $c_1(L) = h$ and with a unitary connection on L having curvature $-i\omega$. We will consider sections of L^k, where k is a parameter throughout the discussion which will eventually be chosen to be sufficiently large. Fix an almost-complex structure J on V, compatible with ω. This differential-geometric data gives us a way to define the $(0,1)$ and $(1,0)$ parts $\overline{\partial}s$, ∂s of the covariant derivative of a section s of L^k. Of course in the classical case of a complex Kähler manifold the line bundle L is holomorphic and the sections with $\overline{\partial}s = 0$ are just the holomorphic sections. Let g_k be the Riemannian metric on V associated to the almost-complex structure and the form $k\omega$. Thus the diameter of (V,g_k) is $O(k^{1/2})$. In standard local co-ordinates on balls of g_k-diameter $O(1)$ the almost-complex structure differs from the flat model by $O(k^{-1/2})$. We can define the C_r-norms $\|s\|_{C^r}, \|\overline{\partial}s\|_{C^r}$ in the usual way, using the metric g_k, and its Levi-Civita connection. Since the geometry of the data (V, L^k, g_k) is locally bounded—in fact close to the standard model as $k \to \infty$—we get equivalent norms (uniformly with respect to k) by working with representations of the section in terms of standard local co-ordinates on V, and local trivialisations of L^k. The crux of our work is the interplay between the notions of "approximate holomorphicity" and "controlled transversality" which we pin down with the following definitions.

First, for $C > 0$, we say that a section s of L^k is *C-bounded* if

$$\|s\|_{C^3} < C \ , \ \|\overline{\partial} s\|_{C^2} < Ck^{-1/2}. \tag{3}$$

The C-bounded sections, for a suitable constant C, will serve as our substitute for the holomorphic sections in the classical case.

The discussion of transversality requires a little more space (cf. [1]). Consider a linear map $T : \mathbf{R}^p \to \mathbf{R}^q$. Define $\nu(T) \geq 0$ to be the square root of the least positive eigenvalue of $TT^* : \mathbf{R}^q \to \mathbf{R}^q$, so $\nu(T) = 0$ if and and only if T fails to be surjective. If T is surjective then $\nu(T)^{-1}$ is the minimal operator norm of a left-inverse to T. Let ϕ be a smooth map from a neighbourhood of a set $K \subset \mathbf{R}^p$ to $\mathbf{R}^q$, and y be a point in $\mathbf{R}^q$. We say that ϕ is *ϵ-transverse* to y over K if there is no point x in K where

$$|\phi(x) - y| < \epsilon \ \text{ and } \ \nu(d\phi_x) < \epsilon. \tag{4}$$

Notice that we retain a control of transversality under perturbations that are small in C^1: if ϕ is ϵ-transverse to y and $\|\psi - \phi\|_{C^1} < \epsilon/10$ (say), then ψ is $\epsilon/2$-tranverse to y. The notion generalises in an obvious way to sections of a bundle, given a conection on the bundle and metrics on the base and fibre. We say that a section σ is ϵ-transverse to 0 if there is no point where $|\sigma| < \epsilon$ and $\nu(\nabla\sigma) < \epsilon$.

Definition 5. For $\epsilon > 0$ we say that a pair (s_0, s_1) of sections of $L^k \to V$ satisfies the ϵ-transversality conditions if the following hold:

1. the section s_0 of L^k is ϵ-transverse to 0,
2. the section (s_0, s_1) of $L^k \oplus L^k$ is ϵ-transverse to 0,
3. the $(1,0)$ derivative ∂F of the complex-valued function $F = s_1/s_0$, defined on the complement in V of the zero-set W_∞ of s_0, is ϵ-transverse to 0 over $V \setminus W_\infty$.

The main result of this section is

Proposition 6. *For any $C, \epsilon > 0$, if k is sufficiently large, and (s_0, s_1) are C-bounded sections of L^k satisfying the ϵ-transversality conditions of Definition 5, then there is a topological Lefschetz pencil on V (with this value of k) satisfying the conditions of Theorem 2.*

First, we define $A \subset V$ to be the common zero-set of s_1, s_0: this is a codimension 4-submanifold by the transversality condition (2) of Definition 5. The map f required in the definition of a topological Lefschetz pencil will be obtained as a small perturbation of the map $F : V \setminus A \to \mathbf{CP}^1$ defined by the ratio $F = s_1/s_0$, which we also think of as a complex-valued function on $V \setminus W_\infty$. It follows from the transversality condition (1) of Definition 5 that F is a submersion on a neighbourhood of W_∞, and moreover that, when k is large, W_∞ is a symplectic submanifold—just as in [3]. Let $\Gamma \subset V \setminus W_\infty$ be the set of points where $|\partial F| \leq |\overline{\partial} F|$ so that, as in [3], F is a submersion with symplectic fibres away from Γ.

Lemma 7. *There is a constant $\chi > 0$, depending only on C and ϵ, such that if k is sufficiently large, then $|s_0| \geq \chi$ on Γ.*

In fact we can take $\chi = \min(\epsilon/\sqrt{2}, \epsilon^2/4\sqrt{2}C)$.

Proof of Lemma 7. Suppose that $x \in \Gamma$ and $|s_0(x)| < \epsilon/\sqrt{2}$. There are two cases to consider. First suppose that $|s_1(x)| < \epsilon/\sqrt{2}$; so we are in the regime where the ϵ-transversality of (s_0, s_1) comes into play. By condition (2) of Definition 5 there is a right inverse R to the linear map $\nabla s_0 \oplus \nabla s_1 : TV_x \to L^k_x \oplus L^k_x$ with norm less that ϵ^{-1}. Then we can write

$$s_0^2 \partial F = s_0 \partial s_1 - s_1 \partial s_0,$$
$$s_0^2 \overline{\partial} F = s_0 \overline{\partial} s_1 - s_1 \overline{\partial} s_0,$$

so the definition of Γ gives, at the point x,

$$|s_0 \partial s_1 - s_1 \partial s_0| \leq |s_0 \overline{\partial} s_1 - s_1 \overline{\partial} s_0|.$$

Now fix a Hermitian isomorphism of the fibre L^k_x with $\mathbf{C}$ and let $\xi \in TV_x$ be the tangent vector $R(1,0)$. Then, writing $\nabla s = \partial s + \overline{\partial} s$, we have

$$\partial s_0(\xi) + \overline{\partial} s_0(\xi) = 1,$$
$$\partial s_1(\xi) + \overline{\partial} s_1(\xi) = 0.$$

Then we get

$$\begin{aligned} s_1 &= s_1\partial s_0(\xi) + s_1\overline{\partial} s_0(\xi) \\ &= s_0\partial s_1(\xi) + (s_1\partial s_0 - s_0\partial s_1)(\xi) + s_1\overline{\partial} s_0(\xi) \\ &= -s_0\overline{\partial} s_1(\xi) + (s_1\partial s_0 - s_0\partial s_1)(\xi) + s_1\overline{\partial} s_0(\xi). \end{aligned}$$

Hence

$$\begin{aligned} |s_1| &\le (|s_0\overline{\partial} s_1| + |s_0\partial s_1 - s_1\partial s_0| + |s_1\overline{\partial} s_0|)|\xi| \\ &\le (|s_0\overline{\partial} s_1| + |s_0\overline{\partial} s_1 - s_1\overline{\partial} s_0| + |s_1\overline{\partial} s_0|)|\xi|. \end{aligned}$$

Now, from the bound on the norm of R, we have $|\xi| \le \epsilon^{-1}$, and we also know that $|\overline{\partial} s_i| \le Ck^{-1/2}$. Putting this together, we get

$$|s_1| \le 2\epsilon^{-1}Ck^{-1/2}(|s_0| + |s_1|).$$

When k is sufficiently large, more precisely if $2\epsilon^{-1}Ck^{-1/2} < 1/2$, we deduce from this that $|s_1| < |s_0|$. But symmetrically we can also show that $|s_0| < |s_1|$, thus giving a contradiction. So in sum we see that when k is sufficiently large, there are no points of Γ with $|s_0|, |s_1| < \epsilon/\sqrt{2}$.

So now suppose that x is a point in Γ at which $|s_0| \le \epsilon/\sqrt{2}$ but $|s_1| > \epsilon/\sqrt{2}$. We are in the regime where the transversality condition (1) of Definition 5 comes into play. We write the defining condition for Γ in the form

$$|\partial s_0 - \frac{s_0}{s_1}\partial s_1| \le |\overline{\partial} s_0 - \frac{s_0}{s_1}\overline{\partial} s_1|.$$

This gives

$$\begin{aligned} |\partial s_0| &\le |\frac{s_0}{s_1}|\ |\partial s_1| + |\overline{\partial} s_0| + |\frac{s_0}{s_1}|\ |\overline{\partial} s_1| \\ &\le \sqrt{2}C\epsilon^{-1}|s_0| + Ck^{-1/2} + \sqrt{2}C\epsilon^{-1}k^{-1/2}|s_0|, \end{aligned}$$

and so

$$|\nabla s_0| \le 2\sqrt{2}C\epsilon^{-1}|s_0| + 2Ck^{-1/2}.$$

But we know by the condition (1) of Definition 5 that $|\nabla s_0| \ge \epsilon$, so we get

$$|s_0| \ge \frac{\epsilon^2}{2\sqrt{2}C} - \frac{\epsilon}{\sqrt{2}}k^{-1/2}.$$

Thus if k is sufficiently large, so that $\frac{\epsilon^2}{4C} > \epsilon k^{-1/2}$ say, we get a lower bound, $|s_0| \ge \frac{\epsilon^2}{4\sqrt{2}C}$. In conclusion we have shown that, when k is sufficiently large, $|s_0|$ is bounded below on Γ by the minimum of $\epsilon/\sqrt{2}$ and $\frac{\epsilon^2}{4\sqrt{2}C}$.

With this value of $\chi = \chi(C, \epsilon)$, define Ω_χ to be the set of points in V where $|s_0| > \chi/2$. Thus if $\rho \leq \chi/2C$, the ρ-neighbourhood of Γ is contained in Ω_χ, hence Γ is compact. The lower bound on the denominator s_0 gives uniform estimates

$$\|F\|_{C^3} \leq C', \|\overline{\partial}F\|_{C^2} \leq C'k^{-1/2},$$

over Ω_χ, for some C' depending on C, ϵ. Let Δ be the set of zeros of ∂F: these form a discrete subset of $V \setminus A$ by the transversality condition (3) of Definition 5. From the definition, Δ is a subset of the compact set Γ, so Δ is finite. We know that if $\rho \leq \chi/2C$, the ρ-balls centred on the points of Δ are contained in Ω_χ. We want to show that for small enough ρ (and when k is sufficiently large, as usual) these balls are disjoint and the set Γ is contained in their union. The point here is that ρ should be independent of k. The main tool we need is an explicit form of the Inverse Function Theorem.

Lemma 8. *Let $\phi : B^n \to \mathbf{R}^n$ be a C^2 map with $\|\phi\|_{C^2} \leq c$. Suppose that the derivative $d\phi_0$ is invertible, with $\nu(d\phi_0) \geq \epsilon$. Then if $|\phi(0)| \leq \epsilon^2/10c$ there is a unique $x \in B^n$ with $|x| \leq \epsilon/4c$ and $\phi(x) = 0$.*

Proof. This is just a matter of reviewing the ordinary proof of the Inverse Function Theorem, keeping track of the constants. We leave the reader to fill in the details.

We apply this result to the maps ϕ given by the derivative ∂F, written in standard local trivialisations over ρ-balls centred on points of Γ. At a point x of Γ we have

$$|\partial F| \leq |\overline{\partial}F| \leq C'k^{-1/2},$$

so we may suppose first that k is large enough for $|\partial F| \leq \epsilon$. Thus the transversality condition (3) of Definition 5 on the ϵ-transversality of ∂F comes into force. We may further suppose that k is so large that $|\partial F| \leq c\epsilon^{-2}$, where c is the constant derived from C', depending on the C^2-norm of ∂F in the standard trivialisation. Then we can apply Lemma 8 to find a zero of ∂f close to x. In sum we have

Proposition 9.

1. *There is a constant ρ_0, independent of k, such that the balls of radius ρ_0 centred on points of Δ are disjoint and are contained in Ω_χ.*
2. *For any $\rho \leq \rho_0$, once $k > k(\rho)$ is sufficiently large, the set Γ is contained in the union of the balls of radius ρ centred on the points of Δ.*

We can now move on to the last stage of the proof. In this we modify the map F over the ρ-balls, and over a tubular neighbourhood of A, to get a new map f which satisfies all the hypotheses of Theorem 2. Consider first the points of Δ. There are two issues to address: first we want to obtain a map given by the standard quadratic model near each of these points. Second, and less substantial, we want to arrange that the images of these points are distinct in $\mathbf{C}$. Choose standard complex co-ordinates z_α centred on a point x of Δ. Let β_ρ be a standard cut-off function, with

$$\beta_\rho(x') = 1 \quad \text{if} \quad d(x,x') \leq \rho/2, \;\; \beta_\rho(x') = 0 \quad \text{if} \quad d(x,x') \geq \rho,$$

so we may suppose that $|\nabla \beta_\rho| = O(\rho^{-1})$. The derivative $\nabla(\partial F)$ of ∂F at x decomposes into complex-linear and anti-linear parts

$$\nabla(\partial F) = \partial_\nabla(\partial F) + \overline{\partial}_\nabla(\partial F).$$

The "complex Hessian" $H = \frac{1}{2}\partial_\nabla(\partial F)$ is a complex quadratic form on the tangent space of V at x. So, using the local coordinates, we may regard it as a complex valued function

$$H(z) = \sum H_{\alpha\beta} z_\alpha z_\beta$$

on a neighbourhood of x, and in particular on the ρ-ball B_ρ. Let $w = F(x) \in \mathbf{C}$. We modify F, near x to

$$f(z) = \beta_\rho(w' + H(z)) + (1 - \beta_\rho)F(z),$$

where $w' \in \mathbf{C}$.

Lemma 10. *If ρ is sufficiently small, $k > k(\rho)$ is sufficiently large, and $|w - w'|$ is sufficiently small compared with ρ, then x is the only point in B_ρ where $|\partial f| \leq |\overline{\partial} f|$.*

Proof. There are two cases; first consider a point x' with $d(x,x') < \rho/2$, so $\beta_\rho(x') = 1$. Near to x' we can write $f = w' + H$, whence

$$\partial\tilde{f} = \partial H \qquad \overline{\partial}\tilde{f} = \overline{\partial}H.$$

Recall (see the discussion in [3]) that in our local complex coordinates we can write

$$\partial = \partial_0 + \overline{\mu}\overline{\partial}_0 \ , \ \overline{\partial} = \overline{\partial}_0 + \mu\partial_0,$$

where $\partial_0, \overline{\partial}_0$ are the standard operators defined by the complex coordinates and $|\mu(z)| \leq c|z|k^{-1/2}$. Now, since H is a holomorphic function in these co-ordinates, we have $\overline{\partial}_0 H = 0$. The ϵ-transversality condition for ∂F (applied at the origin of the coordinate system) tells us that

$$|\partial H(z)| \geq \epsilon|z| - |\overline{\partial}_\nabla(\partial F)_{z=0}||z|.$$

Now the term $\overline{\partial}_\nabla\partial F(0)$ is controlled by the C^1 norm of $\overline{\partial}F$ (note that $(\partial\overline{\partial} + \overline{\partial}_\nabla\partial) = 0$ on functions, even in the non- integrable case). Hence we have

$$|\partial_0 H(z)| \geq \epsilon|z| - C'k^{-1/2}|z|.$$

Thus if $|\partial H(z)| \leq |\overline{\partial}H(z)|$ we have

$$\epsilon|z| - C'k^{-1/2}|z| \leq |\partial H(z)| \leq |\overline{\partial}H(z)| \leq C''|z|^2k^{-1/2},$$

for a suitable constant C'', and once k is suffiently large we can deduce that $z = 0$.

Now consider the points x' in the annulus containing the support of $\nabla\beta_\rho$. We can write

$$\overline{\partial}f = \overline{\partial}(\beta_\rho)(w' + H - F) + \beta_\rho\overline{\partial}H + (1-\beta)\overline{\partial}F.$$

The Taylor expansion of F about the origin, up to terms of quadratic order, has the form

$$F(z) = w + \sum l_\alpha\overline{z}_\alpha + \sum H_{\alpha\beta}z_\alpha z_\beta + \sum m_{\alpha\beta}z_\alpha\overline{z}_\beta + \sum n_{\alpha\beta}\overline{z}_\alpha\overline{z}_\beta + O(|z|^3).$$

The remainder term is uniformly controlled by the C^2 bound on F and the co-efficients $l_\alpha, m_{\alpha\beta}, n_{\alpha\beta}$ are all $O(k^{-1/2})$ from the C^2 bound on $\overline{\partial}F$. So we have

$$|w' + H - F| \leq c(\rho^3 + \rho k^{-1/2} + |w - w'|).$$

Since $|\nabla\beta_\rho| = O(\rho^{-1})$ this gives

$$|\overline{\partial} f| \leq c(\rho^2 + k^{-1/2} + |w - w'|\rho^{-1}),$$

since $|\overline{\partial} H| \leq |\mu||\partial H| \leq ck^{-1/2}$.

On the other hand in the corresponding calculation for ∂f,

$$\partial f = \partial\beta(w' + H - F) + \beta\partial H + (1-\beta)\partial F,$$

we can write $\beta\partial H + (1-\beta)\partial F = \partial H + (1-\beta)(\partial F - \partial H)$. We know that $|\partial H| \geq \frac{\epsilon\rho}{2}$ while $|\partial F - \partial H| \leq c(\rho k^{-1/2} + \rho^2)$. Putting everything together we have, on the annulus containing the support of $\nabla\beta_\rho$,

$$|\partial f| - |\overline{\partial} f| \geq \frac{\epsilon\rho}{2} - c(\rho^2 + k^{-1/2} + |w - w'|\rho^{-1}).$$

If ρ is sufficiently small, k sufficiently large and $|w-w'|$ sufficiently small compared with ρ, then the lower bound is strictly positive.

We use this lemma to modify the function F over each of the ρ-balls centred on the points of Δ to obtain a new function f. By construction f is a submersion with symplectic fibres away from the finite set of points Δ, and the function conforms to the local model required around these points (using the fact that the nondegenerate quadratic form H can be diagonalised). We can make the critical values $f(x), x \in \Delta$ distinct using the parameter w' in the construction above.

The only further issue is to modify the function to conform to the standard model of Definition 1 around a point $x \in A$. We will do this modification in a tubular neighbourhood of A contained in the complement of the set $\Omega_\chi \subset V \setminus W_\infty$, so there is no interaction with the modifications we have discussed above.

Consider the linear algebra of the situation around a point $x \in A$. The tangent space TA_x is the kernel of the $\mathbf{R}$-linear map

$$D_x = \nabla s_0 \oplus \nabla s_1 : TV_x \to L^k_x \oplus L^k_x.$$

Suppose that TA_x is a symplectic subspace of TV. Then its anhilliator TA^0_x with respect to the symplectic form is a complementary subspace, $TV_x = TA_x \oplus TA^0_x$, and D_x gives an $\mathbf{R}$-linear isomorphism from TA^0_x to the complex vector space $L^k_x \oplus L^k_x$. Another way of saying this is that D_x induces a complex structure on TA^0_x. (Notice that, at this stage, the construction has nothing to do with the almost complex structure on V.)

Lemma 11. *For a point x in A, $F = s_1/s_0$ can be represented in the standard model of Definition 1 at x if and only if TA_x is a symplectic subspace, and the restriction of ω to TA_x^0 is a positive form of type $(1,1)$ with respect to the complex structure on TA_x^0 induced by D_x.*

Proof. In one direction it is clear by inspection that if a standard co-ordinate system centred at x exists, then D_x has the property stated. In the other direction, suppose that A is a symplectic submanifold near to x. We can choose complex co-ordinates $z_2, \dots, z_{n-1}$ for A centred on x, so that

$$\omega|_A = \frac{i}{2}(dz_2 d\overline{z}_2 + \dots dz_{n-1} d\overline{z}_{n-1})$$

at the origin. Now extend the functions $z_2, \dots z_{n-1}$ to a neighbourhood of x in V, in such a way that their derivatives vanish in the direction of the complementary subspace TA^0. Next choose a local trivialisation of L^k by a section σ and define functions z_0, z_1 by $s_i = z_i \sigma$. Then $z_0, \dots, z_{n-1}$ define local co-ordinates on V in which, by construction, the symplectic form at the origin has the shape

$$\sum_{\alpha,\beta=0,1} \Omega_{\alpha\beta} dz_\alpha d\overline{z}_\beta + dz_2 d\overline{z}_2 + \dots + dz_{n-1} d\overline{z}_{n-1},$$

where $(\Omega_{\alpha\beta})$ is positive, while $F = z_1/z_0$.

The original map $F = s_1/s_0$ will not normally satisfy the hypothesis of Lemma 11 along A, since there is no reason why ω should have type $(1,1)$ on TA_x^0. However from the ϵ-transversality of (s_0, s_1), and the usual bound on $\overline{\partial} s_0, \overline{\partial} s_1$ it follows that this is approximately true, in the sense that there is a smooth family of linear maps $\tilde{D}_x : TV_x \to L_x^k \oplus L_x^k$ with

$$|(\tilde{D}_x - D_x)\xi| \leq ck^{-1/2} |D_x(\xi)|$$

and such that the hypothesis of the lemma holds with $\tilde{D}_x$ in place of D_x. Now it is straightforward to modify (s_0, s_1) in a tubular neighboorhood of width τ of A, for some τ independent of k, to get a new pair of sections $(\tilde{s}_0, \tilde{s}_1)$ with derivatives $\tilde{D}_x$. Then we define f in this tubular neighbourhood to be the ratio $\tilde{s}_1/\tilde{s}_0$.

We call a map f obtained by following the steps described above a *modification* of F.

3. Construction of the sections

Our task now is to construct sections s_0, s_1 of L^k satisfying the conditions of Proposition 6. We will make use of the following version of Sard's Theorem for holomorphic maps, which is proved in the second half of the paper. We consider a holomorphic map f from the unit ball in $\mathbf{C}^n$ to the unit ball in $\mathbf{C}^m$. Given $\eta, \epsilon > 0$ and a point $w \in \mathbf{C}^m$ we define the subset $U(f, w, \eta, \epsilon)$ of the ball $B_\eta(w)$ of radius η centred on w to consist of the points w' such that f is ϵ-transverse to w' over the interior ball $\frac{1}{2}B^{2n} \subset B^{2n} \subset \mathbf{C}^n$. As in [3] for $\delta > 0$, and an integer p we write

$$Q_p(\delta) = \log(\delta^{-1})^{-p}.$$

Theorem 12. *For any n, m and γ with $0 < \gamma < 1$ there is a $p = p(m, n, \gamma)$ such that for all maps f as above, $w \in \mathbf{C}^m$ and $0 < \eta < 1/2$; if $\epsilon = \eta Q_p(\eta)$ then there is a connected component of $U(f, w, \eta, \epsilon)$ whose volume is at least γ times the volume of $B_\eta(w)$.*

To put this in context, one knows that the set of critical values of a holomorphic map form a proper complex subvariety, so their complement is connected. Theorem 12 is an extension of this principle to the "near-critical values". Our proof of the existence of the Lefschetz pencils uses only the fact that $U(f, w, \eta, \epsilon)$ is non-empty, the refinement involving the "large" connected component is used later in the discussion of asymptotic uniqueness. In the case where $m = 1$ the first assertion is Proposition 26 of [3], and the refinement is proved by Auroux in [1]. The result for the case where $n < m$ is proved by Auroux in [2]. The case we will need in the present paper is that where $n = m$.

Of the three transversality conditions in Definition 6 that we need to achieve the third one, involving the transversality of the derivative, is the most significant. The first conditions can be achieved by applying the main theorem of [3]. The second condition can be obtained from [1], or by copying the argument we give below for condition (3). We will now review the basic construction of "approximately holomorphic" sections in [3], which we need to extend slightly in the present paper. For any point $q \in V$ we defined in [3] an approximately holomorphic section σ_q of L^k suported in a ball of g_k-radius $O(k^{1/6})$. We work in standard complex co-ordinates z_α centred at q and a standard local trivialisation of the bundle L^k, in which the connection form is

$$A = \frac{1}{4}\left(\sum z_\alpha d\overline{z}_\alpha - \overline{z}_\alpha dz_\alpha\right).$$

In this trivialisation, the section σ_q is the product

$$\sigma_q(z) = \beta_k e^{-|z|^2/4},$$

where β_k is a cut-off function, whose derivative is supported on an annulus of radius $O(k^{1/6})$. Now—going beyond the discussion in [3]— given a linear form $\pi(z) = \sum \pi_\alpha z_\alpha$ on $\mathbf{C}^n$ we may consider the section given, in these trivialisations, by

$$\sigma_{q,\pi}(z) = \pi(z)\sigma_q(z) = \sum \pi_\alpha z_\alpha e^{-|z|^2/4}\beta_k(z). \tag{13}$$

We extend $\sigma_{q,\pi}$ by zero over the remainder of V, in the fashion of [3]. The point of the construction is that the derivative $\partial\sigma_{q,\pi}$ evaluated at the origin is π. (Here we are identifying the fibre L_q^k with $\mathbf{C}$; more invariantly we should think of π as an element of $T^*V_q \otimes_{\mathbf{C}} L_q^k$.) For points x, y in V write

$$\begin{aligned} E_k(x,y) &= \exp(-d_k(x,y)^2/5) \quad \text{if} \quad d_k(x,y) < k^{1/4} \\ &= 0 \quad \text{if} \quad d_k(x,y) \geq k^{1/4}, \end{aligned}$$

where d_k is the distance function of the metric g_k.

Lemma 14. *There is a constant M, depending only on (V, ω, J), such that*

$$\begin{aligned} |\sigma_q(x)| + |\nabla^3\sigma_q(x)| &\leq M E_k(x,q), \\ |\sigma_{q,\pi}(x)| + |\nabla^3\sigma_{q,\pi}(x)| &\leq M|\pi| E_k(x,q), \\ |\overline{\partial}\sigma_q(x)| + |\nabla^2\overline{\partial}\sigma_q(x)| &\leq M k^{-1/2} E_k(x,q), \\ |\overline{\partial}\sigma_{q,\pi}(x)| + |\nabla^2\overline{\partial}\sigma_{q,\pi}(x) &\leq M k^{-1/2}|\pi| E_k(x,q). \end{aligned}$$

The proof is a straightforward extension of that in [3] which we omit.

Now suppose, following [1], [3], that we have constants $\tilde{C}, \tilde{\epsilon}$ and $\tilde{C}$-bounded sections s_0, s_1 which satisfy the transversality conditions (1),(2) of Definition 5, with $\tilde{\epsilon}$ in place of ϵ. We can now get on to the main issue, modifying the sections to achieve the ϵ-transversality of the derivative for a suitable ϵ. In fact we will only need to modify the section s_1, so s_0 is fixed from now on. Write $F = s_1/s_0$, as usual.

Lemma 15. *For a suitable $\chi = \chi(\tilde{C}, \tilde{\epsilon})$, and sufficiently large k, any point x where $|\partial F| < 2\tilde{\epsilon}$ lies in the set Ω_χ where $|s_0| > \chi$.*

The proof is almost identical to that of Lemma 7. (The value of χ here will be somewhat different from that in Lemma 7, but for simplicity we keep the same symbol. Note that, as far as the $\tilde{\epsilon}$-dependence goes, χ is $O(\tilde{\epsilon}^2)$.)

Now let x be a point in $\Omega_{\chi/2}$, r be a small number and p be a large integer (r and p will be specified below). Given a $\tilde{C}$-bounded section s_1 and a vector π in the fibre of $T^*V \otimes L^k$ at x we obtain a complex-valued function

$$F_\pi = (s_1 + \sigma_{x,\pi})/s_0,$$

over $V \setminus W_\infty$. For $\delta, r > 0$ let $U(x, s_1, \delta)$ be the subset of the δ-ball in $(T^*V \otimes L^k)_x$ consisting of the vectors π such that ∂F_π is $\delta Q_p(\delta)$-transverse to zero over the r-ball in V centred on x.

Proposition 16. *For small r and a suitable choice of p, for any $\delta < \tilde{\epsilon}$ and for sufficiently large $k > k(\delta, \tilde{C})$, the set $U(x, s_1, \delta)$ has a connected component of volume at least $3/4$ the volume of the δ-ball in $TV^* \otimes L^k$.*

Proof. As usual we work in standard co-ordinates around the point x. Let e_α be the corresponding orthonormal basis for $T^*V \otimes L^k$ and write σ_α for σ_{x,e_α}. Thus $\sigma_{x,\pi} = \sum \pi_\alpha \sigma_\alpha$, where π_α are the components of the vector $\pi = \sum \pi_\alpha e_\alpha$. Let h_α be the complex-valued function σ_α / s_0. Thus h_α vanishes at p and the derivative of h_α at x is $e_\alpha / s_0(x)$. So the derivatives ∂h_α form a basis for the cotangent space at x. It follows that if we choose r small enough (where the bound depends only on the section s_0) then the same is true over the $4r$-ball centred on x. More precisely, we may suppose that the $n \times n$ complex matrix $[\partial h_\alpha]$ formed from these partial derivatives has inverse of norm less that 10, say, over the $4r$-ball. We may also suppose that $|s_0| \geq \chi/2$, say, on this $4r$-ball.

Now write

$$G(z, \pi) = \partial F_\pi(z) = \partial F + \sum \pi_\alpha \partial h_\alpha.$$

We are regarding this as a map from a neighbourhood of $(0,0)$ in $\mathbf{C}^n \times \mathbf{C}^n$ to $\mathbf{C}^n$, using our local co-ordinates. We can also think of this as a family of maps with π as a parameter $G(z, \pi) = G_\pi(z)$. Our task is to choose π to make G_π transverse to zero by a controlled amount. To do this we follow the standard strategy to put the problem into the framework of Theorem 12. (That is, we are mimicking a standard procedure by which one deduces transversality theorems from Sard's Theorem.) Since the ∂h_α form a basis for the cotangent spaces (by the discussion of the previous paragraph) the equation $G(z, \pi) = 0$ defines π implicitly as a function $\pi = \phi(z)$ of z, for $|z| < 4r$. We get a smooth map $\phi(z)$ on the smaller $3r$-ball which is uniformly bounded in C^3. Moreover ϕ is approximately holomorphic,

$$\|\overline{\partial}_0\phi\|_{C^1} \le ck^{-1/2}.$$

Here the constant c depends on C, but is independent of k. Thus, using the argument in Lemma 28 of [3], we can find a holomorphic map $\tilde{\phi}$ with

$$\|\tilde{\phi} - \phi\|_{C^2} \le ck^{-1/2}, \tag{17}$$

over the $2r$-ball. Now, rescaling the balls, we can obviously deduce from Theorem 12 that there is a p such that for any $\theta > 0$ the set $U(\tilde{\phi}, 0, \theta, \theta Q_p(\theta))$ has a component of volume larger than 3/4-the volume of the θ ball. Now once k is large—depending on θ—and at the expense of re-choosing p, the same statement holds with ϕ in place of $\tilde{\phi}$ by (17).

The final step is to go from the transversality of ϕ to that of G_π, with appropriate estimates.

Lemma 17. *For a suitable constant c' and all $\alpha > 0$, if ϕ is α transverse to π over the r-ball, then G_π is $c'\alpha$-transverse to 0 over the same ball.*

Suppose z is a point in the r-ball with $|G_\pi(z)| = |G(z,\pi)| \le \beta$. Then there is a nearby parameter value π' with $G(z,\pi') = 0$ and $|\pi - \pi'| \le 10\beta$: this follows from the fact that G is linear in the π variable and our assumption on the inverse of the matrix $[\partial h_\alpha]$. Now by definition $\pi' = \phi(z)$, so $|\phi(z) - \pi| \le 10\beta$. If $10\beta < \alpha$ above, we know that the derivative of ϕ at z has an inverse of norm at most α^{-1}. Let us write $A_{z,\pi}, B_{z,\pi}$ for the partial derivatives of the map G in the z and π variables respectively, evaluated at a point π, z. Then the derivative of ϕ at z is the linear map

$$D\phi_z = -B_{z,\pi'}^{-1} A_{z,\pi'}.$$

Thus the operator norm $\|A_{z,\pi'}^{-1}\|$ is bounded by

$$\|D\phi_z^{-1}\|\ \|B_{z,\pi'}^{-1}\| \le c\|D\phi_z^{-1}\| \le c\alpha^{-1}.$$

The derivative of the map G_π at z is given by $A_{z,\pi}$, so we need to compare the two points $(z,\pi), (z,\pi')$. By the usual estimate on the norm of an inverse we see that if $\|A_{z,\pi} - A_{z,\pi'}\| \le \frac{1}{2}\|A_{z,\pi'}^{-1}\|^{-1}$, then

$$\|A_{z,\pi}^{-1}\| \le 2\|A_{z,\pi'}^{-1}\| \le 2c\alpha^{-1}.$$

(To see this, write

$$A_{z,\pi}^{-1} = \left(1 + A_{z,\pi'}^{-1}(A_{z,\pi} - A_{z,\pi'})\right)^{-1} A_{z,\pi'}^{-1},$$

and then use the series expansion for the inverse of $1+\alpha$.) Now the map G is bounded in C^2 so that

$$\|A_{z,\pi} - A_{z,\pi'}\| \leq c\|\pi - \pi'\| \leq 10c\beta.$$

In sum we see that G_π is β-transverse to 0 where $\beta = \min(\alpha/10, \alpha/20c^2)$.
Now Proposition 16 follows from Lemma 17 and the preceding discussion (we can change the parameter p to absorb the constant c').

Given Proposition 16, the remainder of the proof of Theorem 2, the "local to global argument" follows the lines of [3]. With r fixed as in Proposition 16 we choose for each k and any $D > 0$ a covering of V by balls B_i, $i \in I$, of g_k-radius r, centred on points x_i, such that

1. any ball which meets Ω_χ lies in $\Omega_{\chi/2}$;
2. $\sum_i E_k(x, x_i)$ is bounded by some fixed constant independent of k;
3. there is a partition of the index set I into $N = N(D)$ subsets $I_1, \ldots, I_{N(D)}$ such that the distance between balls belonging to the same subset is at least D.

To each index $i \in I$ we associate n sections $\sigma_{i,\alpha}, \alpha = 1, \ldots, n$ of L^k with

1. if the ball B_i has non-empty intersection with Ω_χ, then $\sigma_{i,\alpha} = \sigma_{x_i, e_{i,\alpha}}$ where $e_{i,\alpha}$ is arbitrary orthonormal basis for the fibre of $T^*V \otimes L^k$ at x_i.
2. if B_i does not intersect Ω_χ, then we let $\sigma_{i,\alpha}$ be identically zero.

Remark 18. Of course the sections $\sigma_{i,\alpha}$ corresponding to the balls which do not meet Ω_χ play no role in the construction. The point of setting up the construction in this way is that the covering by balls need not depend on the section s_0. Lemma 15 gives automatic transversality outside Ω_χ.

We now proceed to modify the section s_1 by adding linear combinations of the $\sigma_{i,\alpha}$, following the scheme of [3]. At stage j in the construction we add suitable multiples of the sections corresponding to the set I_j, and we achieve control of the transversality over the balls centred on the points $x_i, i \in I_j$. The upshot is that, with an arbitrarily small perturbation in the section s_1, we achieve control of the transversality over all balls which meet Ω_χ. Using Lemma 15 we obtain sections satisfying condition (3) of Definition 5, for a suitable ϵ.

4. Asymptotic uniqueness

We will now formulate the uniqueness property of our Lefschetz fibrations. Given a compatible almost-complex structure J on (V, ω), real constants C, ϵ and a positive integer k let $\mathcal{L}_k(J, C, \epsilon)$ be the set of pairs of C-bounded sections (s_0, s_1) satisfying the ϵ-transversality hypotheses of Definition 5. So far we have seen that, if k is sufficiently large, the set $\mathcal{L}_k(J, C, \epsilon)$ is non-empty and any element in it yields a topological Lefschetz pencil $f : V \setminus A \to S^2$, a modification of the ratio s_1/s_0.

Definition 19. We say that two topological Lefschetz pencils $f : V \setminus A \to S^2, f' : V \setminus A' \to S^2$ are *isotopic* if there are smooth isotopies $\Phi_t : V \to V,\ \phi_t : S^2 \to S^2$ for $t \in [0, 1]$ such that

1. Φ_0 is the identity map on V and ϕ_0 is the identity map on S^2;

2. $\Phi_1(A') = A$;

3. $f \circ \Phi_1 = \phi_1 \circ f' : V \setminus A' \to S^2$.

Theorem 20. *Suppose we are given $\epsilon, C > 0$, a pair of compatible almost-complex structures J, J' and a sequence of integers $k_i \to \infty$. Suppose for each i we have topological Lefschetz pencils $f_i : V \setminus A_i \to S^2, f'_i : V \setminus A'_i \to S^2$ which are modifications of elements of $\mathcal{L}_{k_i}(J, C, \epsilon), \mathcal{L}_{k_i}(J', C, \epsilon)$ respectively. Then, for sufficiently large i, f_i and f'_i are isotopic.*

A particularly notable case is the "classical" situation when J is integrable, so defines a Kähler structure. Then we may consider Lefschetz pencils in the usual sense, defined by a ratio of holomorphic sections. It is a standard fact that, for each k, any two such are isotopic (because the set of pencils is a complex Zariski-open subset of a Grassmannian).

Proposition 21. *If J is a compatible complex structure on (V,ω), then there are constants C,ϵ such that for large enough k there are holomorphic sections (s_0,s_1) in $\mathcal{L}_k(J,C,\epsilon)$.*

The proof of this follows closely that of Proposition 34 in [3]. One projects the approximately holomorphic sections $\sigma_p, \sigma_{\pi,p}$ to the space of genuine holomorphic sections of L^k, and then runs the whole construction above using these. We leave the details to the reader. Combining Theorem 20 with Proposition 21 we obtain one of the main results of this paper which asserts, roughly speaking, that the topology of Lefschetz pencils on complex algebraic varieties is a symplectic invariant:

Corollary 22. *Let V_1, V_2 be compact complex manifolds, and $L_i \to V_i$ be positive line bundles. Let ω_1, ω_2 be Kähler metrics on V_1, V_2 representing $c_1(L_1), c_1(L_2)$. If there is a symplectomorphism $f:(V_1,\omega_1)\to(V_2,\omega_2)$ with $f^*(c_1(L_2)) = c_1(L_1)$, then for sufficiently large k the Lefschetz pencils defined by generic homolomorphic sections of L_1^k, L_2^k are isotopic.*

More precisely, we mean that if we regard L_1 and $f^*(L_2)$ as two holomorphic line bundles over the same manifold V_1, with respect to two different complex structures, then the pencils are isotopic in the sense of Definition 19.

The essential step in the proof of Theorem 20 is a parametrised version of the main construction.

Proposition 23. *Given C,ϵ there are $\tilde{C},\tilde{\epsilon}$ such that if k is sufficiently large, and $(s_0,s_1),(s_0',s_1')$ are pairs in $\mathcal{L}_k(J,C,\epsilon)$, then there is a continuous 1-parameter family $(s_0^{(t)}, s_1^{(t)})$ in $\mathcal{L}_k(J,\tilde{C},\tilde{\epsilon})$, for $t\in[0,1]$, running from (s_0,s_1) at $t=0$ to (s_0',s_1') at $t=1$.*

The argument is essentially the same as that of Auroux in the corresponding statements in [1], [2], but we will include the crucial step here for completeness. The full argument requires a number of stages, corresponding to the stages in the existence proof in Section 3. But in each case the construction is essentially the same, so we will focus on one of them. Suppose first then that

1. $s_0 = s_0'$;

2. we have constructed a path σ_t such that for each t the pair (s_0,σ_t) is $\tilde{C}$-bounded, and that (s_0,σ_t) is $\tilde{\epsilon}$-transverse to 0, for some $\tilde{C},\tilde{\epsilon}$.

We want to change the path to achieve the third condition of Definition 5 (transversality of the derivative), with possibly different values of $\tilde{C}, \tilde{\epsilon}$. We construct the new path by following the overall scheme of Section 3, with t as a parameter. Consider a single step in this construction, over a fixed r-ball centred on $x \in V$. For suitable $\delta > 0$ and each $t \in [0,1]$ we want to choose a vector π_t in the δ-ball B_δ in the fibre of $TV^* \otimes L^k$ at x. We want to make this choice in such a way that the derivative of the ratio

$$(\sigma_t + \sigma_{x,\pi_t})/s_0$$

is $\delta Q_p(\delta)$-transverse to zero over the half-sized ball. We get a family of subsets U_t of the δ-ball, just as in Section 3, and write $U = \{(t,\pi) : \pi \in U_t\} \subset [0,1] \times B_\delta$. Now we use an elementary observation (in essence the argument of Auroux at the end of Section 2 of [1]):

Lemma 24. *Suppose $U \subset [0,1] \times B_\delta$ is a subset with the properties:*

1. *U is open;*
2. *For each t the "slice" U_t has a connected component whose volume is at least $3/4$ of the volume of B_δ;*
3. *The slices U_0, U_1 at the ends are both the whole ball B_δ.*

Then there is a continuous section $\pi_t \in U_t$, with $\pi_0 = \pi_1 = 0$.

Proof of Lemma 24. For each time t we can choose some arbitrary ρ_t in the "large" component of U_t. Since U is open, there is a connected open neighbourhood I_t of $t \in [0,1]$ such that $I_t \times \{\rho_t\}$ lies in U. Moreover we can suppose that for $s \in I_t$, ρ_t lies in the large component of U_s. (For we can choose a compact, connected, subset K of U_t which contains ρ_t and whose volume is at least 0.6 times the volume of B_δ. Then if I_t is sufficiently small the product $I_t \times K$ lies in U.) Now take a finite cover of $[0,1]$ by a collection of these intervals. We achieve a finite subdivision

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1,$$

and a collection of vectors $\rho_0, \ldots, \rho_{N-1}$ such that ρ_i lies in U_t for all parameters t in an open neighbourhood of $[t_i, t_{i+1}]$. At a parameter value t_i, for $i = 1, \ldots, N-1$, we have two points ρ_{i-1}, ρ_i in the large component of U_{t_i} and we join these by a path $\gamma_i : [-1,1] \to U_{t_i}$. Then for sufficiently small α the path $[t_i - \alpha, t_i + \alpha] \mapsto [0,1] \times B_\delta$ defined by

$t \mapsto (t, \gamma_i((t-t_i)/\alpha)$ maps into U. We construct the desired section by piecing together these paths and the constant paths $t \to (t, \rho_i)$.

We can apply this Lemma in our situation. The first hypothesis is satisfied by continuity in the t-parameter. The second follows from Prop. 16, for suitable choice of δ, and the third is achieved by choosing δ small compared with ϵ. Then we may extend the whole argument of Section 3 to these 1-parameter families.

We should now return to discuss the hypotheses (1),(2) we have assumed above. In the case where $s_0 = s_0'$ we can construct a path σ_t satisfying (2) by repeating the discussion above to obtain the transversality condition in the family (Indeed, this is precisely the situation considered by Auroux in the main result of [1], so we may quote this part of the construction directly from that reference.) If we have general pairs $(s_0, s_1), (s_0', s_1')$ we can first construct a path $s_0^{(t)}$ from s_0 to s_0' which is $\tilde{C}$-bounded and $\tilde{\epsilon}$-transverse. For each t we can construct a companion $\tau^{(t)}$ such that $(s_0^{(t)}, \tau^{(t)}) \in \mathcal{L}_k(\tilde{C}, \tilde{\epsilon})$ once k is sufficiently large. The crucial thing here is that all constants are uniform in t; see Remark 18. The argument now is similar to the proof of Lemma 24. For each t there is, by continuity, a small neighbourhood I_t such that $(s_0(t'), \tau(t)) \in \mathcal{L}_k(\tilde{C}, \tilde{\epsilon})$ for $t' \in I_t$. We choose a finite cover of $[0,1]$ by these intervals. At a point t' of overlap between two of these intervals we have two pairs

$$(s_0(t'), \tau_{t_i}), (s_0(t'), \tau_{t_{i+1}}),$$

say. We can apply our result above to construct a path between these pairs and then construct a path $s_1(t)$ by piecing these together. The main thing to note is that we can achieve a uniform estimate $(s_0^{(t)}, s_1^{(t)}) \in \mathcal{L}_k(J, \tilde{C}, \tilde{\epsilon})$ for $k > k(C, \epsilon)$, independent of the number of intervals needed to cover $[0,1]$.

Proof of Theorem 20. Assume first that the almost-complex structure J' is the same as J. We apply Prop. 23 to find a path $(s_0^{(t)}, s_1^{(t)})$ in some $\mathcal{L}_k(J, \tilde{C}, \tilde{\epsilon})$ joining (s_0, s_1) to (s_0', s_1'). We want to construct a corresponding continuous family of modifications. Here "continuous" has the following meaning. The zero sets of $s_i^{(t)}$ define a codimension-4 submanifold $\underline{A} \subset [0,1] \times V$ and we seek a map $F : ([0,1] \times V) \setminus \underline{A} \to S^2$ which restricts at each time t to a modification of s_1/s_0. There are two issues involved here, corresponding to the similar two steps in Section 2 . At each time t we have a discrete, finite set $\Delta_t \subset V \setminus A_t$ of zeros of ∂F_t, and the ρ-balls surrounding this set are disjoint for some fixed $\rho = \rho(\tilde{C}, \tilde{\epsilon})$ and large enough k. Let $\underline{\Delta} \subset [0,1] \times ([0,1] \times V) \setminus \underline{A}$ be the corresponding set, the union of a finite number of sections $b_1(t), b_2(t), \dots, b_r(t))$. There is no difficulty in carrying out the construction of Section 2 in this family, so we may deform $s_1/s_0 : ([0,1] \times V) \setminus \underline{A} \to S^2$ over a ρ-neighbourhood of $\underline{\Delta}$ to get a map f which conforms to the standard quadratic model at all critical points $b_i(t)$. The other issue is that we want the critical values $f(t, b_i(t))$ to be a set of r distinct points in $\mathbf{C}$, for each time t. By an ordinary transversality argument we can choose further generic small perturbations to achieve this. Notice that this last step depends crucially on the fact that we are considering a 1-parameter family so that generically the critical values will be distinct throughout the family.

Lemma 25. *Let $f_t : V \setminus A_t \to S^2, t \in [0,1]$ be a continuous 1-parameter family of topological Lefschetz pencils. Then the f_t are isotopic for all t.*

Since isotopy is an equivalence relation, it suffices to prove this for sufficiently small t. The proof is almost routine. First we can choose isotopies of V moving A_t to A_0, so without loss of generality we may suppose that $A_t = A_0$ for all t. Similarly we may suppose that the critical set Δ_t is independent of t, and—by applying an isotopy—of S^2 that the same is true of the critical values. (Here we use the hypothesis that the critical values are distinct.) Now around each critical point the map f_t is given in a suitable complex co-ordinate system by the standard quadratic model. But these co-ordinate systems differ by a local isotopy, so we may reduce to the case where the maps f_t are fixed (independent of t) near the critical points. Similarly for each t we can choose an identification of a tubular neighbourhood of A_0 with its normal bundle, and a complex structure on the normal bundle, so that in this neighbourhood the map f_t is given by the fibration of the normal bundle minus its zero section over the projectivised normal bundle. By the uniqueness of tubular neighbourhoods, up to isotopy, we may reduce to the case where the f_t are fixed (independent of t) in a neighbourhood of A_0. We come then to consider the case where $f_t = f_0$ outside a set $U \subset V \setminus (A_0 \cup \Delta_0)$, and f_t is a submersion on U. The time-derivative $\frac{\partial f_t}{\partial t}$ is a section of the pull-back $f_t^*(TS^2)$. We can use the orthogonal complement to the fibre to identify this with a subspace of TV, thus getting a time-dependent vector field on V, supported in U. The integral curves of this time-dependent vector field yield the desired isotopy Φ_t.

Lemma 25 and the preceding discussion completes the proof of Theorem 20 in the case where $J = J'$. Finally we go on to the general case, with different almost-complex structures. Any two compatible almost-complex structures can be joined by a 1-parameter family $J_t, t \in [0,1]$ with $J_0 = J, J_1 = J'$. By compactness of the family we can find C, ϵ such that $\mathcal{L}_k(J_t, C, \epsilon)$ is non-empty for all $k \geq k_0$ and all $t \in [0,1]$. It suffices to show that for t sufficiently close to 0 the pencil defined by a modification of any pair in $\mathcal{L}_k(J_t, C, \epsilon)$ is isotopic to f_0. But if a pair (s_0, s_1) lies in $\mathcal{L}_k(J_0, C, \epsilon)$, it also lies in $\mathcal{L}_k(J_t, C, \epsilon)$ for small enough t, and it is clear that the modifications required for the different complex structures J_t can be joined by a continuous family for small t. So our result follows by applying Lemma 25 again.

5. The local theorem

The remainder of this paper is taken up with the proof of Theorem 12. The proof depends upon various subsidiary results which we obtain in this section and the next: these are put together in Section 7 to give the proof of Theorem 12. The overall strategy is as follows. First, by truncating the Taylor series, we reduce to the case where the holomorphic map f has components which are polynomials. As the parameter η in Theorem 12 is made smaller, we need to take polynomials of higher and higher degree. Thus the main focus of the work is to obtain estimates which take account of the dependence on the degree. We show first, in this Section 5, that the set of "near critical" values of such a polynomial map is contained in a small neighbourhood of a codimension-2 real algebraic variety, whose degree we can estimate. In Section 6 we study small neighbourhoods of such varieties, and show that their complement, in a standard ball, has a component of large volume.

Consider a map $F : \mathbf{C}^n \to \mathbf{C}^m$ whose components are polynomial functions of degree at most d. Given $\epsilon > 0$, let $\Sigma_\epsilon \subset \mathbf{C}^n$ be the set

$$\Sigma_\epsilon = \{x \in B^{2n} \subset \mathbf{C}^n : \nu(\partial F_x) \leq \epsilon\},$$

and let Δ_ϵ be the image $F(\Sigma_\epsilon)$. Thus Δ_ϵ is the set of near-critical values of F, and the entire thrust of the work is to show that Δ_ϵ is "small" in a suitable sense. The main result of the present section is

Theorem 26. *There is a $p > 0$, depending only on the dimensions m, n, such that for any F, ϵ as above, there is a real-algebraic subvariety*

$A \subset \mathbf{C}^m$ of real codimension 2 and degree D such that Δ_ϵ is contained in the $C\epsilon$-neighbourhood of A, where $C, D \leq (d+1)^p$.

In the case where $m = 1$, the set A is a finite set of D points in $\mathbf{C}$, and the result is just what is used in [3] (page 689). The statement for larger values of m will be proved by induction, but to make the induction work we need to formulate a more general statement involving families of maps. Let $G : \mathbf{C}^n \times \mathbf{C}^r \to \mathbf{C}^s$ and $\Phi : \mathbf{C}^n \times \mathbf{C}^r \to \mathbf{C}^m$ be polynomial maps, with all components of degree at most d. Here we think of $\mathbf{C}^r$ as a parameter space. For $\tau \in \mathbf{C}^r$ let Z_τ be the complex algebraic variety

$$Z_\tau = \{x \in \mathbf{C}^n : G(x, \tau) = 0\}.$$

We suppose that there is a non-empty, Zariski-open set $T \subset \mathbf{C}^r$ such that the partial derivative of G with respect to x is surjective for all points in $Z_\tau, \tau \in T$. Hence Z_τ is smooth for τ in T. Define, for each τ, a map $F_\tau : Z_\tau \to \mathbf{C}^m$ by $F_\tau(x) = \Phi(x, \tau)$. For $\epsilon > 0$, and each fixed parameter value τ in T, let $\Delta_{\tau,\epsilon}$ be the set of near-critical values defined as above, for the map $F_\tau : Z_\tau \to \mathbf{C}^m$. That is:

$$\Delta_{\tau,\epsilon} = F_\tau(\{x \in Z_\tau \cap B^{2n} \subset \mathbf{C}^n : \nu(\partial F_\tau)_x < \epsilon\}),$$

where ∂F_τ is regarded as a linear map from the Zariski tangent space of Z_τ to $\mathbf{C}^m$. (To make the definition precise we must take account of possible singular points in Z_τ—although in fact these will not play any real role in our results.) Now define

$$\underline{\Delta}_\epsilon = \{(\tau, w) \in T \times \mathbf{C}^m : w \in \Delta_{\tau,\epsilon}\} \subset \mathbf{C}^r \times \mathbf{C}^m.$$

Proposition 27. *With notation as above, there is a codimension-2 real-algebraic subset $\underline{A} \subset \mathbf{C}^r \times \mathbf{C}^m$ of degree $\underline{D}$ such that $\underline{\Delta}_\epsilon$ is contained in the $\underline{C}\epsilon$-neighbourhood of $\underline{A}$, where $\underline{C}, \underline{D} \leq (d+1)^p$, for some p depending only on m, n, r, s.*

We will now prove Prop. 27, by induction on m. The first step is to establish the case where $m = 1$, with any value of r. This is a refinement of the argument on page 689 of [3], so we need to review that argument briefly. First, fix a parameter value $\tau \in T$. We are then in just the position considered in [3], except that our map F_τ is being considered to have domain the intersection of the ball B^{2n} with the smooth algebraic variety Z_τ, whereas in [3] the domain of the map is simply a ball. We consider the real-valued function $P = |\partial F_\tau|^2$ on Z_τ, and suppose for the moment that ϵ^2 is a regular value of this function and of the restriction of P to the boundary $Z_\tau \cap S^{2n-1}$. Thus the open set in Z_τ

$$M_\epsilon(\tau) = \{x \in Z_\tau \cap B^{2n} : P(x) < \epsilon^2\}$$

is the interior of a manifold-with-corners. On each path-connected component of $M_\epsilon(\tau)$ we have a path-length metric: the distance between two points is the infimum of the lengths of paths in $M_\epsilon(\tau)$ which connect them.

Lemma 28. *There is a number q, depending only on the dimension n, such that the number of connected components of $M_\epsilon(\tau)$ and the diameters of the components in the path-length metric are bounded by $(d+1)^q$.*

In the case where $Z_\tau = \mathbf{C}^n$ this is deduced from Proposition 29 in [3]. The proof in this slightly more general situation is a simple modification of the proof of that Proposition, and we leave the details to the reader.

Now consider the function $|x|^2$ on $Z_\tau \cap B^{2n}$ and on ∂M_ϵ. Let $B(\tau)$ be the union of the sets of critical points for the two cases. We expect that generically this is a finite set. In any event, each component of $M_\epsilon(\tau)$ contains a point of $B(\tau)$ in its closure (since we may minimise $|x|^2$ over the closure of the component). Integrating the derivative of F_τ along a path and using the defining condition of $M_\epsilon(\tau)$ we see that the near-critical set $\Delta_{\tau,\epsilon}$ is contained in the $C\epsilon$-neighbourhood of the image $A(\tau) = F(B(\tau))$, where C is bounded by a polymomial in d by Lemma 28. Now let τ vary over the Zariski-open set $T \subset \mathbf{C}^r$. Assume for the moment that the transversality assumptions we have made are satisfied for each parameter value. We define $\underline{A} \subset \mathbf{C} \times \mathbf{C}^r$ to be the closure of

$$\{(w,\tau) \in \mathbf{C} \times T : w \in A(\tau)\}.$$

In follows immediately from the discussion above that $\underline{\Delta}$ is contained in the $C\epsilon$-neighbourhood of $\underline{A}$.

The next task is to see that $\underline{A}$ is a codimension-2 real-algebraic subvariety of degree $\underline{D}$ where $\underline{D}$ is bounded by a polynomial function of d. Consider the critical points on Z_τ, for fixed $\tau \in T$. These critical points are defined by the usual Lagrange multiplier condition: the derivative of $|x|^2$ should be a linear combination of the derivatives of the real and imaginary parts of the components of $\Phi(-,\tau)$. Since these latter derivatives are lineary independent, by hypothesis, this is equivalent to saying that a certain $2n$ by $2s+1$ matrix of partial derivatives has rank $2s$.

We may then cover Z_τ by a fixed number of open sets, on each of which the required condition is equivalent to the vanishing of the determinants of $2(n-s)$ minors of the matrix. Thus in each of these open sets the critical set is cut out by $2(n-s)$ real polynomial equations. Similarly for the critical points on ∂M_τ. Now if we let the parameter τ vary, the critical points are described as the solutions of polynomial equations in the x and τ variables. Thus if we define

$$\underline{B}_0 = \{(\tau, x) \in \mathbf{C}^r \times \mathbf{C}^n : \tau \in T, x \in B(\tau)\},$$

then $\underline{B}_0$ is a dense open subset of a real-algebraic variety $\underline{B}$. The set $\underline{A}$ is the image of $\underline{B}$ under a polynomial map, so is also an algebraic variety. Moreover it is easy to see that the degree of $\underline{A}$ is bounded by some polynomial in d, working through the construction to keep track of the equations involved. To finish the argument we just need some discussion of general position. Up to now we have assumed that the boundaries ∂M_τ are smooth. We can easily arrange this for a single value of τ, by changing the parameter ϵ slightly, but in a family we will generically be forced to allow singularities. Similarly, by changing the origin slightly, we can arrange that the critical points are nondegenerate for a single value of τ, but not necessarily in a family. Thus we suppose that there is a (real) Zariski-open subset $T' \subset T \subset \mathbf{C}^r$ such that ∂M_τ is smooth and the critical points are non-degenerate for τ in T'. In particular the sets $A(\tau)$ are finite, for τ in T'. We change the definition of $\underline{B}_0$ above by restricting τ to lie in T', and then take the closure of this to define an algebraic variety $\underline{B}$ with image $\underline{A} \subset \mathbf{C}^r \times \mathbf{C}$. The intersection of $\underline{A}$ with a generic slice $\{\tau\} \times \mathbf{C}$, $\tau \in T'$ is finite and it follows that $\underline{A}$ has real codimension 2. On the other hand, if σ is a point in $T \setminus T'$ we can choose a sequence τ_i in T' converging to σ, and thus from the definition and an obvious continuity argument we see that the points $(\sigma, w), w \in \Delta_{\epsilon,\sigma}$ are still in the $\underline{C}\epsilon$-neighbourhood of $\underline{A}$. This completes the proof of Proposition 27 in the case where $m = 1$.

We now come to the inductive step. Suppose we have proved Prop. 27 for smaller values of m and all r. Begin by fixing a parameter value τ, so we have a polynomial map $F_\tau : Z_\tau \to \mathbf{C}^m$. At each point of Z_τ we have a linear map, the partial derivative $\partial F_\tau : TZ_\tau \to \mathbf{C}^m$. Decompose this into components $(a_1, \ldots, a_m)$ say, so the a_i are elements of the cotangent space. We have

$$|\partial F_\tau|^2 = |a_1|^2 + \cdots + |a_m|^2,$$

so if we define open sets $Z_\tau^{(i)} \subset Z_\tau$ by the condition that $|a_i| > \frac{1}{2\sqrt{m}}|\partial F_\tau|$ on $Z_\tau^{(i)}$, the sets $Z_\tau^{(i)}$ form an open cover of Z_τ. Now write $\mathbf{C}^m = \mathbf{C} \times \mathbf{C}^{m-1}$, splitting off the first component. Let $F_\tau = (f_\tau, g_\tau)$, in this decomposition, and for $u \in \mathbf{C}$ let $Z_{\tau,u} = f^{-1}(u) \subset Z_\tau$. Write $g_{\tau,u}$ for the restriction of g_τ to $Z_{\tau,u}$

Lemma 29. *If $x \in Z_{\tau,u} \cap Z_\tau^{(1)}$ and if at x*

$$|\partial f_\tau| > \epsilon \quad , \quad \nu(\partial g_{\tau,u}) > \epsilon,$$

then $\nu(\partial F_\tau) > c\epsilon$ where $c = \frac{1}{\sqrt{2}(1+2\sqrt{m})}$.

This just involves linear algebra at the point x. We consider right-inverses R_1, R_2 to the linear maps $\partial f_\tau, \partial g_{\tau,u}$ respectively. So R_2 is right inverse to the restriction of ∂g_τ to the kernel of $\partial f_\tau = a_1$. The hypotheses state that we can choose R_i with $\|R_i\| \leq \epsilon^{-1}$. Suppose (ξ, η) is a point in $\mathbf{C}^m = \mathbf{C} \times \mathbf{C}^{m-1}$. Define

$$R(\xi, \eta) = R_1(\xi) + R_2(\eta - (\partial g_\tau)(R_1(\xi))).$$

Then R is a right inverse to $\partial F_\tau = (\partial f_\tau, \partial g_\tau)$, and we need to estimate its norm. Using the hypotheses, we have

$$\|R(\xi, \eta)\| \leq \epsilon^{-1}(|\xi| + |\eta| + |\partial g_\tau R_1(\xi)|).$$

Now we use the assumption that the point x lies in $Z_\tau^{(1)}$. We have

$$\|R_1(\xi)\| \leq |a_1|^{-1}|\xi|,$$

and $|\partial g_\tau| \leq |\partial F_\tau|$. So $|\partial g_\tau R_1(\xi)| \leq 2\sqrt{m}|\xi|$. Hence

$$|R(\xi, \eta)| \leq \epsilon^{-1}(1 + 2\sqrt{m})(|\xi| + |\eta|) \leq \epsilon^{-1}\sqrt{2}(1 + 2\sqrt{m})|(\xi, \eta)|.$$

Now write $\Delta_{\tau,\epsilon}^{(i)} = F_\tau(\Sigma_{\tau,\epsilon} \cap Z_\tau^{(i)})$, and $\underline{\Delta}_\epsilon^{(i)} = \{(w, \tau) : w \in \Delta_{\tau,\epsilon}^{(i)}$ so that

$$\Delta_{\tau,\epsilon} = \Delta_{\tau,\epsilon}^{(1)} \cup \cdots \cup \Delta_{\tau,\epsilon}^{(m)},$$
$$\underline{\Delta}_\epsilon = \underline{\Delta}_\epsilon^{(1)} \cup \cdots \cup \underline{\Delta}_\epsilon^{(m)}.$$

It suffices to prove Prop. 27 with the $\underline{\Delta}_\epsilon^{(i)}$ in place of $\underline{\Delta}_\epsilon$, so we may fix attention on $\underline{\Delta}_\epsilon^{(1)}$. First apply the result with $m = 1$ to the family of maps $f_\tau : Z_\tau \to \mathbf{C}$, with the parameter ϵ replaced by $c^{-1}\epsilon$, where c is the constant in Lemma 29. We get an algebraic variety $A_1 \subset \mathbf{C} \times \mathbf{C}^r$ such that the near-critical values for the f_τ are contained in a small neighbourhood of A_1. Now consider $\mathbf{C} \times \mathbf{C}^r$ as a new parameter space, with a family of spaces $Z_{\tau,u}$ and maps $g_\tau : Z_{\tau,u} \to \mathbf{C}^{m-1}$. For a dense set $T \subset \mathbf{C} \times \mathbf{C}^r$ the $Z_{\tau,u}$ are smooth so we are in the situation considered in Prop. 27 , with r, m replaced by $r+1, m-1$. By inductive hypothesis there is a codimension-2 real-algebraic variety $A_2 \subset \mathbf{C}^{m-1} \times \mathbf{C} \times \mathbf{C}^r$ such that the near critical values for this family are contained in a small neighbourhood of A_2. Again we want to change the parameter ϵ to $c^{-1}\epsilon$. To sum up we achieve the following. Suppose a point $(u, v, \tau) \in \mathbf{C} \times \mathbf{C}^{m-1} \times \mathbf{C}^r$ is *not* in the $C.\epsilon$ neighbourhood of A_2, and (u, τ) is *not* in the $C\epsilon$-neighbourhood of A_1. Then we can apply Lemma 29 to any point $x \in Z_\tau^{(1)}$ with $F_\tau(x) = (u, v)$ to see that (u, v, τ) is not in $\Delta_\epsilon^{(1)}$. Turning this around, the set $\underline{\Delta}_\epsilon^{(1)}$ is contained in the $C\epsilon$-neighbourhood of the algebraic variety $A = \pi^{-1}(A_1) \cup A_2$, where π is the projection $\pi(u, v, \tau) = (u, \tau)$. This completes the proof of Theorem 26.

6. Neighbourhoods of real algebraic varieties

The purpose of this Section is to prove the following result about real-algebraic sets.

Proposition 30. *For each integer $N > 0$ and real number κ with $0 < \kappa < 1$ there is a $\nu = \nu(\kappa, N)$ with the following property. For any codimension-2 real-algebraic variety $A \subset \mathbf{R}^N$ of degree D and $\epsilon \leq (D+1)^{-\nu}$, the complement of the ϵ-neighbourhood A_ϵ in the ball B^N has a conected component whose volume exceeds $\kappa\, Vol\ (B^N)$.*

This will be deduced from another result of a similar nature:

Proposition 31. *For each integer N and real number $\theta > 0$ there is a $\mu = \mu(\theta, N)$ with the following property. For any real-algebraic hypersurface $A \subset \mathbf{R}^N$ of degree D and $\epsilon \leq (D+1)^{-\mu}$,*

$$Vol\ (B^N \cap A_\epsilon) \leq \theta.$$

To deduce (30) from (31) we use

Lemma 32. *For each N there is a constant c_N such that if Ω_1 and Ω_2 are disjoint open subsets of the ball $B^N \subset \mathbf{R}^N$, then there is a* *hyperplane* $H \cong \mathbf{R}^{N-1}$ *such that, if* π_H *is the orthogonal projection to* H*, then*

$$Vol\,(\Omega_1)\;\; Vol\,(\Omega_2) \leq c_N\, Vol\,\Big(\pi_H\big(B^N \setminus (\Omega_1 \cup \Omega_2)\big)\Big).$$

Proof of Lemma 32. This is equivalent to a certain isoperimetric inequality and so could be considered as a standard result, but we give a proof for completeness. We identify the sphere S^{N-1} with the set of (co-oriented) hyperplanes through the origin in $\mathbf{R}^N$ and the tangent space TS^{N-1} with pairs (x, H) where H is a co-oriented hyperplane and x is a point in H. Define a map

$$\Psi : TS^{N-1} \times \mathbf{R}^2 \to \mathbf{R}^N \times \mathbf{R}^N$$

by $\Psi((x, H), (a, b)) = (x + av_H, x + bv_H)$, where v_H is the unit normal to H. This gives a 2-1 covering map from $TS^{N-1} \times (\mathbf{R}^2 \setminus \Delta_{\mathbf{R}})$ to $\mathbf{R}^N \times \mathbf{R}^N \setminus \Delta_{\mathbf{R}^N}$, where $\Delta_{\mathbf{R}}, \Delta_{\mathbf{R}^N}$ are the diagonals. Let ρ_0 be the standard volume form on $\mathbf{R}^N \times \mathbf{R}^N$, multiplied by the characteristic function of $\Omega_1 \times \Omega_2$, and let $\rho = \Psi^*(\rho_0)$. So ρ is a bounded form on $TS^{N-1} \times \mathbf{R}^2$ and

$$2\text{Vol }(\Omega_1) \text{ Vol }(\Omega_2) = \int_{TS^{N-1} \times \mathbf{R}^2} \rho_0.$$

Now consider the projection map $p : TS^{N-1} \times \mathbf{R}^2 \to TS^{N-1}$ and the form $p_*(\rho)$ obtained by integrating over the fibres. This form is bounded, because ρ is supported in the compact set $TS^{N-1} \times [0,1]^2 \subset TS^{N-1} \times \mathbf{R}^2$, since $\Omega_i \subset B^N$. Thus we have $p_*(\rho) \leq k\sigma$, for some k depending only on N, where σ is the standard volume form on TS^{N-1}. (In fact a short calculation shows that we can take

$$k = \int_{-1}^{1} \int_{-1}^{1} (a-b)^N da db.)$$

Suppose a point $(x, H) \in TS^{N-1}$ lies in the support of $p_*(\rho)$. Then the line $\{x + tv_H : t \in \mathbf{R}\}$ in $\mathbf{R}^N$ meets both of the open sets Ω_1, Ω_2 and so must meet the complement $C = B^N \setminus (\Omega_1 \cup \Omega_2)$. In other words, x lies in the image of $\pi_H(C)$. Let $A(H)$ be the $(N-1)$-dimensional volume of $\pi_H(C)$. Then we have

$$2\text{Vol }(\Omega_1)\text{Vol }(\Omega_2) = \int_{TS^{N-1}} p_*(\rho) \leq k\text{Vol }(S^{N-1})\text{Max}_H(A(H)),$$

which is the statement in the lemma, with $C_N = k\mathrm{Vol}\,(S^{N-1})/2$.

Proof of Proposition 30 assuming Proposition 31. Let $A \subset \mathbf{R}^N$ be a codimension-2 variety, as in the statement of Proposition 30. Without loss of generality suppose that $\kappa > 1/3$, and choose θ so that

1. $2\theta \leq \frac{1-\kappa}{2}\mathrm{Vol}\,(B^N)$;
2. $c_N\theta < \frac{(1-\kappa)^2}{16}\mathrm{Vol}\,(B^N)$.

Let H be a hyperplane and A_H be the projection $\pi_H(A)$. This is a hypersurface of degree at most $D = \deg(A)$. Obviously the projection $\pi_H(A^\epsilon \cap B^N)$ is contained in the intersection of the neighbourhood A_H^ϵ and B^{N-1}, and $\mathrm{Vol}\,(A^\epsilon \cap B^N) \leq 2\mathrm{Vol}\,(\pi_H(A^\epsilon) \cap B^{N-1})$. By Proposition 31 (with $N-1$ in place of N), if $\epsilon \leq (1+D)^\mu$, then

$$\mathrm{Vol}\,(B^N \setminus A^\epsilon) \geq \mathrm{Vol}\,(B^N) - 2\mathrm{Vol}\,(A_H^\epsilon) \geq \frac{1+\kappa}{2}\mathrm{Vol}\,(B^N),$$

using condition (1). Suppose that $B^N \setminus A^\epsilon$ does not have a component of volume at least $\kappa\mathrm{Vol}\,(B^N)$. Either all components of $B^N \setminus A^\epsilon$ have volume less than $\frac{\kappa+1}{8}\mathrm{Vol}\,(B^N)$, or there is a component of volume at least $\frac{\kappa+1}{8}\mathrm{Vol}\,(B^N)$. In the first case we can (by an elementary argument) choose a finite union of components whose total volume lies between $\frac{\kappa+1}{8}\mathrm{Vol}\,(B^N)$ and $\frac{\kappa+1}{4}\mathrm{Vol}\,(B^N)$. In either case there is a decomposition

$$B^N \setminus A^\epsilon = \Omega_1 \cup \Omega_2,$$

into disjoint open sets with $\mathrm{Vol}\,(\Omega_1) \geq \frac{1-\kappa}{8}\mathrm{Vol}\,(B^N)$ and

$$\mathrm{Vol}\,(\Omega_2) \geq \min(\frac{1-\kappa}{2}, \frac{1+\kappa}{4})\mathrm{Vol}\,(B^N) = \frac{1-\kappa}{2}\mathrm{Vol}\,(B^N),$$

(since $\kappa > 1/3$). Then $\mathrm{Vol}\,(\Omega_1)\ \mathrm{Vol}\,(\Omega_2) \geq \frac{(1-\kappa)^2}{16}\mathrm{Vol}\,(B^N)$, but

$$\mathrm{Vol}\,(\pi_H(B^N \setminus (\Omega_1 \cup \Omega_2))) \leq \mathrm{Vol}\,(A_H^\epsilon \cap B^{N-1}) \leq \theta,$$

so (2) and Lemma 32 give a contradiction.

Proof of Proposition 31. The proof is by induction on the dimension N. The result is obviously true in the case where $N = 1$, and the inductive step will, in a sense, reduce to this 1-dimensional case. Let $A \subset \mathbf{R}^N$ be a hypersurface of degree D and consider a fixed projection $\pi : \mathbf{R}^N \to \mathbf{R}^{N-1}$. Let $S \subset \mathbf{R}^{N-1}$ be the *discriminant locus* of this projection. Explicitly, if A is the zero-set of a polynomial $P(x_1, \dots, x_N)$, there is a resultant polynomial $R(x_1, \dots, x_{N-1})$ which vanishes just when P, thought of as a polynomial in x_N, has a multiple root, and S is the zero-set of R. So S is an algebraic hypersurface, and one sees from the explicit formula for the resultant that the degree of S is bounded by a polynomial function of D. Now introduce an integer parameter M, to be chosen later, and for integers i between $-(M+1)$ and $(M+1)$ let S_i be the hypersurface in $\mathbf{R}^{N-1}$ defined by the equation $P(x_1, \dots, x_{N-1}, i/M) = 0$; so S_i has degree at most D. Let $X \subset \mathbf{R}^{N-1}$ be the union of all the $2M+4$ hypersurfaces $S, S_0, S_{\pm 1}, \dots S_{\pm(M+1)}$. The main step in the proof is the following elementary Lemma.

Lemma 33. *Let U be a connected component of the open set $\mathbf{R}^{N-1} \setminus X$ in $\mathbf{R}^{N-1}$. If $\epsilon < 1/M$ then*

$$A^\epsilon \cap (U \times [-1,1]) \subset (X^\epsilon \times [-1,1]) \cup \overline{H_U},$$

where H_U is the union of at most $3D$ of the regions

$$U \times (i/M, (i+1)/M) \subset U \times [0,1].$$

Here, as earlier, A^ϵ, X^ϵ are the ϵ-neighbourhoods in $\mathbf{R}^N, \mathbf{R}^{N-1}$ respectively.

Proof of Lemma 33. Pick some point b in U. There are at most D points in $\pi^{-1}(b) \subset A$, each lying in one of the regions $U \times (i/M, (i+1)/M)$. Let J_U be the union of these regions. Now define H_U to be the union of these regions plus the immediately adjacent ones. For example, if $\pi^{-1}(b)$ contains a point in $U \times (1/M, 2/M)$ we include $U \times [0, 3/M]$ in the closure $\overline{H_U}$. Now let p be a point in $A^\epsilon \cap (U \times [-1,1]$, so there is a point $q \in A$ with $|q - p| \leq \epsilon$. Suppose first that $\pi(q)$ does not lie in U. Then the line segment joining $\pi(p)$ to $\pi(q)$ must meet X, so $\pi(p)$ lies in the ϵ-neighourhood of X and $p \in X^\epsilon \times [-1,1]$. Suppose on the other hand that $\pi(q)$ does lie in U. The projection $\pi : \pi^{-1}(U) \to U$ is a covering map, so if we choose a path γ from $\pi(q)$ to b in U, there is a lifting to a path $\tilde{\gamma}$ starting at q and ending at a point of $\pi^{-1}(b)$. By construction the path $\tilde{\gamma}$ does not cross any of the hyperplanes $\{x_N = i/M\}$, so q lies in one of the same collection of regions determined by b: that is, $q \in J_U$. But, since

$\epsilon < 1/M$, the ϵ-neighbourhood of J_U is contained in $X^\epsilon \times \mathbf{R} \cup \overline{H_U}$. So we conclude that in either case p lies in $X^\epsilon \times [-1,1] \cup \overline{H_U}$ as required.

We now complete the proof of Prop. 30. Let U_α , $\alpha = 1, \dots$ be the components of $\mathbf{R}^{N-1} \setminus X$ and write $V_\alpha = U_\alpha \cap B^{N-1}$. Let

$$K_\alpha = \overline{H_{U_\alpha}} \cap (B^{N-1} \times \mathbf{R}).$$

Then

$$\mathrm{Vol}\ (K_\alpha) \leq \frac{3D}{M} \mathrm{Vol}\ (V_\alpha).$$

If $\epsilon \leq 1/M$, we get

$$A^\epsilon \cap B^N \subset A^\epsilon \cap (B^{N-1} \times [-1,1]) \subset (X^\epsilon \times [-1,1]) \cup \bigcup_\alpha K_\alpha,$$

so

$$\mathrm{Vol}\ (A^\epsilon \cap B^N) \leq 2\mathrm{Vol}\ (X^\epsilon) + \frac{3D}{M} \mathrm{Vol}\ B^{N-1},$$

since $\sum \mathrm{Vol}\ (V_\alpha) = \mathrm{Vol}\ (B^{N-1})$.

Now, given $\theta > 0$ and the degree D, we first choose M so that $\frac{3D}{M}\mathrm{Vol}\ (B^{N-1}) \leq \theta/2$. Clearly we can do this with $M = (D+1)^n$ for some $n = n(\theta, N)$. Once M is fixed the degree, D' say, of X is fixed:

$$D' = (2M+1)D + \deg(S) \leq (d+1)^{n'},$$

for some n'. We apply the inductive hypothesis to say that there is a μ' so that $\mathrm{Vol}\ (X^\epsilon \cap B^{N-1}) \leq \theta/4$ if $\epsilon < (D'+1)^{-\mu'}$. Then we have $\mathrm{Vol}\ (A^\epsilon \cap B^N) \leq \theta$ once $\epsilon \leq ((d+1)^{n'} + 1)^{-\mu'}$ and this is plainly less than $(d+1)^{-\mu}$ for a suitable μ.

7. Proof of Theorem 12

Let $f : B^{2n} \to B^{2m}$ be a holomorphic map, as in the statement of Theorem 12. Given $\epsilon > 0$ we can approximate f by a polynomial map F, truncating the Taylor series of f, and as in Lemma 27 of [3] we can arrange that $\|f - F\|_{C^1} < \epsilon/10$ over the interior ball $\frac{1}{2}B^{2n}$ with a map F of degree $d \leq c\log(\epsilon^{-1})$, for some constant c. Now it follows from the definition that for any η and w

$$U(F, w, \eta, 2\epsilon) \subset U(f, w, \eta, \epsilon).$$

We now apply Theorem 26 to the map F, replacing ϵ by 2ϵ. This tells us that the complement of $U(F, w, \eta, \epsilon)$ in the ball $B_\eta(w)$ is contained in the intersection of $B_\eta(w)$ with a $(2C+1)\epsilon$-neighbourhood of a codimension-2 real-algebraic subvariety A of degree D, where $C, D \leq (d+1)^q$. Next we apply Proposition 30 to this subvariety, but rescaling the ball $B_\eta(w)$ to the unit ball. Proposition 30 tells us that for any fixed $\gamma < 1$ there is a component of $U(F, w, \eta, 2\epsilon)$ whose volume exceeds γ times $\mathrm{Vol}(B_\eta(w)$ provided $(2C+1)\epsilon \leq (D+1)^{-\nu}\eta$. Putting all this together, $U(F, w, \eta, \epsilon)$—and so *a fortiori* $U(f, w, \eta, \epsilon)$—has the desired large component so long as

$$\eta \geq ((d+1)^q + 1)^\nu (2(d+1)^q + 1)\epsilon.$$

Clearly we can choose a p so that this occurs if $\eta \geq \epsilon Q_p(\epsilon)^{-1}$. Finally, by the discussion of the inverse function to $x/Q_p(x)$ on page 689 of [3], we see that, with a suitable choice of p, for any small η the set $U(f, w, \eta, \epsilon)$ has a large component with $\epsilon = \eta Q_p(\eta)$.

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